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Degeneracy theorems for holomorphic mappings from a complex disc with finite growth index

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Abstract. In this paper, we prove degeneracy theorems for holomorphic mappings from a complex disc $\Delta(R) \subset \mathbb{C}$ with finite growth index into $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes in general position. We further consider the case that intersecting points of the mappings and the hyperplanes with multiplicities more than a certain number do not need to be counted. These results generalize the previous degeneracy theorems for meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$.

1. Introduction

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly non-degenerate over $\mathbb{C}$. Let d be a positive integer and $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq q).$$

For each point z, let $\nu_{H_j(f)}(z)$ denote the intersecting multiplicity of the mapping f with the hyperplane H_j at z. We consider $\mathcal{F}_{\mathcal{C}}(f, \{H_i\}_{i=1}^q, d)$ the set of all linearly non-degenerate over $\mathbb{C}$ meromorphic mappings $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ that satisfies

- $\min\{\nu_{(f,H_j)}(z), d\} = \min\{\nu_{(g,H_j)}(z), d\} \quad (1 \leq j \leq q)$,

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- $f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $d = 1$, we will say that f and g share q hyperplanes $\{H_j\}_{j=1}^{q}$ regardless of multiplicity.

Degeneracy problems find conditions to give the relation between these mappings. S. Ji [2] was the first one who proved a degeneracy theorem for three meromorphic mappings sharing $3n+1$ hyperplanes regardless of multiplicity. After that, H. Fujimoto [1] gave a degeneracy theorem for $n+2$ meromorphic mappings sharing $2n+2$ hyperplanes with multiplicities are counted to level $\frac{n(n+1)}{2} + n$. The results of L. Smiley and H. Fujimoto have been extended by many authors such as S. D. Quang - L. N. Quynh [4, 8], Q. Yan and Z. Chen [10]. In 2012, S. D. Quang [5] showed a finite theorem for meromorphic mappings sharing $2n+2$ hyperplanes regardless of multiplicity.

We note that, all the results mentioned above, the number of hyperplanes is always assumed to be at least $2n+2$. To show the relation between mappings sharing less than $2n+2$ hyperplanes, N. T. Nhung and L. N. Quynh [3] proved the following theorem.

Theorem A. *Let f be a linearly non-degenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that*

$$\dim f^{-1}(H_i) \cap f^{-1}(H_j) \leq m-2, \ \forall 1 \leq i < j \leq q.$$

Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_{\mathcal{C}}(f, \{H_i\}_{i=1}^{q}, 1)$. Assume that $q \geq \frac{n+6+\sqrt{7n^2+2n+4}}{2}$. Then there exist $[\frac{q}{2}]$ hyperplanes $H_{i_1}, \ldots, H_{i_{[\frac{q}{2}]}}$ among H_i's such that:

$$\frac{(f_1, H_{i_j})}{(f_1, H_{i_1})} = \frac{(f_2, H_{i_j})}{(f_2, H_{i_1})} \ \textit{or} \ \frac{(f_2, H_{i_j})}{(f_2, H_{i_1})} = \frac{(f_3, H_{i_j})}{(f_3, H_{i_1})} \ \textit{or} \ \frac{(f_3, H_{i_j})}{(f_3, H_{i_1})} = \frac{(f_1, H_{i_j})}{(f_1, H_{i_1})},$$

for every $j \in \{2, \ldots, [\frac{q}{2}]\}$.

For the degeneracy result related to a family of k mappings, S. D. Quang [7] proved a degeneracy theorem for a family of meromorphic mappings of a complete Kähler manifold into $\mathbb{P}^n(\mathbb{C})$. We state here his result in case mappings from $\mathbb{C}^m$.

Theorem B. *Let f be a linearly non-degenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(n \geq 2)$. Let $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general possition such that*

$$\dim f^{-1}(H_i) \cap f^{-1}(H_j) \leq m-2 \quad (1 \leq i < j \leq q).$$

Let $f_1, \ldots, f_k$ be k mappings in $\mathcal{F}_{\mathcal{C}}(f, (H_i)_{i=1}^{q}, n)$. Assume that

$$q > n+1+\frac{knq}{kn+(k-1)q-k}.$$

Then $f_1 \times \cdots \times f_k$ *is algebraic degenerate.*

Now, we turn our consideration to a case that f is a holomorphic mapping from a complex disc $\Delta(R) := \{z; |z| \leq R\} \subset \mathbb{C}$ in to $\mathbb{P}^n(\mathbb{C})$. In 2020, M. Ru and N. Sibony [9] introduced the definition of growth index for f as follows.

$$c_f = \inf \left\{ c > 0 \,\middle|\, \int_0^R \exp(cT_f(r))dr = +\infty \right\},$$

and in case $\{c > 0 \mid \int_0^R \exp(cT_f(r))dr = +\infty\} = \varnothing$, we set $c_f = +\infty$, here $T_f(r)$ is the characteristic function of f (defined in Section 2). They also studied a new class of holomorphic mappings from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$, which has finite growth index and they showed the Second Main Theorem for these ones involving hyperplanes in general position.

The goal of this paper is generalizing Theorem A and Theorem B to the case that the mappings from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ with finite growth index. In addition, we extend ours in the case that intersecting points of the mappings and the hyperplanes with multiplicities more than a certain number can be omited. Because of the difference in error term in the Second Main Theorem for holomorphic curves from $\Delta(R)$, some quantities can not be estimated like ones from $\mathbb{C}$. To overcome the difficulty, we give some new ways to evaluate these inequalities in this case.

In order to state our results, we need some notations.

For a hyperplane H in $\mathbf{P}^n(\mathbf{C})$ and a positive integer k or $k = \infty$, we set

$$\nu_{(f,H),\leqslant k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,H)}(z) > k, \\ \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \leq k, \end{cases}$$
$$\text{and } \nu_{(f,H),>k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,H)}(z) \leq k, \\ \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) > k, \end{cases}$$

for every $z \in \Delta(R)$.
Let $H_1, \ldots, H_q$ be q hyperplanes of $\mathbf{P}^n(\mathbb{C})$ in general position such that

$$\{z \mid \nu_{(f,H_i),\leq k_i} > 0\} \cap \{z \mid \nu_{(f,H_j),\leq k_j} > 0\} = \emptyset, \forall 1 \leq i < j \leq q.$$

Let $k_1, k_2, \ldots, k_q$ be q positive integers or $+\infty$ and let d be an integer. Denote by $\mathcal{F}_\Delta\left(f, \{H_i, k_i\}_{i=1}^q, d\right)$ the family of all linearly nondegenerate holomorphic curves g of $\Delta(R)$ into $\mathbf{P}^n(\mathbf{C})$ with finite growth index satisfying:

(a) $\min\left\{d, \nu_{(g,H_i),\leq k_i}\right\} = \min\left\{d, \nu_{(f,H_i),\leq k_i}\right\}, \forall i = 1, \ldots, q;$

(b) $g = f$ on $\{z \mid \nu_{(f,H_i),\leq k_i} > 0\}$.

Theorem 1.1 and Theorem 1.2 stated below are our generalizations of Theorem A and Theorem B respectively to the case of mappings with finite growth index.

Theorem 1.1. *Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ $(0 < R \leq +\infty)$ be a holomorphic mapping. Let $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that*

$$\{z \mid \nu_{(f,H_i),\leq k_i} > 0\} \cap \{z \mid \nu_{(f,H_j),\leq k_j} > 0\} = \emptyset, \forall 1 \leq i < j \leq q.$$

Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i, k_i\}_{i=1}^q, 1)$. Assume that

$$q > n + 1 + \frac{3nq}{2q + 3n - 6} + \frac{3n + 7}{4} \sum_{i=1}^{q} \frac{n}{k_i + 1} + \frac{n(n+7)}{2}\big(c_{f_1} + c_{f_2} + c_{f_3}\big).$$

and

$$q > n + 3 + \frac{n(q-2)}{q + n - 3} + \big(1 + \frac{5q}{6}\big) \sum_{i=1}^{q} \frac{n}{k_i + 1} + \frac{n(n+1)}{2}\big(c_{f_1} + c_{f_2} + c_{f_3}\big).$$

Then there exist $[\frac{q}{2}]$ hyperplanes $H_{i_1}, \ldots, H_{i_{[\frac{q}{2}]}}$ among $H_i's$ such that:

$$\frac{(f_1, H_{i_j})}{(f_1, H_{i_1})} = \frac{(f_2, H_{i_j})}{(f_2, H_{i_1})} \text{ or } \frac{(f_2, H_{i_j})}{(f_2, H_{i_1})} = \frac{(f_3, H_{i_j})}{(f_3, H_{i_1})} \text{ or } \frac{(f_3, H_{i_j})}{(f_3, H_{i_1})} = \frac{(f_1, H_{i_j})}{(f_1, H_{i_1})},$$

for every $j \in \{2, \ldots, [\frac{q}{2}]\}$.

Remark. When $R = +\infty$ and f_i is not constant, then $c_{f_i} = 0$. Letting $k_i = +\infty$ then the result of the theorem obtained if $q > n + 1 + \dfrac{3nq}{2q + 3n - 6}$ and $q > n + 3 + \dfrac{n(q-2)}{q + n - 3}$. It is easy to check that if $q \geq \frac{n+6+\sqrt{7n^2+2n+4}}{2}$ as in Theorem A, then q satisfies these inequalities. So Theorem 1.1 implies Theorem A.

Theorem 1.2. *Let f and $H_1, \ldots, H_q$ be as in Theorem 1.1. Let $f_1, \ldots, f_k$ be k mappings in $\mathcal{F}_\Delta\left(f, \{H_i, k_i\}_{i=1}^q, n\right)$. Assume that*

$$q > n + 1 + \frac{knq}{kn + (k-1)q - k} + \sum_{i=1}^{q} \frac{n}{k_i + 1} + \frac{kn(n+1)}{2} \min_{1 \leq i \leq k} \{c_{f_i}\}.$$

Then $f_1 \times \cdots \times f_k$ is algebraic degenerate.

Remark. When $R = +\infty$ and f_i is not constant, then $c_{f_i} = 0$. If we further consider a special case that $k_i = +\infty$, then we can see directly that Theorem 1.2 implies Theorem B.

2. Basic notions and auxiliary results from Nevanlinna theory

We denote by $\Delta(R)$ a disc in $\mathbb{C}$, $\Delta(R) := \{z \in \mathbb{C}; |z| < R\}$, $(0 < R \leq +\infty)$. Let ν be a divisor on $\Delta(R)$. We consider ν as a function on $\Delta(R)$ with values in $\mathbb{Z}$ such that $\mathrm{Supp}\,(\nu) := \{z; \nu(z) \neq 0\}$ is a discrete subset of $\Delta(R)$. Let k be a positive integer or $+\infty$. The truncated counting function of ν is defined by:

$$n^{[k]}(t) = \sum_{|z|\leq t} \min\{k, \nu(z)\}\ (0 \leq t \leq R) \text{ and } N^{[k]}(r,\nu) = \int_{r_0}^{r} \frac{n^{[k]}(t) - n^{[k]}(0)}{t} dt.$$

We will omit the character $^{[k]}$ if $k = +\infty$.

Let $\varphi : \Delta(R) \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function. We denote by ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ and set $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$. As usual, we will write $N^{[k]}_\varphi(r)$ and $N^{[k]}_{1/\varphi}(r)$ for $N^{[k]}(r, \nu^0_\varphi)$ and $N^{[k]}(r, \nu^\infty_\varphi)$ respectively. The proximity function of φ with respect to the point ∞ is defined by

$$m(r, \varphi) = \int_0^{2\pi} \log^+ |\varphi(re^{t\theta})| d\theta.$$

We consider φ as a holomorphic map into $\mathbb{P}^1(\mathbb{C})$ and denote by Ω_1 the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. The characteristic function of φ is defined by

$$T_\varphi(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} \varphi^* \Omega_1.$$

By Jensen's formula, we have

$$T_\varphi(r) = m(r, \varphi) + N_{1/\varphi}(r) + O(1).$$

Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with a reduced representation $(f_0 : \cdots : f_n)$, where $f_0, \ldots, f_n$ are holomorphic functions on $\Delta(R)$ without common zeros. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ defined by $H := \{(\omega_0 : \cdots : \omega_n); \sum_{i=0}^n a_i \omega_i = 0\}$, where a_i $(0 \leq i \leq n)$ are constants, not all zero. We define

$$(f, H) = \sum_{i=0}^n a_i f_i.$$

The function (f, H) depends on the choices of the reduced representation of f and the presentation of H, but the divisor $\nu_{(f,H)}$ is well-defined, not depending on these choices. The proximity function of f with respect to H is defined by

$$m_f(r, H) = \int_0^{2\pi} \log \frac{\|f\|(re^{t\theta}) \cdot \|H\|}{|(f, H)(re^{t\theta})|} d\theta,$$

where $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$ and $\|H\| = (|a_0|^2 + \cdots + |a_n|^2)^{1/2}$. The characteristic function of f (with respect to the Fubini-Study form Ω_n on $\mathbb{P}^n(\mathbb{C})$) is defined by

$$T_f(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\Omega_n.$$

The first main theorem states that

$$T_f(r) = m_f(r, H) + N_{(f,H)}(r) + O(1).$$

Theorem 2.1. *(Lemma on logarithmic derivatives [9, Theorem 5.1]). Let $0 < R \le +\infty$ and let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $f(z)$ be a meromorphic function on $\Delta(R)$. Then, for every $\varepsilon > 0$, we have the following inequality*

$$\Big\|_E \; m\left(r, \frac{f'}{f}\right) \le (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + O\left(\log T_f(r)\right).$$

Here and throughout this paper, we use the notation $\|_E P$ to say that the proposition P holds for all $r \in (0; R)$ outside a subset E of $(0; R)$ with $\int_E \gamma(r)dr < +\infty$.

Theorem 2.2. *(see [9, Theorem 1.7]). Let f be a linearly non-degenerate holomorphic map from $\Delta(R)$ $(0 < R \le +\infty)$ into $\mathbb{P}^n(\mathbb{C})$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$ and let $H_1, \ldots, H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then, for every $\varepsilon > 0$, we have*

$$\begin{aligned}\Big\|_E (q-n-1)T_f(r) \le \sum_{i=1}^{q} N^{[n]}_{(f,H_i)}(r) &+ \frac{n(n+1)}{2}(1+\varepsilon)\log\gamma(r) \\ &+ O(\log T_f(r)) + \frac{n(n+1)}{2}\varepsilon \log r.\end{aligned}$$

In case f has finite growth index (i.e., $c_f < +\infty$), then in Theorem 2.1 and Theorem 2.2, we may take $\gamma(r) = \exp\left((c_f + \varepsilon) T_f(r)\right)$.

In degeneracy theorems, Cartan's function play an essential role in our proofs. We recall here its definition as well as some necessary properties.

For meromorphic functions F, G, H on $\Delta(R)$, the Cartan's function is defined by

$$\Phi(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ \left(\frac{1}{F}\right)' & \left(\frac{1}{G}\right)' & \left(\frac{1}{H}\right)' \end{vmatrix}.$$

From the definition, we can deduce easily that

$$\Phi(F,G,H)=F.G.H\begin{vmatrix} \dfrac{1}{F}-\dfrac{1}{G} & \dfrac{1}{F}-\dfrac{1}{H} \\ \Big(\dfrac{1}{F}-\dfrac{1}{G}\Big)' & \Big(\dfrac{1}{F}-\dfrac{1}{H}\Big)' \end{vmatrix}. \tag{2.3}$$

Also, Cartan's function can be written as follows.

$$\begin{aligned}\Phi(F,G,H)=&\begin{vmatrix} F & G & H \\ 1 & 1 & 1 \\ F\Big(\dfrac{1}{F}\Big)' & G\Big(\dfrac{1}{G}\Big)' & H\Big(\dfrac{1}{H}\Big)' \end{vmatrix}\\ =&F\left(H\Big(\frac{1}{H}\Big)'-G\Big(\frac{1}{G}\Big)'\right)+G\left(F\Big(\frac{1}{F}\Big)'-H\Big(\frac{1}{H}\Big)'\right)\\ &+H\left(G\Big(\frac{1}{G}\Big)'-F\Big(\frac{1}{F}\Big)'\right).\end{aligned} \tag{2.4}$$

From the property of proximity function and Theorem 2.1, we can show the following inequality:

$$\begin{aligned}\|_E\, m(r,\Phi)\le\ & m(r,F)+m(r,G)+m(r,H)+2m(r,(1/F)'/(1/F))\\ &+2m(r,(1/G)'/(1/G))+2m(r,(1/H)'/(1/H))+O(1)\\ \le\ & m(r,F)+m(r,G)+m(r,H)+6(1+\varepsilon)\log\gamma(r)\\ &+6\varepsilon\log r+O\left(\log T_F(r)+\log T_G(r)+\log T_H(r)\right).\end{aligned} \tag{2.5}$$

In addition, for every meromorphic function h, we have

$$\Phi(hF,hG,hH)=h\cdot\Phi(F,G,H).$$

Lemma 2.6 (see [3], Lemma 3.2)**.** *If $\Phi(F,G,H)\equiv 0$ then there exist constants α_0,β_0, not all zeros, such that*

$$\alpha_0\left(\frac{1}{F}-\frac{1}{G}\right)=\beta_0\left(\frac{1}{F}-\frac{1}{H}\right).$$

3. Proof of Theorems

Proof of Theorem 1.1 For convenience, with three mappings $f_1,f_2,f_3\in\mathcal{F}_\Delta(f,\{H_i,k_i\}_{i=1}^q,1)$, we set

- $F_k^{ij}=\dfrac{(f_k,H_i)}{(f_k,H_j)}$ $(1\le k\le 3,\ 1\le i<j\le q)$;

- $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon\log r$;
- $T(r) = T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r)$;
- $T_i = \big\{z \mid \nu_{(f,H_i),\leqslant k_i}(z) > 0\big\}$;
- $S_i = \bigcup_{u=1}^3 \big\{z \mid \nu_{(f_u,H_i),>k_i}(z) > 0\big\}$;
- $R_i = \bigcap_{u=1}^3 \big\{z \mid \nu_{(f_u,H_i),>k_i}(z) > 0\big\}$.

Now we need some preparation lemmas before going into main proofs.

Lemma 3.1. *Let $g \in \mathcal{F}_\Delta(f, \{H_i, k_i\}_{i=1}^q, 1)$. Suppose that*

$$q > n + 1 + \sum_{i=1}^q \frac{n}{k_i+1} + \frac{n(n+1)}{2}(c_f + c_g).$$

Then $\big\|_E\, T_g(r) = O(T_f(r))$ and $\big\|_E\, T_f(r) = O(T_g(r))$.

Proof. By the Second Main Theorem, we have

$$\begin{aligned}
&\big\|_E\, (q-n-1)T_g(r) \le \sum_{i=1}^q N^{[n]}_{(g,H_i)}(r) + \\
&\frac{n(n+1)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + o(T_g(r)) \\
&\le \sum_{i=1}^q n\ N^{[1]}_{(g,H_i)}(r) + \frac{n(n+1)}{2}S(r) + o(T_g(r)) \\
&\le \sum_{i=1}^q n\big(N^{[1]}_{(g,H_i),\le k_i}(r) + N^{[1]}_{(g,H_i),>k_i}(r)\big) + \frac{n(n+1)}{2}S(r) + o(T_g(r)) \\
&\le \sum_{i=1}^q n\big(N^{[1]}_{(f,H_i),\le k_i}(r) + \frac{1}{k_i+1}T_g(r)\big) + \frac{n(n+1)}{2}S(r) + o(T_g(r)) \\
&\le \sum_{i=1}^q n\big(T_f(r) + \frac{1}{k_i+1}T_g(r)\big) + \frac{n(n+1)}{2}S(r) + o(T_g(r)).
\end{aligned}$$

By taking $\gamma(r) = exp((c_g+\varepsilon)T_g(r))$ and letting $\varepsilon \to 0$, we get

$$\big\|_E\, (q-n-1)T_g(r) \le \sum_{i=1}^q nT_f(r) + \sum_{i=1}^q \frac{n}{k_i+1}T_g(r) + \frac{n(n+1)}{2}c_gT_g(r) + o(T_g(r)).$$

This implies that

$$(q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{n(n+1)}{2}c_g)T_g(r)\le nqT_f(r)+o(T_g(r)).$$

Hence $\big\|_E\, T_g(r)=O(T_f(r))$. Similarly, we get $\big\|_E\, T_f(r)=O(T_g(r))$.

The following lemma is proved similarly Lemma 3.6 in [4] for the case mappings from complex discs into $P^n(\mathbb{C})$.

Lemma 3.2. *Let f_1,f_2,f_3 be three mappings in $\mathcal{F}_\Delta(f,\{H_i,k_i\}_{i=1}^q,1)$. Assume that there exist $i,j\in\{1,2,\dots,q\}$ $(i\neq j)$ such that $\Phi_{ij}:=\Phi(F_1^{ij},F_2^{ij},F_3^{ij})\not\equiv 0$. Then for every $\varepsilon>0$, the following assertions hold:*

i. We have

$$\begin{aligned}T(r)+6S(r)+o(T(r))\geqslant&\sum_{u=1}^{3}N^{(n)}_{(f_u,H_i),\leqslant k_i}(r)+\sum_{u=1}^{3}N^{(n)}_{(f_u,H_j),\leqslant k_j}(r)\\&+2\sum_{\substack{t=1\\t\neq i,j}}^{2n+2}N^{(1)}_{(f,H_t),\leqslant k_t}(r)-(2n+1)N^{(1)}_{(f,H_i),\leqslant k_i}(r)-(n+1)N^{(1)}_{(f,H_j),\leqslant k_j}(r)\\&-\sum_{u=1}^{3}\left(\left(1+\frac{n-1}{3}\right)N^{(1)}_{(f_u,H_j),>k_j}-\left(1+\frac{2n-2}{3}\right)N_{(f_u,H_i),>k_i}\right).\end{aligned}$$

ii. Furthermore, if we also have $\Phi(F_1^{ji},F_2^{ji},F_3^{ji})\not\equiv 0$ then

$$\begin{aligned}\Big\|_E\, T(r)+6S(r)+o(T(r))\geqslant\sum_{t=i,j}\Big(&\sum_{u=1}^{3}N^{[n]}_{(f_u,H_t),\leq k_t}(r)-\frac{3n+6}{2}N^{[1]}_{(f,H_t),\leq k_t}(r))\\&+2\sum_{t=1}^{q}N^{[1]}_{(f,H_t),\leq k_t}(r)-\sum_{u=1}^{3}\frac{n+1}{2}N_{(f_u,H_t),>k_t}\Big).\end{aligned}$$

Proof.

(i) From Inequality 2.5, we have

$$\Big\|_E\, m(r,\Phi_{ij})\le\sum_{v=1}^{3}m(r,F_v^{ij})+6S(r)+O\left(\sum_{u=1}^{3}\log T_{F_u^{ij}}(r)\right). \tag{3.3}$$

Therefore, we have

$$\begin{aligned}
T(r) &\geqslant \sum_{u=1}^{3} T\left(r, F_u^{ij}\right) + O(1) = \sum_{u=1}^{3}\left(m\left(r, F_u^{ij}\right) + N_{\frac{1}{F_u^{ij}}}(r)\right) + O(1)\\
&\geq m\left(r, \Phi_{ij}\right) + \sum_{u=1}^{3} N_{\frac{1}{F_u^{ij}}}(r) - 6S(r) + O\left(\sum_{u=1}^{3}\log T_{F_u^{ij}}(r)\right)\\
&\geqslant T\left(r, \Phi_{ij}\right) - N_{\frac{1}{\Phi_{ij}^{\alpha}}}(r) + \sum_{u=1}^{3} N_{\frac{1}{F_u^{ij}}}(r) - 6S(r) + O\left(\sum_{u=1}^{3}\log T_{f_u}(r)\right)\\
&\geqslant N_{\Phi_{ij}}(r) - N_{\frac{1}{\Phi}}(r) + \sum_{u=1}^{3} N_{\frac{1}{F_u^{ij}}}(r) - 6S(r) + o(T(r))\\
&= N\left(r, \nu_{\Phi_{ij}}\right) + \sum_{u=1}^{3} N_{\frac{1}{F_u^{ij}}}(r) - 6S(r) + o(T(r)).
\end{aligned}$$

On the other words, it is easy to see that

$$\begin{aligned}
&N\left(r, S_i\right) + N\left(r, S_j\right) + (2n-2)N\left(r, R_i\right) + (n-1)N\left(r, R_j\right)\\
&\leq \sum_{u=1}^{3}\left(\left(1 + \frac{n-1}{3}\right)N^{(1)}_{(f_u, H_j), > k_j} + \left(1 + \frac{2n-2}{3}\right)N^{(1)}_{(f_u, H_i), > k_i}\right).
\end{aligned}$$

Then, in order to prove Lemma 3.2, it is sufficient for us to show

$$\begin{aligned}
N\left(r, \nu_{\Phi}\right) &\geqslant \sum_{u=1}^{3} N^{(n)}_{(f_u, H_i), \leqslant k_i}(r) + \sum_{u=1}^{3} N^{(n)}_{(f_u, H_j), \leqslant k_j}(r) + 2\sum_{\substack{t=1\\ t\neq i,j}}^{2n+2} N^{(1)}_{(f, H_t), \leqslant k_t}(r)\\
&- (2n+1)N^{(1)}_{(f, H_i), \leqslant k_i}(r) - (n+1)N^{(1)}_{(f, H_j), \leqslant k_j}(r) - \sum_{u=1}^{3} N_{\frac{1}{F_u^{ij}}}(r)\\
&- N\left(r, S_i\right) - N\left(r, S_j\right) - (2n-2)N\left(r, R_i\right) - (n-1)N\left(r, R_j\right) + o(T(r)).
\end{aligned}$$

We also find that the above inequality follows from the truth of the following one

(3.4)

$$\begin{aligned}
P \overset{Def}{:=} &\sum_{u=1}^{3} \nu^{(n)}_{(f_u, H_i), \leqslant k_i} + \sum_{u=1}^{3} \nu^{(n)}_{(f_u, H_j), \leqslant k_j} + 2\sum_{\substack{t=1\\ t\neq i,j}}^{2n+2} \chi_{T_t} - (2n+1)\chi_{T_i} - (n+1)\chi_{T_j}\\
&- \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}} - \chi_{S_i} - \chi_{S_j} - 2(n-1)\chi_{R_i} - (n-1)\chi_{R_j} \leqslant \nu_{\Phi_{ij}^{\alpha}}
\end{aligned}$$

on $\Delta(R)$.
Indeed, for $z \in \Delta(R)$, we consider these cases:
Case 1. $z \in T_t \backslash S_i \cup S_j (t \neq i, j)$. We see that $P(z) = 2$. From (2.3), we write Φ_{ij} in the form

$$\Phi_{ij} = F_1^{ij} \cdot F_2^{ij} \cdot F_3^{ij} \times \begin{vmatrix} \left(F_1^{ji} - F_2^{ji}\right) & \left(F_1^{ji} - F_3^{ji}\right) \\ \left(F_1^{ji} - F_2^{ji}\right)' & \left(F_1^{ji} - F_3^{ji}\right)' \end{vmatrix}.$$

Then by the assumption that f_1, f_2, f_3 are identify on T_t, we have $F_1^{ji} = F_2^{ji} = F_3^{ji}$ on $T_t \backslash S_i$. The property of the wronskian implies that $\nu_\Phi(z) \geqslant 2 = P(z)$.
Case 2. $z \in T_t \cap (S_i \cup S_j)\, (t \neq i, j)$. We see that $P(z) \leqslant -\sum_{u=1}^3 \nu^\infty_{F_u^{ij}}(z) - 1$. By computing Φ_{ij} as in (2.4)

$$\Phi_{ij} = F_1^{ij}\left(\frac{\left(F_3^{ji}\right)'}{F_3^{ji}} - \frac{\left(F_2^{ji}\right)'}{F_2^{ji}}\right) + F_2^{ij}\left(\frac{\left(F_1^{ji}\right)'}{F_1^{ji}} - \frac{\left(F_3^{ji}\right)'}{F_3^{ji}}\right) + F_3^{ij}\left(\frac{\left(F_2^{ji}\right)'}{F_2^{ji}} - \frac{\left(F_1^{ji}\right)'}{F_1^{ji}}\right),$$

we find that

$$\nu_{\Phi_{ij}}(z) \geqslant \min\left\{\nu_{F_1^{ij}}(z) - 1, \nu_{F_2^{ij}}(z) - 1, \nu_{F_3^{ij}}(z) - 1\right\} \geqslant P(z).$$

Case 3. $z \in T_i \backslash S_j$. We have

$$P(z) = \sum_{u=1}^{3} \nu^{(n)}_{(f_u, H_i), \leqslant k_i}(z) - (2n+1) \leqslant \min_{1 \leqslant u \leqslant 3}\left\{\nu^{(n)}_{(f_u, H_i), \leqslant k_i}(z)\right\} - 1.$$

We may assume that $\nu_{(f_1, H_i)}(z) \leqslant \nu_{(f_2, H_i)}(z) \leqslant \nu_{(f_3, H_i)}(z)$. We write

$$\Phi_{ij} = F_1^{ij}\left[F_2^{ij}\left(F_1^{ji} - F_2^{ji}\right) F_3^{ij}\left(F_1^{ji} - F_3^{ji}\right)' - F_3^{ij}\left(F_1^{ji} - F_2^{ji}\right) F_2^{ij}\left(F_1^{ji} - F_2^{ji}\right)'\right].$$

It is easy to see that $F_2^{ij}\left(F_1^{ji} - F_2^{ji}\right)$ and $F_3^{ij}\left(F_1^{ji} - F_3^{ji}\right)$ are holomorphic on a neighborhood of z and

$$\nu^\infty_{F_3^{ij}\left(F_1^{ji} - F_3^{ji}\right)'}(z) \leqslant 1$$

and

$$\nu^\infty_{F_2^{ij}\left(F_1^{ji} - F_2^{ji}\right)'}(z) \leqslant 1.$$

Therefore, it follows that

$$\nu_\Phi(z) \geqslant \nu^{(n)}_{(f_1, H_i), \leqslant k_i}(z) - 1 \geqslant P(z).$$

Case 4. $z \in T_i \cap S_j$. The assumption that f_1, f_2, f_3 are identity on T_i implies that $z \in R_j$. We get

$$P(z) \leqslant \sum_{u=1}^{3} \nu^{(n)}_{(f_u,H_i),\leqslant k_i}(z) - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - (2n+1) - n \leqslant -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1.$$

We have

$$\nu_{\Phi_{ij}}(z) \geqslant \min\left\{\nu_{F_1^{ij}}(z) - 1, \nu_{F_2^{ij}}(z) - 1, \nu_{F_3^{ij}}(z) - 1\right\} \geqslant -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1 \geqslant P(z).$$

Case 5. $z \in T_j$. We may assume that

$$\nu_{F_1^{ji}}(z) = d_1 \geqslant \nu_{F_2^{ji}}(z) = d_2 \geqslant \nu_{F_3^{ji}}(z) = d_3.$$

Choose a holomorphic function h on $\Delta(R)$ with multiplicity 1 at z such that $F_u^{ji} = h^{d_u}\varphi_u (1 \leqslant u \leqslant 3)$, where φ_u are meromorphic on $\Delta(R)$ and holomorphic on a neighborhood of z. Then

$$\begin{aligned}\Phi &= F_1^{ij} \cdot F_2^{ij} \cdot F_3^{ij} \cdot \begin{vmatrix} F_2^{ji} - F_1^{ji} & F_3^{ji} - F_1^{ji} \\ \left(F_2^{ji} - F_1^{ji}\right)' & \left(F_3^{ji} - F_1^{ji}\right)' \end{vmatrix} \\ &= F_1^{ij} \cdot F_2^{ij} \cdot F_3^{ij} \cdot h^{d_2+d_3} \cdot \begin{vmatrix} \varphi_2 - h^{d_1-d_2}\varphi_1 & \varphi_3 - h^{d_1-d_3}\varphi_1 \\ \frac{\left(h^{d_2-d_3}\varphi_2 - h^{d_1-d_3}\varphi_1\right)'}{h^{d_2-d_3}} & \left(\varphi_3 - h^{d_1-d_3}\varphi_1\right)' \end{vmatrix}.\end{aligned}$$

This yields that

$$\nu_\Phi(z) \geqslant \sum_{u=1}^{3} \nu_{F_u^{ij}}(z) + d_2 + d_3 - \max\left\{0, \min\left\{1, d_2 - d_3\right\}\right\}.$$

Now we put

$\nu_i = \big\{z; k_i \geq \nu_{(f_u,H_i)}(z) \geq \nu_{(f^v,H_i)}(z) = \nu_{(f^t,H_i)}(z)$ for a permutation (u, v, t) of $(1, 2, 3)$.

If $z \notin S_i$ then

$$P(z) = -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + \sum_{u=1}^{3} \min\left\{n, d_u\right\} - (n+1) + \chi_{\nu_j}$$

and

$$\begin{aligned}\nu_\Phi(z) &\geqslant -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + \sum_{u=1}^{3} \nu^{0}_{F_u^{ij}}(z) + d_2 + d_3 - 1 + \chi_{\nu_j} \\ &\geqslant -\sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + d_2 + d_3 - 1 + \chi_{\nu_j} \geqslant P(z).\end{aligned}$$

Otherwise, if $z \in S_i$ then $z \in R_i$, we have

$$P(z) \leqslant \sum_{u=1}^{3} \nu^{(n)}_{(f_u,H_j),\leqslant k_j} - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 3n - 1 + \chi_{\nu_j} \leqslant - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 3n$$

and

$$\begin{aligned}\nu_\Phi(z) &\geqslant - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + \sum_{u=1}^{3} \nu^{0}_{F_u^{ij}}(z) + d_2 + d_3 - 1 \\ &\geqslant - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 \geqslant P(z)\end{aligned}$$

Case 6. $z \in (S_i \cup S_j) \setminus \left(\bigcup_{t=1}^{2n+2} T_t\right)$. Similarly as Case 5, we obtain

$$\begin{aligned}\nu_\Phi(z) &\geqslant - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 \\ &\geq - \sum_{u=1}^{3} \nu^{\infty}_{F_u^{ij}}(z) - 1 \geqslant - \sum_{u=1}^{3} \nu^{\infty}_{Fu}(z) - \chi_{S_i} - \chi_{S_j} \geqslant P(z).\end{aligned}$$

From the above six cases, we see that the inequality (3.4) holds. Hence the first assertion of the lemma is proved.

(ii). Now we assume that $\Phi(F_1^{ji}, F_2^{ji}, F_3^{ji}) \not\equiv 0$. From (i) we see that

$$\begin{aligned}&\Big\|_E\, T(r) + 6S(r) + o(T(r)) \geqslant \sum_{u=1}^{3} N^{(n)}_{(f_u,H_i),\leqslant k_i}(r) + \sum_{u=1}^{3} N^{(n)}_{(f_u,H_j),\leqslant k_j}(r) \\ &+ 2 \sum_{\substack{t=1 \\ t\neq i,j}}^{2n+2} N^{(1)}_{(f,H_t),\leqslant k_t}(r) - (2n+1) N^{(1)}_{(f,H_i),\leqslant k_i}(r) - (n+1) N^{(1)}_{(f,H_j),\leqslant k_j}(r) \\ &- \sum_{u=1}^{3} \left(\left(1 + \frac{n-1}{3}\right) N_{(f_u,H_j),>k_j} - \left(1 + \frac{2n-2}{3}\right) N_{(f_u,H_i),>k_i} \right).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\Big\|_E\, T(r) + 6S(r) + o(T(r)) \geqslant \sum_{u=1}^{3} N^{(n)}_{(f_u,H_j),\leqslant k_j}(r) + \sum_{u=1}^{3} N^{(n)}_{(f_u,H_i),\leqslant k_i}(r) \\ &+ 2 \sum_{\substack{t=1 \\ t\neq i,j}}^{2n+2} N^{(1)}_{(f,H_t),\leqslant k_t}(r) - (2n+1) N^{(1)}_{(f,H_i),\leqslant k_j}(r) - (n+1) N^{(1)}_{(f,H_i),\leqslant k_i}(r) \\ &- \sum_{u=1}^{3} \left(\left(1 + \frac{n-1}{3}\right) N^{(1)}_{(f_u,H_i),>k_i} - \left(1 + \frac{2n-2}{3}\right) N^{(1)}_{(f_u,H_j),>k_j} \right).\end{aligned}$$

Summing up both sides of the above two inequalities, we get

$$\begin{aligned}&\|_E\, 2T(r)+12S(r)+o(T(r))\geqslant \sum_{t=i,j}\Big(2\sum_{u=1}^{3}N^{[n]}_{(f_u,H_t),\le k_t}(r)\\&-(3n+6)N^{[1]}_{(f,H_t),\le k_t}(r)\Big)+4\sum_{t=1}^{q}N^{[1]}_{(f,H_t),\le k_t}(r)-\sum_{u=1}^{3}(n+1)N^{(1)}_{(f_u,H_t),>k_t}\Big).\end{aligned}$$

Dividing both sides of the above inequality to 2, we get the desired inequality of the assertion (ii). The lemma is proved.

Lemma 3.5. *Let f and $H_1,\dots,H_q$ be as in Theorem 1.1. Let f_1,f_2,f_3 be three mappings in $\mathcal{F}(f,\{H_i,k_i\}_{i=1}^q,1)$. Assume that*

$$q>n+1+\frac{3nq}{2q+3n-6}+\frac{3n+7}{4}\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{n(n+7)}{2}\big(c_{f_1}+c_{f_2}+c_{f_3}\big).$$

Then there exist $([\frac{q}{2}]+1)$ hyperplanes $H_{i_0},\dots,H_{i_{[\frac{q}{2}]}}$ among H_i's such that for each j $(1\le j\le[\frac{q}{2}])$ there exist two constants α_j,β_j, not all zeros, satisfying

$$\begin{aligned}&\alpha_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}\right)=\beta_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}\right)\\ \text{or}\quad &\alpha_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_2,H_{i_0})}{(f_2,H_{i_j})}\right)=\beta_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_3,H_{i_0})}{(f_3,H_{i_j})}\right).\end{aligned}$$

Proof. We take an arbitrary i $(1\le i\le q)$ and set

$$T_i(r)=\sum_{u=1}^{3}N^{[n]}_{(f_u,H_i),\le k_i}(r)-\frac{3n+6}{2}N^{[1]}_{(f,H_i),\le k_i}(r)-\sum_{u=1}^{3}(n+1)N^{(1)}_{(f_u,H_i),>k_t}.$$

Let $\mathcal{I}$ denote the set of all permutations of the $q-$tupe $(1,\dots,q)$, i.e.,

$$\mathcal{I}=\{(i_0,\dots,i_{q-1})\mid\{i_0,\dots,i_{q-1}\}=\{1,\dots,q\}\}.$$

For every permutation $I=(i_0,\dots,i_{q-1})$, let A_I denote the set containing all r in $(0,R)$ satisfying $T_{i_0}(r)\ge T_{i_1}(r)\ge\cdots\ge T_{i_{q-1}}(r)$.

By the assumption $\int_0^\infty\gamma(r)=+\infty$, we find that there is a set $I_0\in\mathcal{I}$, for instance $I_0=(i_0,\dots,i_{q-1})$ such that $\int_{A_{I_0}}\gamma dr=+\infty$.

Fix j $(1\le j\le[\frac{q}{2}])$. We prove that $\Phi(F_1^{i_0i_j},F_2^{i_0i_j},F_3^{i_0i_j})\equiv 0$
or $\Phi(F_1^{i_ji_0},F_2^{i_ji_0},F_3^{i_ji_0})\equiv 0$.

Indeed, we suppose that $\Phi(F_1^{i_0i_j}, F_2^{i_0i_j}, F_3^{i_0i_j}) \not\equiv 0$
and $\Phi(F_1^{i_ji_0}, F_2^{i_ji_0}, F_3^{i_ji_0}) \not\equiv 0$. Then by Lemma 3.2(ii), we have

$$(3.6) \qquad || \; T_{i_0}(r) + T_{i_j}(r) + 2\sum_{t=1}^{q} N^{[1]}_{(f,H_t)}(r) \le T(r) + 6S(r) + o(T(r)).$$

On the other side, for all $r \in A_{I_0}$, we get

$$\begin{aligned} T_{i_0}(r) + T_{i_j}(r) \ge T_{i_0}(r) + T_{i_{[\frac{q}{2}]}}(r) &\ge \frac{1}{[q/2]} \sum_{t=0}^{2[\frac{q}{2}]-1} T_{i_t}(r) \\ &\ge \frac{1}{[q/2]}\frac{2[q/2]}{q}\sum_{t=0}^{q-1} T_{i_t}(r) = \frac{2}{q}\sum_{i=1}^{q} T_i(r). \end{aligned}$$

From the above inequality and (3.6) together with the Second Main Theorem 2.2, we obtain

$$\begin{aligned} &\|_E \; T(r) + 6S(r) + o(T(r)) \ge T_{i_0}(r) + T_{i_j}(r) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) \\ &\ge \frac{2}{q}\sum_{i=1}^{q} T_i(r) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) \\ &= \frac{2}{q}\sum_{i=1}^{q}\sum_{u=1}^{3} N^{[n]}_{(f_u,H_i),\le k_i}(r) + (2 - \frac{3n+6}{q})\sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) \\ &- \sum_{i=1}^{q}\sum_{u=1}^{3}\frac{(n+1)}{2} N^{(1)}_{(f_u,H_i),>k_t} \\ &= \frac{2}{q}\sum_{i=1}^{q}\sum_{u=1}^{3} N^{[n]}_{(f_u,H_i),\le k_i}(r) + \frac{2q-3n-6}{3q}\sum_{i=1}^{q}\sum_{u=1}^{3} N^{[1]}_{(f_u,H_i),\le k_i}(r) \\ &- \sum_{i=1}^{q}\sum_{u=1}^{3}\frac{(n+1)}{2} N^{(1)}_{(f_u,H_i),>k_t} \\ &\ge \left(\frac{2}{q} + \frac{2q-3n-6}{3nq}\right)\sum_{u=1}^{3}\sum_{i=1}^{q} N^{[n]}_{(f_u,H_i),\le k_i}(r) \\ &- \sum_{i=1}^{q}\sum_{u=1}^{3}\frac{(n+1)}{2} N^{(1)}_{(f_u,H_i),>k_t} \end{aligned}$$

$$\geq \left(\frac{2}{q} + \frac{2q-3n-6}{3nq}\right) \sum_{u=1}^{3} \sum_{i=1}^{q} \big(N^{[n]}_{(f_u,H_i)}(r) - N^{[n]}_{(f_k,H_i),>k_i}(r)\big)$$
$$- \sum_{i=1}^{q} \sum_{u=1}^{3} \frac{(n+1)}{2} N^{(1)}_{(f_u,H_i),>k_t}$$
$$\geq \left(\frac{2}{q} + \frac{2q-3n-6}{3nq}\right) \sum_{u=1}^{3} \Big(\sum_{i=1}^{q} N^{[n]}_{(f_u,H_i)}(r)$$
$$- \sum_{i=1}^{q} \frac{n}{k_i+1} T_{f_u}(r) \Big) - \sum_{i=1}^{q} \sum_{u=1}^{3} \frac{(n+1)}{2(k_i+1)} T_{f_u}(r)$$
$$\geq \left(\frac{2}{q} + \frac{2q-3n-6}{3nq}\right) \sum_{u=1}^{3} \Big((q-n-1) T_{f_u}(r) - \frac{n(n+1)}{2} S(r)$$
$$- \sum_{i=1}^{q} \frac{n}{k_i+1} T_{f_u}(r) \Big) - \sum_{i=1}^{q} \sum_{u=1}^{3} \frac{(n+1)}{2(k_i+1)} T_{f_u}(r)$$
$$\geq \frac{2q+3n-6}{3nq} \Big((q-n-1-\sum_{i=1}^{q} \frac{n}{k_i+1}) T(r) - \frac{3n(n+1)}{2} S(r) \Big)$$
$$- \sum_{i=1}^{q} \frac{(n+1)}{2(k_i+1)} T(r).$$

By taking $\gamma(r) = \exp\{(\min\{c_{f_1}, c_{f_2}, c_{f_3}\} + \varepsilon) T(r)\}$, we have

$$\left(6 + \frac{(2q+3n-6)(n+1)}{2q}\right) \Big((1+\varepsilon)(\min\{c_{f_1}, c_{f_2}, c_{f_3}\} + \varepsilon) T(r) + \varepsilon \log r \Big) + o(T(r))$$
$$\geq \left(\frac{2q+3n-6}{3nq} (q-n-1-\sum_{i=1}^{q} \frac{n}{k_i+1}) - \sum_{i=1}^{q} \frac{(n+1)}{2(k_i+1)} - 1 \right) T(r).$$

Letting $\varepsilon \to 0, r \to R, r \in A_{I_0} \setminus E$ in this inequality, we get

$$\left(6 + \frac{(2q+3n-6)(n+1)}{2q}\right) \min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{(2q+3n-6)}{3nq} \Big(q-n-1-\sum_{i=1}^{q} \frac{n}{k_i+1} \Big)$$
$$- \sum_{i=1}^{q} \frac{(n+1)}{2(k_i+1)} - 1.$$

This implies that

$$\begin{aligned}
q &\le n+1+\frac{3nq}{2q+3n-6}+\sum_{i=1}^{q}\frac{n}{k_i+1}\Big(1+\frac{3q(n+1)}{2(2q+3n-6)}\Big)\\
&+3\min\{c_{f_1},c_{f_2},c_{f_3}\}\Big(\frac{n(n+1)}{2}+\frac{6nq}{2q+3n-6}\Big)\\
&\le n+1+\frac{3nq}{2q+3n-6}+\frac{3n+7}{4}\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{n(n+7)}{2}\big(c_{f_1}+c_{f_2}+c_{f_3}\big).
\end{aligned}$$

This is a contradiction.
Thus, we have $\Phi(F_1^{i_0i_j},F_2^{i_0i_j},F_3^{i_0i_j})\equiv 0$ or $\Phi(F_1^{i_ji_0},F_2^{i_ji_0},F_3^{i_ji_0})\equiv 0$ for every j $(1\le j\le[\frac{q}{2}])$. From Lemma 2.6, we deduce that for each j $(1\le j\le[\frac{q}{2}])$ there exist constants α_j,β_j, not all zeros such that

$$\begin{aligned}
&\alpha_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}\right)=\beta_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}\right)\\
\text{or}\quad &\alpha_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_2,H_{i_0})}{(f_2,H_{i_j})}\right)=\beta_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_3,H_{i_0})}{(f_3,H_{i_j})}\right).
\end{aligned}$$

The theorem is proved.

Proof. Turn back to the Theorem 1.1, we will prove the theorem by contradiction. Assume that the conclusion of the theorem is not true. It follows from Lemma 3.5 that there are $([\frac{q}{2}]+1)$ hyperplanes $H_{i_0},\dots,H_{i_{[\frac{q}{2}]}}$ among H_i's and for each j $(1\le j\le[\frac{q}{2}])$ there are constants α_j,β_j, not all zeros, such that

$$\begin{aligned}
&\alpha_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}\right)=\beta_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}\right)\\
\text{or}\quad &\alpha_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_2,H_{i_0})}{(f_2,H_{i_j})}\right)=\beta_j\left(\frac{(f_1,H_{i_0})}{(f_1,H_{i_j})}-\frac{(f_3,H_{i_0})}{(f_3,H_{i_j})}\right).
\end{aligned}$$

From the premises of the theorem, there exists an index j $(1\le j\le[\frac{q}{2}])$ such that $\alpha_j\ne 0$, $\beta_j\ne 0$ and $\alpha_j\ne\beta_j$, for instance

$$\alpha_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}\right)=\beta_j\left(\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}-\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}\right).$$

Hence

$$(\beta_j-\alpha_j)\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}+\alpha_j\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}=\beta_j\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}. \tag{3.7}$$

For every z and permutations (k,l,s) of $(1,2,3)$, we can deduce from (3.7) that

$$\begin{aligned}\nu_{(f_k,H_{i_j})}(z)&\ge\min\{\nu_{(f_l,H_{i_j})}(z),\nu_{(f_s,H_{i_j})}(z)\},\\ \nu_{(f_k,H_{i_0})}(z)&\le\max\{\nu_{(f_l,H_{i_0})}(z),\nu_{(f_s,H_{i_0})}(z)\},\end{aligned}\tag{3.8}$$

For each $z\in\Delta(R)$, we consider two holomorphic functions h_1,h_2 chosen by

$$\nu_{h_1}(z)=\max_{1\le u\le 3}\nu_{(f_u,H_{i_0})}(z),$$
$$\nu_{h_2}(z)=\min_{1\le u\le 3}\nu_{(f_u,H_{i_j})}(z)$$

and denote by F the meromorphic mapping of $\Delta(R)$ into $\mathbb{P}^1(\mathbb{C})$ with a reduced representation $F=\left(\frac{h_1}{h_2}\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}:\frac{h_1}{h_2}\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}\right)$.
For each $t\in\{1,\dots,q\}$, we define the following divisor

$$\nu_t(z)=\begin{cases}1 & \text{if } \min_{1\le u\le 3}\nu_{(f_u,H_t)}(z)<\max_{1\le u\le 3}\nu_{(f_u,H_t)}(z)\\ 0 & \text{if } \min_{1\le u\le 3}\nu_{(f_u,H_t)}(z)=\max_{1\le u\le 3}\nu_{(f_u,H_t)}(z).\end{cases}$$

By (3.8), we may suppose that $\nu_{(f_k,H_{i_j})}(z)=\nu_{(f_l,H_{i_j})}(z)\le\nu_{(f_s,H_{i_j})}(z)$ for each $z\in\Delta(R)$ and for a permutation (k,l,s) of $(1,2,3)$.
This implies that

$$\begin{aligned}\sum_{u=1}^{3}\min\{1,\nu_{\frac{(f_u,H_{i_j})}{h_2}}(z)\}&=\min\{1,\nu_{(f_s,H_{i_j})}(z)-\nu_{(f_l,H_{i_j})}(z)\}\\ &=\nu_{i_j}(z)\le\min\{1,\nu_{(f_k,H_{i_j}),\le k_{i_j}}(z)\}+\nu_{S_{i_j}}.\end{aligned}$$

This follows that

$$\sum_{u=1}^{3}N^{[1]}_{\frac{(f_u,H_{i_j})}{h_2}}(r)\le N(r,\nu_{i_j})\le N^{[1]}_{(f_k,H_{i_j}),\le k_{i_j}}(r)+N_{S_{i_j}}(r).$$

By the same argument, we also have

$$\sum_{u=1}^{3}N^{[1]}_{\frac{h_1}{(f_u,H_{i_0})}}(r)\le N^{[1]}_{(f_k,H_{i_0}),\le k_{i_0}}(r)+N_{S_{i_0}}.$$

We suppose that F is not constant, then by the Second Main Theorem we have

$$\begin{aligned}\|_E\, T_F(r) &\le N^{[1]}_{\frac{h_1}{h_2}\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}}(r) + N^{[1]}_{\frac{h_1}{h_2}\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}}(r) + N^{[1]}_{\frac{h_1}{h_2}\frac{(f_3,H_{i_j})}{(f_3,H_{i_0})}}(r)\\ &\quad + (1+\varepsilon)\log\gamma(r) + \varepsilon\log r + O(\log T_F(r))\\ &\le \sum_{u=1}^{3}\left(N^{[1]}_{\frac{(f_u,H_{i_j})}{h_2}}(r) + N^{[1]}_{\frac{h_1}{(f_u,H_{i_0})}}(r)\right) + S(r) + O(\sum_{u=1}^{3}\log T_{f_u}(r))\\ &\le N^{[1]}_{(f,H_{i_0}),\le k_{i_0}}(r) + N^{[1]}_{(f,H_{i_j}),\le k_{i_j}}(r) + S(r) + N_{S_{i_j}} + N_{S_{i_0}} + o(T(r)).\end{aligned}$$

On the other hand, applying the first main theorem to the map F and the hyperplane $\{w_0 - w_1 = 0\}$ in $\mathbf{P}^1(\mathbf{C})$, we have

$$\|_E\, T_F(r) \ge N_{\frac{h_1}{h_2}\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})} - \frac{h_1}{h_2}\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}}(r) + O(1) \ge \sum_{\substack{t=1\\ t\ne i_0,i_j}}^{q} N^{[1]}_{(f,H_t),\le k_t}(r) + O(1).$$

Thus

$$\begin{aligned}&\|_E\, N^{[1]}_{(f,H_{i_0}),\le k_{i_0}}(r) + N^{[1]}_{(f,H_{i_j}),\le k_{i_j}}(r) \ge \sum_{\substack{t=1\\ t\ne i_0,i_j}}^{q} N^{[1]}_{(f,H_t),\le k_t}(r) - S(r)\\ &- N_{S_{i_0}}(r) - N_{S_{i_j}}(r) + o(T(r))\\ &\ge \sum_{\substack{t=1\\ t\ne i_0,i_j}}^{q} N^{[1]}_{(f,H_t),\le k_t}(r) - S(r) - \sum_{u=1}^{3}\big(N^{[1]}_{(f_u,H_{i_0}),>k_{i_0}}(r) + N^{[1]}_{(f_u,H_{i_j}),>k_{i_j}}(r)\big) + o(T(r)).\end{aligned}$$

Without loss of generality, we may take $\{i_0, i_j\} = \{q-1, q\}$. For each $i \in \{1, \dots, q-2\}$, we set

$$N_i(r) = N^{[n]}_{(f_1,H_i),\le k_i}(r) + N^{[n]}_{(f_2,H_i),\le k_i}(r) - (n+1)N^{[1]}_{(f,H_i),\le k_i}(r).$$

For each permutation $I = (s_1, \dots, s_{q-2})$ of $(1, \dots, q-2)$, let A_I denote the set containing all r in $(0, R)$ satisfying

$$N_{s_1}(r) \ge N_{s_2}(r) \ge \cdots \ge N_{s_{q-2}}(r).$$

Since $\int_0^R \gamma(r)dr = +\infty$, there exists $I_0 \in \mathcal{I}$ such that $\int_{A_{I_0}} \gamma(r)dr = +\infty$. We assume that $I_0 = (1, \dots, q-2)$.

By the assumption, there is an index $s \in \{2, \dots, [q/2]\}$ such that $\frac{(f_1,H_1)}{(f_1,H_s)} - \frac{(f_2,H_1)}{(f_2,H_s)} \not\equiv 0$. Therefore

$$P = (f_1, H_1)\cdot(f_2, H_s) - (f_1, H_s)\cdot(f_2, H_1) \not\equiv 0.$$

It is easily to see that: If z is a zero of (f, H_1) with multiplicity at most k_1 (resp. zero of (f, H_s) with multiplicity at most k_s) then it is a zero of P with multiplicity at least $\min\{\nu_{(f_1,H_1),\le k_1}(z), \nu_{(f_2,H_1),\le k_1}(z)\}$
(resp. at least $\min\{\nu_{(f_1,H_s),\le k_s}(z), \nu_{(f_2,H_s),\le k_1}(z)\}$) and if z is a zero of some (f, H_i) $(2 \le i \le q, i \ne s)$ with multiplicity at most k_i then it is a zero of P. Thus, we have

$$\nu_P(z) \ge \min\{\nu_{(f_1,H_1),\le k_1}(z), \nu_{(f_2,H_1),\le k_1}(z)\} + \min\{\nu_{(f_1,H_s),\le k_s}(z), \nu_{(f_2,H_s),\le k_s}(z)\} + \sum_{\substack{i=2 \\ i\ne s}}^{q} \min\{1, \nu_{(f,H_i),\le k_i}(z)\}, z \in \Delta(R).$$

Moreover, because $\min\{a, b\} \ge \min\{a, n\} + \min\{b, n\} - n$ for all positive integers a, b, it follows from this inequality that

$$\nu_P(z) \ge \sum_{i=1,s} (\min\{n, \nu_{(f_1,H_i),\le k_i}(z)\} + \min\{n, \nu_{(f_2,H_i),\le k_i}(z)\} - n \min\{1, \nu_{(f,H_i),\le k_i}(z)\}) + \sum_{\substack{i=2 \\ i\ne s}}^{q} \min\{1, \nu_{(f,H_i),\le k_i}(z)\}, z \in \Delta(R).$$

Integrating the both sides of the above inequality and applying the Theorem 2.2, we obtain

$$\begin{aligned}
\|_E\ N_P(r) &\ge \sum_{i=1,s} (N^{[n]}_{(f_1,H_i),\le k_i}(r) + N^{[n]}_{(f_2,H_i),\le k_i}(r) - nN^{[1]}_{(f,H_i),\le k_i}(r)) \\
&+ \sum_{\substack{i=2 \\ i\ne s}}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) \\
&= \sum_{i=1,s} N_i(r) + \sum_{i=1}^{q} N^{[1]}_{(f,H_i),\le k_i}(r) \\
&\ge \frac{2}{q-2} \sum_{i=1}^{q-2} N_i(r) + 2\sum_{i=1}^{q-2} N^{[1]}_{(f,H_i),\le k_i}(r) - \sum_{u=1}^{3} \big(N^{[1]}_{(f_u,H_{q-1}),>k_{q-1}}(r) + N^{[1]}_{(f_u,H_q),>k_q}(r)\big) \\
&= 2\sum_{i=1}^{q-2} \Big(\sum_{u=1,2} \frac{1}{q-2} N^{[n]}_{(f_u,H_i),\le k_i}(r) + \frac{q-n-3}{q-2} N^{[1]}_{(f,H_i),\le k_i}(r)\Big) \\
&- \sum_{u=1}^{3} \big(N^{[1]}_{(f_u,H_{q-1}),>k_{q-1}}(r) + N^{[1]}_{(f_u,H_q),>k_q}(r)\big)
\end{aligned}$$

$$\geq \frac{2}{q-2}\sum_{u=1,2}\sum_{i=1}^{q-2}\frac{q+n-3}{2n}N^{[n]}_{(f_u,H_i),\leq k_i}(r)$$
$$-\sum_{u=1}^{3}\big(N^{[1]}_{(f_u,H_{q-1}),>k_{q-1}}(r)+N^{[1]}_{(f_u,H_q),>k_q}(r)\big)$$
$$\geq \frac{q+n-3}{n(q-2)}\sum_{u=1,2}\sum_{i=1}^{q-2}\big(N^{[n]}_{(f_u,H_i)}(r)$$
$$-N^{[n]}_{(f_u,H_i),>k_i}(r)\big)-\sum_{u=1}^{3}\big(N^{[1]}_{(f_u,H_{q-1}),>k_{q-1}}(r)+N^{[1]}_{(f_u,H_q),>k_q}(r)\big)$$
$$\geq \frac{q+n-3}{n(q-2)}\sum_{u=1,2}\big(\sum_{i=1}^{q-2}N^{[n]}_{(f_u,H_i)}(r)-\sum_{i=1}^{q-2}\frac{n}{k_i+1}T_{f_u}(r)\big)-\sum_{u=1}^{3}\big(\frac{1}{k_{q-1}+1}+\frac{1}{k_q+1}\big)T_{f_u}$$
$$\geq \frac{q+n-3}{n(q-2)}\big(\big(q-n-3-\sum_{i=1}^{q-2}\frac{n}{k_i+1}\big)\sum_{u=1,2}T_{f_u}(r)$$
$$-\frac{(q+n-3)(n+1)}{q-2}S(r)-\sum_{u=1}^{3}\big(\frac{1}{k_{q-1}+1}+\frac{1}{k_q+1}\big)T_{f_u}+o(T_{f_1}(r)+T_{f_2}(r)).$$

Furthermore, we have

$$T_{f_3}(r)\leq \frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{n(n+1)}{2}c_{f_3}}(T_{f_1}(r)+T_{f_2}(r))+o(T_{f_3}),$$

and

$$\big\|_E T_{f_3}(r)=O(T_{f_1}(r)) \quad \text{and} \quad \big\|_E T_{f_3}(r)=O(T_{f_2}(r)).$$

So we see that

$$\big\|_E\ N_P(r)\geq \frac{q+n-3}{n(q-2)}\big(\big(q-n-3-\sum_{i=1}^{q-2}\frac{n}{k_i+1}\big)\sum_{u=1,2}T_{f_u}(r)-\frac{(q+n-3)(n+1)}{q-2}S(r)$$
$$-\big(\frac{1}{k_{q-1}+1}+\frac{1}{k_q+1}\big)\big(1+\frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{n(n+1)}{2}c_{f_3}}\big)\sum_{u=1,2}T_{f_u}(r)$$
$$+o\big(\sum_{u=1,2}T_{f_u}(r).$$

On the other hand, by Jensen formula, we get

$$\begin{aligned} N_P(r) &= \int_{S(r)} \log|P|\eta + O(1) \\ &\le \int_{S(r)} \log((|(f_1,H_1)|^2 + |(f_1,H_s)|^2)\cdot(|(f_2,H_1)|^2 + |(f_2,H_s)|^2))^{1/2}\eta + O(1) \\ &\le \int_{S(r)} \log(||f_1||\cdot||f_2||)\eta + O(1) = T_{f_1}(r) + T_{f_2}(r) + O(1). \end{aligned}$$

Then we have

$$\begin{aligned} &\|_E\ T_{f_1}(r) + T_{f_2}(r) \ge \frac{q+n-3}{n(q-2)}\Big(q-n-3-\sum_{i=1}^{q-2}\frac{n}{k_i+1}\Big)\sum_{u=1,2}T_{f_u}(r) \\ &- \frac{(q+n-3)(n+1)}{q-2}\Big((1+\varepsilon)(\min\{c_{f_1},c_{f_2}\}+\varepsilon)\sum_{u=1,2}T_{f_u}(r) + \varepsilon\log r\Big) \\ &- \Big(\frac{1}{k_{q-1}} + \frac{1}{k_q}\Big)\Big(1+\frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1} - \frac{n(n+1)}{2}c_{f_3}}\Big)\sum_{u=1,2}T_{f_u}(r) \\ &+ o\Big(\sum_{u=1,2}T_{f_u}(r)\Big). \end{aligned}$$

Letting $r \longrightarrow R, r \in A_{I_0}\backslash E$ and letting $\varepsilon \to 0$, we get

$$\begin{aligned} &1 + \frac{(q+n-3)(n+1)}{q-2}\min\{c_{f_1},c_{f_2}\} \ge \frac{(q+n-3)}{n(q-2)}\Big(q-n-3-\sum_{i=1}^{q-2}\frac{n}{k_i+1}\Big) \\ &- \Big(\frac{1}{k_{q-1}+1} + \frac{1}{k_q+1}\Big)\Big(1+\frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1} - \frac{n(n+1)}{2}c_{f_3}}\Big). \end{aligned}$$

This follows that

$$\begin{aligned} &q \le n+3+\sum_{i=1}^{q-2}\frac{n}{k_i+1} + \frac{n(q-2)}{q+n-3} + n(n+1)\min\{c_{f_1},c_{f_2}\} \\ &+ \frac{n(q-2)}{q+n-3}\Big(\frac{1}{k_{q-1}+1} + \frac{1}{k_q+1}\Big)\Big(1+\frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1} - \frac{n(n+1)}{2}c_{f_3}}\Big). \end{aligned}$$

On the other side, we have

$$q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{n(n+1)}{2}c_{f_3}\geq q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}$$
$$-\frac{n(n+1)}{2}(c_{f_1}+c_{f_2}+c_{f_3})$$
$$\geq \frac{3nq}{2q+3n-6}+\frac{3q(n+1)}{2(2q+3n-6)}\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{6nq}{2q+3n-6}(c_{f_1}+c_{f_2}+c_{f_3})$$
$$\geq \frac{3n}{5}+\frac{3n}{10}\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{6n}{5}(c_{f_1}+c_{f_2}+c_{f_3})\geq\frac{3n}{5}.$$

This implies that

$$\frac{n(q-2)}{q+n-3}\Big(\frac{1}{k_{q-1}+1}+\frac{1}{k_q+1}\Big)\Big(1+\frac{1}{2}\frac{qn}{q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{n(n+1)}{2}c_{f_3}}\Big)$$
$$\leq\Big(\frac{n}{k_{q-1}+1}+\frac{n}{k_q+1}\Big)\Big(1+\frac{5q}{6}\Big).$$

Thus

$$q\leq n+3+\frac{n(q-2)}{q+n-3}+\big(1+\frac{5q}{6}\big)\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{n(n+1)}{2}\big(c_{f_1}+c_{f_2}+c_{f_3}\big).$$

Therefore, the mapping F must be constant map. Then there exists nonzero constant γ so that

$$\frac{(f_1,H_{i_j})}{(f_1,H_{i_0})}=\gamma\frac{(f_2,H_{i_j})}{(f_2,H_{i_0})}.$$

If $\gamma=1$ then $\beta_j=0$, this is a contradiction. If $\gamma\neq 1$, then $\bigcup_{\substack{t=1\\ t\neq i_0,i_j}}^{q} f^{-1}(H_t)=\emptyset$, since f_1 and f_2 agree on $\bigcup_{\substack{t=1\\ t\neq i_0,i_j}}^{q} f^{-1}(H_t)$. This follows that

$$\Big\|_E\,(q-n-3)T_f(r)\leq\sum_{\substack{t=1\\ t\neq i_0,i_j}}^{q}N_{(f,H_t)}(r)+o(T_f(r))=o(T_f(r)).$$

This is a contradiction. Therefore, the conslusion of the theorem holds. This finishes the proof of Theorem 1.1.

Proof of the Theorem 1.2 In order to prove the Theorem 1.2, we need the following lemma which based on Lemma 3.1 in [7].

Lemma 3.9. *Let f be a horomorphic mapping from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$. Let $f_1, f_2, \dots, f_k$ be k mappings in $\mathcal{F}_\Delta\left(f, \{H_i, k_i\}_{i=1}^q, n\right)$. Suppose that each f_u has a reduced representation $f_u = \left(f_u^0 : \cdots : f_u^n\right), 1 \le u \le k$. Assume that there are integers $1 \le i_1 < i_2 < \cdots < i_k \le q$ such that*

$$P := \det \begin{pmatrix} (f_1, H_{i_1}) & (f_1, H_{i_2}) & \cdots & (f_1, H_{i_k}) \\ \vdots & \vdots & \dots & \vdots \\ (f_k, H_{i_1}) & (f_k, H_{i_2}) & \cdots & (f_k, H_{i_k}) \end{pmatrix} \not\equiv 0.$$

Then we have for any $z \in \Delta(R)$,

$$\nu_P(z) \ge \sum_{j=1}^{k} \left(\min_{1 \le u \le k} \left\{ \nu_{(f_u, H_i), \le k_i}(z) \right\} - \nu^{[1]}_{(f, H_i), \le k_i}(z) \right) + (k-1) \sum_{i=1}^{q} \nu^{[1]}_{(f, H_i), \le k_i}(z).$$

Proof. For simplicity, we may choose $i_1 = 1, \dots, i_k = k$. We take a point $z \in \Delta(R)$ and consider the following cases.

If z is a zero of a function $(f, H_j)\,(1 \le j \le k)$ with the multiplicity at most k_j, for instance z is a zero of (f, H_1) with the multiplicity at most k_1, then z is a zero of (f_u, H_1) with multiplicity at least $\min_{1 \le u \le k} \left\{ \nu_{(f_u, H_1), \le k_1}(z) \right\}$ and also is a zero of all $\frac{(f_u, H_j)}{(f_u, H_q)} - \frac{(f_1, H_j)}{(f_1, H_q)}$. Put

$$A = \begin{pmatrix} \frac{(f_1, H_1)}{(f_1, H_q)} & \frac{(f_2, H_1)}{(f_2, H_q)} - \frac{(f_1, H_1)}{(f_1, H_q)} & \cdots & \frac{(f_k, H_1)}{(f_k, H_q)} - \frac{(f_1, H_1)}{(f_1, H_q)} \\ \vdots & \vdots & \dots & \vdots \\ \frac{(f_1, H_k)}{(f_1, H_q)} & \frac{(f_2, H_k)}{(f_2, H_q)} - \frac{(f_1, H_k)}{(f_1, H_q)} & \cdots & \frac{(f_k, H_k)}{(f_k, H_q)} - \frac{(f_1, H_k)}{(f_1, H_q)} \end{pmatrix}.$$

We see that $P = \left(\prod_{u=1}^{k} (f_u, H_q) \right) \det A$. Therefore, z is a zero of all members in the columns $2, 3, \dots, k$. In addition, it is easy to find that z is a zero of all members in the first row of the matrix A with multiplicities at least $\min_{1 \le u \le k} \left\{ \nu_{(f_u, H_1), \le k_1}(z) \right\}$. This follows that

$$\begin{aligned} \nu_P(z) &\ge (k-1) + \left(\min_{1 \le u \le k} \left\{ \nu_{(f_u, H_1), \le k_1}(z) \right\} - 1 \right) \\ &= \sum_{i=1}^{k} \left(\min_{1 \le u \le k} \left\{ \nu_{(f_u, H_i)}(z) \right\} - \nu^{[1]}_{(f, H_i), \le k_i}(z) \right) + (k-1) \sum_{i=1}^{q} \nu^{[1]}_{(f, H_i), \le k_i}(z). \end{aligned}$$

Otherwise, if z is a zero of a function (f, H_j) with the multiplicity at most k_j with $j > k$ and we may suppose that $k < q$, then z is a zero of all members in the columns $2, \ldots, k$ of the matrix A. This implies that

$$\nu_P(z) \geq (k-1) = \sum_{j=1}^{k} \left(\min_{1 \leq u \leq k} \left\{ \nu_{(f_u, H_i), \leq k_i}(z) \right\} - \nu^{[1]}_{(f, H_i), \leq k_i}(z) \right) + (k-1) \sum_{i=1}^{q} \nu^{[1]}_{(f, H_i), \leq k_i}(z).$$

The lemma is complete.

We now turn to prove Theorem 1.2.

Let $\mathcal{I}$ denote the set of all k-tuples $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq q$, and put $p = \sharp \mathcal{I}$. We suppose that $f_1 \times f_2 \times \cdots \times f_k$ is not algebraically degenerate. Then for every $I = (i_1, \ldots, i_k) \in \mathcal{I}$,

$$P_I := \det \left((f^s, H_{i_t}) \, ; 1 \leq s, t \leq k \right) \not\equiv 0$$

It follows from Lemma 3.9 that

$$\begin{aligned} \nu_{P_I} &\geq \sum_{s=1}^{k} \left(\min \left\{ \nu_{(f_u, H_{i_s}), \leq k_{i_s}} ; 1 \leq u \leq k \right\} - \nu^{[1]}_{(f, H_{i_s}), \leq k_{i_s}} \right) + (k-1) \sum_{i=1}^{q} \nu^{[1]}_{(f, H_i), \leq k_i} \\ &= \sum_{s=1}^{k} \left(\nu^{[n]}_{(f, H_{i_s}), \leq k_{i_s}} - \nu^{[1]}_{(f, H_{i_s}), \leq k_{i_s}} \right) + (k-1) \sum_{i=1}^{q} \nu^{[1]}_{(f, H_i), \leq k_i}. \end{aligned}$$

Setting $P = \prod_{I \in \mathcal{I}} P_I$ and summing up both sides of the above inequality over all $I \in \mathcal{I}$, we get

$$\begin{aligned} \nu_P &\geq \sum_{i=1}^{q} \left(\frac{pk}{q} \nu^{[n]}_{(f, H_i, \leq k_i)} + \frac{p((k-1)q-k)}{q} \nu^{[1]}_{(f, H_i), \leq k_i} \right) \\ &\geq \left(\frac{pk}{q} + \frac{p((k-1)q-k)}{nq} \right) \sum_{i=1}^{q} \nu^{[n]}_{(f, H_i), \leq k_i} \\ &= \left(\frac{p}{q} + \frac{p((k-1)q-k)}{knq} \right) \sum_{u=1}^{k} \sum_{i=1}^{q} \nu^{[n]}_{(f_u, H_i), \leq k_i}. \end{aligned}$$

Applying the Second Main Theorem, we have

$$\begin{aligned} \|_E (q-n-1) \sum_{u=1}^{k} T_{f_u}(r) &\leq \sum_{u=1}^{k} \sum_{i=1}^{q} N^{[n]}_{(f_u, H_i)}(r) + o\left(\sum_{u=1}^{k} T_{f_u}(r) \right) + \frac{kn(n+1)}{2} S(r) \\ &\leq \sum_{u=1}^{k} \sum_{i=1}^{q} \left(N^{[n]}_{(f_u, H_i, \leq k_i)}(r) + N^{[n]}_{(f_u, H_i), > k_i}(r) \right) + o\left(\sum_{u=1}^{k} T_{f_u}(r) \right) + \frac{kn(n+1)}{2} S(r) \end{aligned}$$

$$\leq \sum_{u=1}^{k}\sum_{i=1}^{q}\left(N^{[n]}_{(f_u,H_i,\leq k_i)}(r)+\frac{n}{k_i+1}T_{f_u}(r)\right)+o\left(\sum_{u=1}^{k}T_{f_u}(r)\right)+\frac{kn(n+1)}{2}S(r)$$

$$\leq \frac{knq}{pkn+p(k-1)q-k}N_P(r)+\sum_{i=1}^{q}\frac{n}{k_i+1}\sum_{u=1}^{k}T_{f_u}(r)$$

$$+o\left(\sum_{u=1}^{k}T_{f_u}(r)\right)+\frac{kn(n+1)}{2}S(r)$$

$$\leq \left(\frac{knq}{kn+(k-1)q-k}+\sum_{i=1}^{q}\frac{n}{k_i+1}\right)\sum_{u=1}^{k}T_{f_u}(r)$$

$$+o\left(\sum_{u=1}^{k}T_{f_u}(r)\right)+\frac{kn(n+1)}{2}S(r).$$

Taking $\gamma(r)=\exp\{(\min_{1\leq i\leq k}\{c_{f_i}\}+\varepsilon)\sum_{u=1}^{k}T_{f_u}(r)\}$
and letting $\varepsilon\longrightarrow 0, r\notin E$, we get

$$q\leq n+1+\frac{knq}{kn+(k-1)q-k}+\sum_{i=1}^{q}\frac{n}{k_i+1}+\frac{kn(n+1)}{2}\min_{1\leq i\leq k}\{c_{f_i}\}.$$

This is a contradiction. Thus we complete the proof.

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