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Upper bound of multiplicity in Cohen-Macaulay rings of prime characteristic

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Abstract. Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and of dimension d with the embedding dimension v, type s and the Frobenius test exponent for parameter ideals $\mathrm{Fte}(R)$. We will give an upper bound for the multiplicity of Cohen-Macaulay rings in prime characteristic in terms of $\mathrm{Fte}(R), d, v$ and s. Our result extends the main results for Gorenstein rings due to Huneke and Watanabe [8].

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian commutative local ring of prime characteristic p and of dimension d. In 2015, Huneke and Watanabe [8] gave an upper bound of the multiplicity $e(R)$ of an F-pure ring R in terms of the dimension d and the embedding dimension v. Namely, Huneke and Watanabe proved that

$$e(R) \leq \binom{v}{d}$$

for any F-pure ring. If R is F-rational, the authors of [8] provided a better bound that $e(R) \leq \binom{v-1}{d-1}$ (cf. [8, Theorem 3.1]). If R is Gorenstein, the upper bound is largely reduced by the duality as follows (cf. [8, Theorem 5.1])
(1) If R is Gorenstein and F-pure then

$$e(R) \leq \begin{cases} 2\binom{v-r-1}{r} & \text{if } \dim(R) = 2r+1 \\ \binom{v-r}{r} + \binom{v-r-1}{r-1} & \text{if } \dim(R) = \quad 2r. \end{cases}$$

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(2) If R is Gorenstein and F-rational then

$$e(R) \leq \begin{cases} \binom{v-r-1}{r} + \binom{v-r-2}{r-1} & \text{if } \dim(R) = 2r+1 \\ 2\binom{v-r-1}{r-1} & \text{if } \dim(R) = 2r. \end{cases}$$

In 2019, Katzman and Zhang tried to remove the F-pure condition in Huneke-Watanabe's theorem by using the Hartshorne-Speiser-Lyubeznik number $\mathrm{HSL}(R)$. Notice that $\mathrm{HSL}(R) = 0$ if R is F-injective (e.g. R is F-pure). If R is Cohen-Macaulay, Katzman and Zhang [13, Theorem 3.1] proved the following inequality

$$e(R) \leq Q^{v-d}\binom{v}{d},$$

where $Q = p^{\mathrm{HSL}(R)}$. They also constructed examples to show that their bound is asymptotically sharp (cf. [13, Remark 3.2]). Recall that the Frobenius test exponent for parameter ideals of R, denoted by $\mathrm{Fte}(R)$, is the least integer (if exists) e satisfying that $(\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]}$ for every parameter ideal $\mathfrak{q}$, where $\mathfrak{q}^F$ is the Frobenius closure of $\mathfrak{q}$. It is asked by Katzman and Sharp that whether $\mathrm{Fte}(R) < \infty$ for every (equidimensional) local ring (cf. [12]). If R is Cohen-Macaulay then $\mathrm{Fte}(R) = \mathrm{HSL}(R)$. Moreover the question of Katzman and Sharp has affirmative answers when R is either generalized Cohen-Macaulay by [7] or F-nilpotent by [18] (see the next section for the details). Recently, we (cf. [10, Theorem 3.]) extended the result for any ring of finite Frobenius test exponent for parameter ideals. Set $Q = p^{\mathrm{Fte}(R)}$, we have:
(1) Suppose $\mathrm{Fte}(R) < \infty$. Then

$$e(R) \leq Q^{v-d}\binom{v}{d}.$$

(2) If R is F-nilpotent then

$$e(R) \leq Q^{v-d}\binom{v-1}{d-1}.$$

The main result of this paper is to give a reduced upper bound for the multiplicity of the ring when the ring is Cohen-Macaulay, that is a natural extension of Huneke and Watanabe's result in Gorenstein cases (cf. [8, Theorem 5.1]).

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of prime characteristic p with the dimension d, the embedding dimension v and the type s. Set $Q = p^{\mathrm{Fte}(R)}$. Then*

(1) We have

$$e(R) \leq \begin{cases} (s+1)Q^{v-d}\binom{v-r-1}{r} & \text{if } \dim(R) = 2r+1 \\ \frac{(s+1)}{2}Q^{v-d}\left(\binom{v-r}{r} + \binom{v-r-1}{r-1}\right) & \text{if } \dim(R) = 2r. \end{cases}$$

(2) If R is F-nilpotent then

$$e(R) \leq \begin{cases} \frac{(s+1)}{2} Q^{v-d} \left(\binom{v-r-1}{r} + \binom{v-r-2}{r-1} \right) & \text{if } \dim(R) = 2r+1 \\ (s+1) Q^{v-d} \binom{v-r-1}{r-1} & \text{if } \dim(R) = 2r. \end{cases}$$

We will prove the above theorem in the last section. In the next section we collect some useful materials.

2. Preliminaries

We firstly give the definition of the tight closure and the Frobenius closure of ideals.

Definition 2.1 ([5, 6]). *Let R have characteristic p. We denote by R° the set of elements of R that are not contained in any minimal prime ideal. Then for any ideal I of R we define*

1. *The Frobenius closure of I, $I^F = \{x \mid x^{p^e} \in I^{[p^e]}$ for some $p^e\}$, where $I^{[p^e]} = (x^{p^e} \mid x \in I)$.*
2. *The tight closure of I, $I^* = \{x \mid cx^{p^e} \in I^{[p^e]}$ for some $c \in R^\circ$ and for all $p^e \gg 0\}$.*

The Frobenius endomorphism of R induces the natural Frobenius action on local cohomology $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ for all $i \geq 0$. By a similar way, we can define the Frobenius closure and tight closure of zero submodule of local cohomology, and denote by $0^F_{H^i_{\mathfrak{m}}(R)}$ and $0^*_{H^i_{\mathfrak{m}}(R)}$ respectively.

Let I be an ideal of R. The *Frobenius test exponent* of I, denoted by $\mathrm{Fte}(I)$, is the smallest number e satisfying that $(I^F)^{[p^e]} = I^{[p^e]}$. By the Noetherianess of R, $\mathrm{Fte}(I)$ exists (and depends on I). In general, there is no upper bound for the Frobenius test exponents of all ideals in a local ring by the example of Brenner [1]. In contrast, Katzman and Sharp [12] showed the existence of a uniform bound of Frobenius test exponents if we restrict to the class of parameter ideals in a Cohen-Macaulay local ring. For any local ring $(R, \mathfrak{m})$ of prime characteristic p we define the *Frobenius test exponent for parameter ideals*, denoted by $\mathrm{Fte}(R)$, is the smallest integer e such that $(\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]}$ for every parameter ideal $\mathfrak{q}$ of R, and $\mathrm{Fte}(R) = \infty$ if we have no such integer. Katzman and Sharp raised the following question.

Question 1. *Is $\mathrm{Fte}(R)$ a finite number for any (equidimensional) local ring?*

The Frobenius test exponent for parameter ideals is closely related to an invariant defined by the Frobenius actions on the local cohomology modules $H^i_{\mathfrak{m}}(R)$, namely *the Hartshorne-Speiser-Lyubeznik number* of $H^i_{\mathfrak{m}}(R)$. The Hartshorne-Speiser-Lyubeznik number of $H^i_{\mathfrak{m}}(R)$ is a nilpotency index of Frobenius action on $H^i_{\mathfrak{m}}(R)$ and it is defined as follows

$$\mathrm{HSL}(H^i_{\mathfrak{m}}(R)) = \min\{e \mid F^e(0^F_{H^i_{\mathfrak{m}}(R)}) = 0\}.$$

By [3, Proposition 1.11] and [14, Proposition 4.4], $\mathrm{HSL}(H^i_{\mathfrak{m}}(R))$ is well defined (see also [19]). The Hartshorne-Speiser-Lyubeznik number of R is

$$\mathrm{HSL}(R) = \max\{\mathrm{HSL}(H^i_{\mathfrak{m}}(R)) \mid i = 0, \ldots, d\}.$$

Remark 2.1. 1. If R is Cohen-Macaulay then $\mathrm{Fte}(R) = \mathrm{HSL}(R)$ by Katzman and Sharp [12]. In general, we proved in [9] that $\mathrm{Fte}(R) \geq \mathrm{HSL}(R)$. Moreover, Shimomoto and Quy [17, Main Theorem B] constructed a local ring satisfying that $\mathrm{HSL}(R) = 0$, i.e. R is F-injective, but $\mathrm{Fte}(R) > 0$.

2. Huneke, Katzman, Sharp and Yao [7] gave an affirmative answer for Question 1 for generalized Cohen-Macaulay rings.

3. In 2019, Quy [18] provided a simple proof for the theorem of Huneke, Katzman, Sharp and Yao. By the same method he also proved that $\mathrm{Fte}(R) < \infty$ if R is F-nilpotent. In 2019, Maddox [15] extended this result for *generalized F-nilpotent* rings (i.e. $H^i_{\mathfrak{m}}(R)/0^F_{H^i_{\mathfrak{m}}(R)}$ has finite length for all $i < d$), and more general in [11] by us.

We next recall some classes of F-singularities mentioned in this paper.

Definition 2.2. *A local ring $(R, \mathfrak{m})$ is called F-rational if it is a homomorphic image of a Cohen-Macaulay local ring and every parameter ideal is tight closed, i.e. $\mathfrak{q}^* = \mathfrak{q}$ for all $\mathfrak{q}$.*

Definition 2.3. *A local ring $(R, \mathfrak{m})$ is called F-pure if the Frobenius endomorphism $F : R \to R, x \mapsto x^p$ is a pure homomorphism. If R is F-pure, then it is proved that every ideal I of R is Frobenius closed, i.e. $I^F = I$ for all I.*

Definition 2.4. *1. A local ring $(R, \mathfrak{m})$ is called F-injective if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective, i.e. $0^F_{H^i_{\mathfrak{m}}(R)} = 0$, for all $i \geq 0$.*

2. A local ring $(R, \mathfrak{m})$ is called F-nilpotent if the Frobenius actions on all lower local cohomologies $H^i_{\mathfrak{m}}(R)$, $i \leq d-1$, and $0^*_{H^d_{\mathfrak{m}}(R)}$ are nilpotent, i.e. $0^F_{H^i_{\mathfrak{m}}(R)} = H^i_{\mathfrak{m}}(R)$ for all $i \leq d-1$ and $0^F_{H^d_{\mathfrak{m}}(R)} = 0^*_{H^d_{\mathfrak{m}}(R)}$.

Remark 2.2. 1. It is well known that an equidimensional local ring R is F-rational if and only if it is Cohen-Macaulay and $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.

2. An excellent equidimensional local ring is F-rational if and only if it is both F-injective and F-nilpotent.

3. Suppose every parameter ideal of R is Frobenius closed. Then R is F-injective (cf. [17, Main Theorem A]). In particular, an F-pure ring is F-injective.

4. An excellent equidimensional local ring R is F-nilpotent if and only if $\mathfrak{q}^* = \mathfrak{q}^F$ for every parameter ideal $\mathfrak{q}$ (cf. [16, Theorem A]).

3. Proof of the main result

This section is devoted to prove the main result of this paper. Without loss of generality we will assume that R is complete with an infinite residue field. We recall Briançon-Skoda's Theorem (cf. [5, Theorem 5.6]) that gives a relation between the tight closure and the integral of an ideal.

Theorem 3.1 (Briançon-Skoda). *Let R be a Noetherian ring of prime characteristic p, I an ideal generated by n elements. Then for all $w \geq 0$ we have*

$$\overline{I^{n+w}} \subseteq (I^{w+1})^*.$$

Corollary 3.1. *Keeping all assumptions of Theorem 3.1, then for all $w \geq 0$ we have*

$$\overline{I^{d+w}} \subseteq (I^{w+1})^*.$$

In particular, if $w = 0$ then

$$\overline{I^d} \subseteq I^*.$$

Proof. Let J be a minimal reduction of I. We have $\mu(J) = \ell(I) \leq d$ (cf. [20, Proposition 8.3.7 and Corollary 8.3.9]) and $\overline{I^k} = \overline{J^k}$ for every positive integer k. Applying Theorem 3.1 for J,

$$\overline{I^{\ell(I)+w}} = \overline{J^{\ell(I)+w}} = \overline{J^{\mu(J)+w}} \subseteq (J^{w+1})^* \subseteq (I^{w+1})^*.$$

Thus, $\overline{I^{d+w}} \subseteq (I^{w+1})^*$. ■

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\mathfrak{q}$ a parameter ideal. We have*

(1) If R is excellent and F-nilpotent then $\overline{\mathfrak{q}^d} \subseteq \mathfrak{q}^F$.

(2) If all associated prime ideals of R are minimal then $\overline{\mathfrak{q}^{d+1}} \subseteq \mathfrak{q}^F$.

Proof. (1) By Corollary 3.1 we have $\overline{\mathfrak{q}^d} \subseteq \mathfrak{q}^*$. Moreover, $\mathfrak{q}^* = \mathfrak{q}^F$ since R is F-nilpotent. Thus the first assertion holds.
(2) Take any $x \in \overline{\mathfrak{q}^{d+1}}$, by [20, Theorem 6.8.12], there exists $c \in R^\circ$ such that $cx^N \in \mathfrak{q}^{(d+1)N}$ for all large N, so $cx^N \subseteq cR \cap \mathfrak{q}^{(d+1)N}$. By Artin-Rees Lemma, there is $k \geq 1$ such that for large N we have

$$cx^N \in cR \cap \mathfrak{q}^{(d+1)N} = \mathfrak{q}^{(d+1)N-k}(\mathfrak{q}^k \cap cR) \subseteq c\mathfrak{q}^{(d+1)N-k}.$$

Because all associated prime ideals of R are minimal, c is a non-divisior of zero so $x^N \in \mathfrak{q}^{(d+1)N-k}$ for large N. Hence, for large $N = p^e$ we have

$$x^{p^e} \in \mathfrak{q}^{(d+1)p^e-k} \subseteq \mathfrak{q}^{dp^e} \subseteq \mathfrak{q}^{[p^e]}.$$

Thus $x \in \mathfrak{q}^F$. ■

We recall the concepts of type and socle of a module (cf. [2, Section 1.2]). Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a R-finitely generated nonzero module with $\operatorname{depth}(M) = t$. Set $k = R/\mathfrak{m}$. Then the type of M is

$$\mathrm{r}(M) := \dim_k(\operatorname{Ext}^t_R(k, M)).$$

The socle of M is

$$\operatorname{Soc}(M) := (0 : \mathfrak{m})_M \cong \operatorname{Hom}_R(k, M).$$

If $\underline{x}$ is a maximal M-sequence then $\mathrm{r}(M) = \dim_k(\operatorname{Soc}(M/\underline{x}M))$.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be an Artinian local ring with infinite residual field $k = R/\mathfrak{m}$, and let M be a finitely generated module over R such that $\dim_k(\operatorname{Soc}(M)) = s$. Then $\ell_R(M) \leq s\ell_R(R)$.*

Proof. Since M is finitely generated over Artinian ring, M is an Artinian module and $\operatorname{Soc}(M) \subseteq M$ is an essential extension. We have

$$M \subseteq E_R(\operatorname{Soc}(M)).$$

Set $k = R/\mathfrak{m}$ and $E = E_R(k)$. By Matlis duality, E has finite length over R and $\ell_R(E) = \ell_R(R)$. Moreover, $\dim_k(\operatorname{Soc}(M)) = s$ so $\operatorname{Soc}(M) \cong k^s$ and $E_R(\operatorname{Soc}(M)) \cong E^s$. Thus $M \subseteq E^s$ and

$$\ell_R(M) \leq s\ell_R(E) = s\ell_R(R).$$

The proof is complete. ■

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay ring with prime characteristic p, of dimension d with the embedding dimension v and the type s. Set $Q = p^{\mathrm{Fte}(R)}$. Then we have*

(1)

$$e(R) \leq \begin{cases} (s+1)Q^{v-d}\binom{v-r-1}{r} & \text{if } \dim(R) = 2r+1 \\ \frac{(s+1)}{2}Q^{v-d}\left(\binom{v-r}{r} + \binom{v-r-1}{r-1}\right) & \text{if } \dim(R) = \quad 2r. \end{cases}$$

(2) If R is F-nilpotent then

$$e(R) \leq \begin{cases} \frac{(s+1)}{2}Q^{v-d}\left(\binom{v-r-1}{r} + \binom{v-r-2}{r-1}\right) & \text{if } \dim(R) = 2r+1 \\ (s+1)Q^{v-d}\binom{v-r-1}{r-1} & \text{if } \dim(R) = \quad 2r. \end{cases}$$

Proof. Because the proofs of two assertions are almost the same, we will only prove (2). Let $\mathfrak{q} = (x_1,\ldots,x_d)$ be a minimal reduction of $\mathfrak{m}$. Since R is F-nilpotent, $\mathrm{Fte}(R) < \infty$. By Corollary 3.2(1), $\mathfrak{m}^d \subseteq \overline{\mathfrak{m}^d} = \overline{\mathfrak{q}^d} \subseteq \mathfrak{q}^F$. On the other hand, we have $(\mathfrak{q}^F)^{[Q]} = \mathfrak{q}^{[Q]}$. Thus $(\mathfrak{m}^d)^{[Q]} \subseteq \mathfrak{q}^{[Q]}$. Set $k = R/\mathfrak{m}$, $A = R/\mathfrak{q}^{[Q]}$, $\mathfrak{n} = \mathfrak{m}/\mathfrak{q}^{[Q]}$. Then $(A,\mathfrak{n})$ is an Artinian local ring and $(\mathfrak{n}^d)^{[Q]} = 0$. Let l be arbitrary positive integer such that $l \leq d$. We have $(\mathfrak{n}^{d-l})^{[Q]} \subseteq 0 :_A (\mathfrak{n}^l)^{[Q]}$ for all $1 \leq l \leq d$. In other words, $(\mathfrak{n}^{d-l})^{[Q]} \subseteq \mathrm{Ann}_A(\mathfrak{n}^l)^{[Q]}$ and we can consider $(\mathfrak{n}^{d-l})^{[Q]}$ as a $A' := A/(\mathfrak{n}^l)^{[Q]}$-module. Moreover, R is Cohen-Macaulay so $x_1,\ldots,x_d$ and $x_1^Q,\ldots,x_d^Q$ are maximal regular sequences. We have

$$\dim_k(\mathrm{Soc}(\mathfrak{n}^{d-l})^{[Q]}) \leq \dim_k(\mathrm{Soc}(A)) = \dim_k(\mathrm{Soc}(R/\mathfrak{q}^{[Q]})) = r_R(R) = s.$$

The first inequality due to $\mathrm{Soc}(\mathfrak{n}^{d-l})^{[Q]} \subseteq \mathrm{Soc}(A)$. By Lemma 3.1, $\ell_A((\mathfrak{n}^{d-l})^{[Q]}) = \ell_{A'}((\mathfrak{n}^{d-l})^{[Q]}) \leq s\ell_{A'}(A') = s\ell_A(A')$. Thus,

$$\begin{aligned} e(R) = e(\mathfrak{m}) = e(\mathfrak{q}) = \frac{1}{Q^d}e(\mathfrak{q}^{[Q]}) &\leq \frac{1}{Q^d}\ell_R(R/\mathfrak{q}^{[Q]}) \\ &\leq \frac{1}{Q^d}\ell_A(A) \\ &\leq \frac{1}{Q^d}(\ell_A((\mathfrak{n}^{d-l})^{[Q]}) + \ell_A(A/(\mathfrak{n}^{d-l})^{[Q]})) \\ &\leq \frac{1}{Q^d}(s\ell_A(A/(\mathfrak{n}^l)^{[Q]}) + \ell_A(A/(\mathfrak{n}^{d-l})^{[Q]})). \end{aligned}$$

Extend $x_1,\ldots,x_d$ to a minimal set of generators $x_1,\ldots,x_d,y_1,\ldots,y_{v-d}$ of $\mathfrak{m}$. Then $\bar{x}_1,\ldots,\bar{x}_d$, $\bar{y}_1,\ldots,\bar{y}_{v-d}$ is a set of generators of $\mathfrak{n}$. Now $A/(\mathfrak{n}^l)^{[Q]}$ is spanned by monomials

$$\bar{x}_1^{\alpha_1}\cdots\bar{x}_d^{\alpha_d}\bar{y}_1^{\beta_1 Q+\gamma_1}\cdots\bar{y}_{v-d}^{\beta_{v-d}Q+\gamma_{v-d}},$$

where $0 \le \alpha_1, \dots, \alpha_d, \gamma_1, \dots, \gamma_{v-d} < Q$ and $0 \le \beta_1 + \dots + \beta_{v-d} < l$. The number of such monomials is less than or equal to $Q^v\binom{v-d+l-1}{l-1}$, thus

$$\ell_A(A/(\mathfrak{n}^l)^{[Q]}) \le Q^v \binom{v-d+l-1}{l-1}.$$

Similarly

$$\ell_A(A/(\mathfrak{n}^{d-l})^{[Q]}) \le Q^v \binom{v-d+d-l-1}{d-l-1}.$$

So we have

$$e(R) \le Q^{v-d}\left(s\binom{v-d+l-1}{l-1} + \binom{v-d+d-l-1}{d-l-1}\right).$$

Choosing $l = r$ if $d = 2r$, choosing $l = r$ and $l = r+1$ if $d = 2r+1$, we obtain that

$$e(R) \le \begin{cases} \frac{(s+1)}{2}Q^{v-d}\left(\binom{v-r-1}{r} + \binom{v-r-2}{r-1}\right) & \text{if } \dim(R) = 2r+1 \\ (s+1)Q^{v-d}\binom{v-r-1}{r-1} & \text{if } \dim(R) = \ \ 2r. \end{cases}$$

The proof is complete. ■

Remark 3.1. If R is Gorenstein then $r(R) = 1$, by Theorem 3.2 we have the result of Huneke and Watanabe [8, Theorem 5.1].

Example 3.3. *Let $S = \mathbb{F}_p[X, Y]$ be a polynomial ring over $\mathbb{F}_p$ with prime p, $\mathfrak{m} = (X, Y)$ maximal ideal S, $f = X^aY^a$. Set $R = S/(f)S$, then R is a Gorenstein ring of dimension $d = 1$, of embedding dimension $v = 2$ with the type $r(R) = s = 1$ and $H^0_{\mathfrak{m}}(R) = 0$.*
Next, we will find $\mathrm{Fte}(R)$. *The Čech cocomplex $\check{C}(X, Y; S)$:*

$$0 \to S \longrightarrow S_X \oplus S_Y \xrightarrow{\phi} S_{XY} \to 0,$$

where $\phi(u, v) = v - u$. For simplification we also use u and v to denote their images in the localizations of R respectively.
The set of the exponents of all monomials of S_X is

$$\{(s, r) \mid s, r \in \mathbb{Z}, r \ge 0\}.$$

The set of the exponents of all monomials of $S_X \oplus S_Y$ is

$$\{(s, r) \mid s, r \in \mathbb{Z}; s \ge 0 \text{ or } r \ge 0\}.$$

The set of the exponents of all monomials of S_{XY} is $\{(s, r) \mid s, r \in \mathbb{Z}\}$. Then

$$H^2_{\mathfrak{m}}(S) = S_{XY}/\mathrm{Im}(\phi) = \oplus_{s,r<0}\mathbb{F}_p X^s Y^r.$$

Thus, $(H^2_{\mathfrak{m}}(S))_{(s,t)} \neq 0$ *if and only if* $s < 0$ *and* $t < 0$. *From an exact sequence*

$$0 \longrightarrow S_1 := S(-a, -a) \xrightarrow{.f} S \longrightarrow R \longrightarrow 0,$$

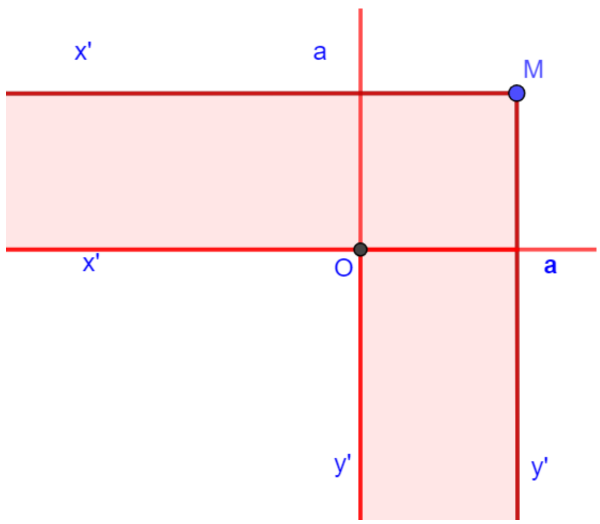
induces the following exact

$$0 = H^1_{\mathfrak{m}}(S) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(S_1) \xrightarrow{\psi} H^2_{\mathfrak{m}}(S) \to 0,$$

where ψ *is the multiplication by* f.
We have $H^1_{\mathfrak{m}}(R) = \mathrm{Ker}(\psi)$ *has the set of exponents* E *defined as follows*

$$E = \{(s, r) \mid s, r < a; s \geq 0 \text{ or } r \geq 0\}.$$

(E *is the coloring area between angle* $\widehat{x'My'}$ *and angle* $\widehat{x'Oy'}$ *including ray* Ox' *and ray* Oy'*.)*



The e-th Frobenius map $F^e : H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R)$ *corresponds to a homothety on set* E *of center* O *and ratio* $k = p^e$. *Then the set of all exponents of* $0^F_{H^1_{\mathfrak{m}}(R)}$ *is*

$$E_1 = (E \setminus (Ox' \cup Oy')) \cup O.$$

So $\mathrm{Fte}(R) = \mathrm{HSL}(R) = \lceil \log_p(a) \rceil$, *where* $\lceil u \rceil$ *is minimal integer such that greater than or equal to* u. *If we choose* $p = a = 2$, *then the Hilbert seri of* R *is*

$$H_R(t) = \frac{1 + t + t^2 + t^3}{1 - t}.$$

Thus, $e(R) = Q(1) = 4$ *where* $Q(t) = 1 + t + t^2 + t^3$ *(cf. [4, Section 6.1.1]). We have the equality in Theorem 3.2(1).*

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