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# New properties on the growth of ultrametric entire functions and applications

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**Abstract.** Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let $f$ be an entire function in $\mathbb{K}$ whose order of growth is finite. We show that the type of growth is finite if and only if so is the cotype. We give bounds for the cotype of growth and also for the lower cotype of growth. We show that the type of growth of $f$ is equal to its lower type if and only if its cotype is equal to its lower cotype and when these are realised, then the cotype is the product of the type by the order of growth and the order of growth (if $> 0$), is then equal to the lower order of growth.
If an entire function $h$ has an order of growth strictly inferior to the lower order of an entire function $f$, then $h$ is a small function with respect to $f$. A similar comparison is made with the type of growth. Conversely, if $h$ is a small function with respect to $f$, then $f+h$ and $f$ have same order, same type and same cotype of growth. Links are showed with the Nevanlinna Theory.
Suppose that $\mathbb{K}$ is of characteristic 0. Given a meromorphic function $f = \frac{g}{h}$, if $f$ admits primitives and if the type or the cotype of $h$ is finite, then $f$ assumes all values infinitely many times.
A counter-example is constructed where the lower order of growth is equal to the order of growth but the lower type of growth is not equal to the type of growth and where the the cotype is not equal to the product of the type by the order of growth.
In complex analysis, a claim was made for complex meromorphic functions stating that if the lower order of growth equals the order, then the lower type equals the type but we contest the proof.

## I. Introduction and main results

**Notations and definitions:** Let $\mathbb{K}$ be a complete ultrametric algebraically closed field whose absolute value is denoted $|\ .\ |$ and let $|\mathbb{K}| = \{\ |x|,\ x \in \mathbb{K}\}$.

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Given $r > 0$, we denote by $d(0, r)$ the disk $\{x \in \mathbb{K} \mid |x| \le r\}$, by $d(0, r^-)$ the disk $\{x \in \mathbb{K} \mid |x| < r\}$ and by $C(0, r)$ the circle $\{x \in \mathbb{K} \mid |x| = r\}$ .

Let $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions with coefficients in $\mathbb{K}$ and let $\mathcal{M}(\mathbb{K})$ be the field of meromorphic functions $\frac{g}{h}$, $g,\ h \in \mathcal{A}(\mathbb{K})$.

Let $f \in \mathcal{M}(\mathbb{K})$. For each $r > 0$, $|f(x)|$ is known to have a limit $|f|(r)$ when $|x|$ tends to $r$ while being different from $r$ and then $|f|(r) = \sup\{|f(x)| \mid |x| \le r\}$ [6], [8].

We denote by $s(r, f)$ the number of zeros of $f$ in $d(0, r)$, each counted with its multiplicity and we denote by $t(r, f)$ the number of poles of $f$ in $d(0, r)$, each counted with its multiplicity.

We denote by $Log$ the Neperian logarithm and by $e$ the number such that $Log(e) = 1$. Let $f \in \mathcal{A}(\mathbb{K})$. As in complex analysis [11], we define

$$\rho(f) = \limsup_{r \to +\infty} \frac{Log(Log(|f|(r)|))}{Log(r)},$$

$$\widetilde{\rho}(f) = \liminf_{r \to +\infty} \frac{Log(Log(|f|(r)|))}{Log(r)}$$

and if $0 < \rho(f) < +\infty$, we put

$$\sigma(f, r) = \frac{Log(|f|(r)}{r^{\rho(f)}},$$

$$\sigma(f) = \limsup_{r \to +\infty} \sigma(f, r),$$

$$\widetilde{\sigma}(f) = \liminf_{r \to +\infty} \sigma(f, r),$$

Moreover, assuming again $0 < \rho(f) < +\infty$, here we put $\psi(f, r) = \frac{s(r, f)}{r^{\rho(f)}}$,
$\psi(f) = \limsup_{r \to +\infty} \frac{s(r, f)}{r^{\rho(f)}}$,

and $\widetilde{\psi}(f) = \liminf_{r \to +\infty} \frac{s(r, f)}{r^{\rho(f)}}$.

$\rho(f)$ is called *the order of growth*, $\widetilde{\rho}(f)$ is called *the lower order of growth*, $\sigma(f)$ is called *the type of growth*, $\widetilde{\sigma}(f)$ is called *the lower type of growth*, $\psi(f)$ is called *the cotype of growth*, $\widetilde{\psi}(f)$ is called *the lower cotype of growth*.

A value $b \in |\mathbb{K}|$ is called *a quasi-exceptional value* of a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ if $f - b$ has finitely many zeros.

Theorem 1 is easy:

**Theorem 1:** *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then $\rho(fg) = \max(\rho(f), \rho(g))$ and $\rho(f^n) = \rho(f)\ \forall n \in \mathbb{N}$. Moreover $\rho(f + g) \le \max(\rho(f), \rho(g))$.*

**Corollary 1.1:** *The set of functions $f \in \mathcal{A}(\mathbb{K})$ of order $\leq t$ is a multiplicative semi-group and (adding 0), an additive group.*

We will now state the following Theorem 2:

**Theorem 2:** *Let $f \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho(f) < +\infty$. Then $\sigma(f) < +\infty$ if anf only if $\psi(f) < +\infty$. Suppose that these hypotheses are satisfied. Then*

$$\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\Big(e\sigma(f) - \widetilde{\sigma}(f)\Big)$$

*and*

$$\rho(f)\Big(\widetilde{\sigma}(f) - \frac{\sigma(f)}{e}\Big) \leq \widetilde{\psi}(f) \leq \rho(f)\widetilde{\sigma}(f).$$

*Further, the hypotheses $\sigma(f) = \widetilde{\sigma}(f)$ and $\psi(f) = \widetilde{\psi}(f)$ are equivalent and if they are satisfied, then $\psi(f) = \rho(f)\sigma(f)$.*

**Definition:** A function $f \in \mathcal{A}(\mathbb{K})$ is said to be *regular* if $\rho(f) = \widetilde{\rho}(f)$ [11] and $f$ is said to be *clean* if $0 < \rho(f) < +\infty$ and $\sigma(f) = \widetilde{\sigma}(f)$.

**Corollary 2.1:** *Let $f \in \mathcal{A}(\mathbb{K})$ be clean. Then $\psi(f) = \rho(f)\sigma(f)$.*

**Remark 1:** In [3] as in [6], it was proved that $\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\Big(e\sigma(f) - \widetilde{\sigma}(f)\Big)$ and that each hypothesis

a) $\sigma(f) = \widetilde{\sigma}(f)$ ,

b) $\psi(f) = \widetilde{\psi}(f)$,

implies $\psi(f) = \rho(f)\sigma(f)$, but it was not proved that the two hypotheses are equivalent.

**Remark 2:** The equality $\psi(f) = \rho(f)\sigma(f)$ holds because $\sigma(f, r)$ was defined with help of the Neperian logarithm.

Let us recall here the following Theorem A from [9] and [3]:

**Theorem A:** *Let $f(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho(f) < +\infty$. Then $e\sigma(f)\rho(f) = \limsup_{n\to+\infty} \big(n \sqrt[n]{|b_n|^{\rho(f)}}\big)$.*

Now Corollary 2.2 is an immediate consequence of Theorem 2 and Theorem A:

**Corollary 2.2:** *Let* $f(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$. *Then*

$$\rho(f)\sigma(f) \leq \psi(f) \leq \limsup_{n\to+\infty} \left(n \sqrt[n]{|b_n|^{\rho(f)}}\right) - \rho(f)\widetilde{\sigma}(f)$$

*and if* $f$ *is clean, then*

$$e\psi(f) = \frac{\limsup_{n\to+\infty} \left(n \sqrt[n]{|b_n|^{\rho(f)}}\right)}{\rho(f)}.$$

**Theorem 3:** *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *be such that* $\rho(g) \leq \rho(f) < +\infty$ *and* $\max(\sigma(f), \sigma(g)) < +\infty$. *Then,* $\sigma(fg) \leq \sigma(f) + \sigma(g)$. *If* $\rho(f) > \rho(g)$ *then* $\sigma(fg) = \sigma(f)$. *If* $f$ *is clean and such that* $\rho(f) > \rho(g)$, *then* $fg$ *is clean. If* $f$ *and* $g$ *are clean and if* $\rho(f) = \rho(g)$, *then* $fg$ *is clean, and* $\sigma(fg) = \sigma(f) + \sigma(g)$ *and* $\psi(fg) = \psi(f) + \psi(g)$.

**Corollary 3.1:** *The set of clean functions* $f \in \mathcal{A}(\mathbb{K})$ *is a multiplicative semi-group. The set* $\mathcal{C}(t, .)$ *of clean functions* $f \in \mathcal{A}(\mathbb{K})$ *of order* $t$ *is a submultiplicative semi-group and* $\sigma$ *and* $\psi$ *are semi-group morphisms from* $\mathcal{C}(t, .)$ *into* $(\mathbb{R}^+, +)$.

Corollary 2.1 suggests a question:

**Question 1:** *Let* $f(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$. *Do we have*

$$\rho(f)\sigma(f) = \psi(f) = \frac{\limsup_{n\to+\infty} \left(n \sqrt[n]{|b_n|^{\rho(f)}}\right)}{e}$$

*when* $f$ *is not clean? The answer is presented through a counter-example at the end of the article.*

Theorem 4 is easy:

**Theorem 4:** *Let* $f \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$ *and* $\widetilde{\sigma}(f) > 0$. *Then* $f$ *is regular.*

**Corollary 4.1:** *Let* $f \in \mathcal{A}(\mathbb{K})$ *be clean, such that* $0 < \rho(f) < +\infty$ *and* $\sigma(f) > 0$. *Then* $f$ *is regular.*

Now, from Theorem 2, we will deduce the following Theorem 5:

**Theorem 5:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$*. Let* $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$*, admitting primitives, be of the form* $\dfrac{g}{h}$ *with* $g,\ h \in \mathcal{A}(\mathbb{K})$ *and be such that* $0 < \rho(h) < +\infty$ *and* $\psi(h) < +\infty$*. Then* $f$ *has no quasi-exceptional value.*

By Theorem 2 we deduce immediately this corollary:

**Corollary 5.1:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$*. Let* $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$*, admitting primitives, be of the form* $\dfrac{g}{h}$ *with* $g,\ h \in \mathcal{A}(\mathbb{K})$ *and be such that* $0 < \rho(h) < +\infty$ *and* $\sigma(h) < +\infty$*. Then* $f$ *has no quasi-exceptional value.*

**Corollary 5.2:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$*. Let* $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$*, be of the form* $\frac{g}{h}$ *with* $g,\ h \in \mathcal{A}(\mathbb{K})$ *and be such that* $0 < \rho(h) < +\infty$ *and* $\sigma(h) < +\infty$*. If* $f$ *has a quasi-exceptional value, then it has a non-zero residue.*

**Corollary 5.3:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$*. Let* $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$*, be of the form* $\dfrac{g}{h}$ *with* $g,\ h \in \mathcal{A}(\mathbb{K})$ *and be such that* $0 < \rho(h) < +\infty$ *and* $\sigma(h) < +\infty$*. Then* $f'$ *has no quasi-exceptional value.*

Let us recall the definition of small functions, applied to entire functions.

**Definition:** Let $f,\ h \in \mathcal{A}(\mathbb{K})$. The function $h$ is said to be *a small function with respect to* $f$ if $\displaystyle\lim_{r\to+\infty} \frac{Log(|h|(r))}{Log(|f|(r))} = 0$. (A more general definition is given for meromorphic functions that we will not use here.)

Given three functions $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$, $f$ and $g$ are said *to share* $h$*, ignoring multiplicity,* if the equality $f(x) = h(x)$ is equivalent to the equality $g(x) = h(x)$.

Now we can state Theorem 6:

**Theorem 6:** *Let* $f,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$ *and* $\rho(h) < \widetilde{\rho}(f)$*. Then* $h$ *is a small function with respect to* $f$*.*

**Corollary 6.1:** *Let* $f,\ h \in \mathcal{A}(\mathbb{K})$*, be such that* $f$ *is regular and* $0 < \rho(f) < +\infty$ *and* $\rho(h) < \rho(f)$*. Then* $h$ *is a small function with respect to* $f$*.*

**Theorem 7:** *Let* $f,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $\rho(h) = \rho(f)$*,* $0 < \rho(f) < +\infty$ *and* $\sigma(h) = 0 < \widetilde{\sigma}(f)$*. Then* $h$ *is a small function with respect to* $f$*.*

**Corollary 7.1:** *Let* $f,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $\rho(h) = \rho(f)$ *and such that* $0 < \rho(f) < +\infty$*. If* $f$ *is clean, and if* $\sigma(h) = 0 < \sigma(f)$*, then* $h$ *is a small function with respect to* $f$*.*

Moreover, we notice that when $r$ is big enough, in each disk $d(0,r)$, the number of zeros of $f+h$ equals this of $f$, therefore $\psi(f+h) = \psi(f)$.

In [5] the following Theorem B was given and it will be useful now. It is also a consequence of results of [10].

**Theorem B:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *share* $3$ *small functions* $h_1, h_2, h_3 \in \mathcal{A}(\mathbb{K})$, *ignoring multiplicity. Then* $f = g$.

Now Corollary 7.2 is an immediate consequence of Theorem 6 and Theorem B.

**Corollary 7.2:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *share* $h_1, h_2, h_3 \in \mathcal{A}(\mathbb{K})$, *ignoring multiplicity, such that* $\max_{1\leq j\leq 3}(\rho(h_j)) < \min(\widetilde{\rho}(f), \widetilde{\rho}(g))$. *Then* $f = g$.

And by Theorem 7, we have Corollary 7.3:

**Corollary 7.3:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *share* $h_1, h_2, h_3 \in \mathcal{A}(\mathbb{K})$, *ignoring multiplicity, such that* $\rho(h_j) = \rho(f) = \rho(g),\ j = 1,2,3,\ \sigma(h_j) = 0,\ j = 1,2,3$ *and* $0 < \min(\widetilde{\sigma}(f), \widetilde{\sigma}(g))$. *Then* $f = g$.

**Corollary 7.4:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *be clean and share* $h_1, h_2, h_3 \in \mathcal{A}(\mathbb{K})$, *ignoring multiplicity, such that* $\rho(h_j) = \rho(f) = \rho(g),\ j = 1,2,3,\ \sigma(h_j) = 0,\ j = 1,2,3$ *and* $0 < \min(\sigma(f), \sigma(g))$. *Then* $f = g$.

**Theorem 8:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f \in \mathcal{A}(\mathbb{K})$ *and let* $h \in \mathcal{A}(\mathbb{K})$ *satisfy, for a certain* $R > 0$, $|h|(r) < |f|(r)\ \forall r > R$. *Then* $\rho(f+h) = \rho(f)$, $\sigma(f+h) = \sigma(f)$ *and* $\psi(f+h) = \psi(f)$.

**Corollary 8.1:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f \in \mathcal{A}(\mathbb{K})$ *and let* $h \in \mathcal{A}(\mathbb{K})$ *be a small function with respect to* $f$. *Then* $\rho(f+h) = \rho(f)$, $\sigma(f+h) = \sigma(f)$ *and* $\psi(f+h) = \psi(f)$.

**Theorem 9:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$. *Then* $\rho(f') = \rho(f)$, $\sigma(f') = \sigma(f)$. *Moreover, if* $f$ *is clean, then* $\psi(f') \geq \psi(f)$ *and if* $f'$ *is clean, then* $\psi(f') \leq \psi(f)$.

**Corollary 9.1:** *Suppose that* $\mathbb{K}$ *is of characteristic* $0$. *Let* $f \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$. *If* $f$ *and* $f'$ *are clean, then* $\psi(f') = \psi(f)$.

By Theorem 9, we can immediately derive Question 2:

**Question 2:** *Suppose that* $\mathbb{K}$ *is of characteristic* 0*. Let* $f \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$*. Do we have* $\psi(f') = \psi(f)$*?*

In [3], the first statement of Theorem C is proved.The second statement is easy:

**Theorem C:** *Suppose that* $\mathbb{K}$ *has residue characteristic* 0*. Then for every* $f \in \mathcal{A}(\mathbb{K})$ *such that* $0 < \rho(f) < +\infty$*, we have* $\psi(f') = \psi(f)$*. Moreover, if* $f$ *is clean, so is* $f'$*.*

Question 2, in the general case, then seems natural, as suggested in [2] and [3]. However, by Theorems 2 and 7, we can write Corollary 9.2:

**Corollary 9.2:** *Suppose that* $\mathbb{K}$ *is of characteristic* 0*. Let* $f \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho(f) < +\infty$*. Then,*

$$|\psi(f) - \psi(f')|_\infty \leq \rho(f)[(e-1)\sigma(f) - \widetilde{\sigma}(f)].$$

In order to prove Theorem 10, we need to recall the second Main Nevanlinna Theorem for p-adic entire functions (for example Theorem C.4.24 in [6]) and first, we will need the counter functions of zeros for an entire function.

Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

**Question 3:** If an entire function $f \in \mathcal{A}(\mathbb{K})$ is clean, is $f'$ clean too?

**Definitions:** Let $f \in \mathcal{A}(\mathbb{K})$. We denote by $Z(r, f)$ the counting function of zeros of $f$ in $d(0, r)$ defined in the following way.

Let $\omega_0(f)$ be the order of multiplicity of 0 if it is a zero of $f$ and let $\omega_0(f) = 0$ else.

Let $(a_n)$, $(1 \leq n \leq q(r))$ be the finite sequence of zeros of $f$ such that $0 < |a_n| \leq r$, of respective order $s_n$.

We set $Z(r, f) = \max(\omega_0(f), 0)Logr + \sum_{n=1}^{q(r)} s_n(Logr - Log|a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of* $f$ *in* $d(0, r)$*, counting multiplicity.*

In order to define the counting function of zeros of $f$ ignoring multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) = 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of $f$ ignoring multiplicity:

$\overline{Z}(r, f) = \overline{\omega_0}(f)Logr + \sum_{n=1}^{q(r)}(Logr - Log|a_n|)$ and so, $\overline{Z}(r, f)$ is called *the counting function of zeros of $f$ in $d(0, r)$ ignoring multiplicity.*

And we denote by $Z^0(f', r)$ the counting function of the zeros of $f'$ that are zeros of $f - a_n$ for any $n \leq q(r)$.

**Theorem N:** *Suppose that $\mathbb{K}$ is of characteristic 0. Let $f \in \mathcal{A}(\mathbb{K})$ and let $a_1, ..., a_q \in \mathbb{K}$. Then*

$$(q-1)Log(|f|(r)) \leq \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r) - Log(r) + O(1).$$

**Theorem 10:** *Suppose that $\mathbb{K}$ is of characteristic 0. Let $f \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho(f) < +\infty$ and let $a_1, ..., a_q \in \mathbb{K}$. Then*

$$(q-1)\sigma(f) \leq \limsup_{r\to+\infty} \Big(\frac{1}{r^{\rho(f)}} \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r)\Big).$$

**Corollary 10.1:** *Suppose that $\mathbb{K}$ is of characteristic 0 and let $f \in \mathcal{A}(\mathbb{K})$ be clean. Let $a_1, ..., a_q \in \mathbb{K}$. Then*

$$(q-1)\sigma(f) \leq \liminf_{r\to+\infty} \Big(\frac{1}{r^{\rho(f)}} \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r)\Big)$$

*and*

$$(q-1)\psi(f) \leq \liminf_{r\to+\infty} \Big(\frac{\rho(f)}{r^{\rho(f)}} \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r)\Big).$$

We will now answer the question 1.

**Theorem 11:** *Suppose that $\mathbb{K}$ is of characteristic 0. There exist regular non-clean functions $f \in \mathcal{A}(\mathbb{K})$ such that $\psi(f) > \rho(f)\sigma(f)$.*

## II. Proofs of theorems

**Proof. of Theorem 1:** All conclusions are easy except that $\rho(fg) = \max(\rho(f), \rho(g))$. It is clear that $\rho(fg) \geq \max(\rho(f), \rho(g))$, since $|fg|(r) = |f|(r)|g|(r)$. Now, let $t = \max(\rho(f), \rho(g)$. Then there exists a function $\omega$ defined in $\mathbb{R}_+$, of limit 0 at $\infty$, such that $\frac{Log(Log(|f|(r)))}{Log(r)} \leq t + \omega(r)$ and $\frac{Log(Log(|g|(r)))}{Log(r)} \leq t + \omega(r)$. Hence, we have

$$Log(|f|(r)) \leq r^{t+\omega(r)},\ Log(|g|(r)) \leq r^{t+\omega(r)}$$

hence

$$Log(|f|(r)) + Log(|g|(r)) \leq 2r^{t+\omega(r)}$$

therefore

$$Log\Big(Log(|f|(r)) + Log(g|(r))\Big) \leq Log(2) + (t + \omega(r))Log(r)$$

hence,

$$Log\Big(Log(|f|(r)) + Log(g|(r))\Big) \leq Log(2) + (t + \omega(r))Log(r)$$

hence

$$\frac{Log(Log(|f|(r).|g|(r)))}{Log(r)} \leq \frac{Log(2)}{Log(r)} + t + \omega(r)$$

and hence

$$\limsup_{r \to +\infty} \frac{Log(Log(|f|(r).|g|(r)))}{Log(r)} \leq t.$$

Consequently, $\rho(fg) \leq \max(\rho(f), \rho(g))$, which ends the proof.

In the proof of Theorem 2, we will use the following Lemma 1 that is classical [6]:

**Lemma 1:** *Let $f(x) \in \mathcal{A}(\mathbb{K}))$ be such that $f(0) \neq 0$, let $r \in ]0, R[$ and let $a_j,\ 1 \leq j \leq q$ be the zeros of $f$ in $d(0, r)$, of respective multiplicity $m_j$. Then*

$$Log(|(f|(r)) = Log(|f(0)|) + \sum_{j=1}^{q} m_j(Log(r) - Log|a_j|).$$

**Proof. of Theorem 2:** Without loss of generality we can assume that $f(0) = 1$. Let $u = \rho(f)$. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of zeros of $f$ with $|a_n| \leq |a_{n+1}|,\ n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let $w_n$ be the multiplicity order of $a_n$. For every $r > 0$, let $k(r)$ be the integer such that $|a_n| \leq r\ \forall n \leq k(r)$

and $|a_n| > r\ \forall n > k(r)$. Then by Lemma 1, $Log(|f|(r))$ is of the form $\sum_{n=0}^{k(r)} w_n(Log(r) - Log(|a_n|))$ hence, we have $\sigma(f,r) = \frac{\sum_{n=0}^{k(r)} w_n(Log(r) - Log(|a_n|))}{r^{\rho(f)}}$.
In the same way, for any $r > 0$ and $n \in \mathbb{N}$, we put $c_n = |a_n|$, $\psi(f,r) = \frac{s(r,f)}{r^{\rho(f)}}$.

We first show the inequality $\rho(f)\sigma(f) \le \psi(f)$. By definition of $\sigma(f,r)$ we can derive

$$\sigma(f,r) = \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(r) - Log(re^{-\alpha})\big)}{r^u}$$

$$+ \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(re^{-\alpha}) - Log(c_n)\big)}{r^u} + \sum_{k(re^{-\alpha})<n\le k(r)} \frac{w_n(Log(r) - Log(c_n)}{r^u},$$

hence

$$\sigma(f,r) \le \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(r) - Log(re^{-\alpha})\big)}{r^u}$$

$$+ \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(re^{-\alpha}) - Log(c_n)\big)}{r^u} + \alpha \sum_{k(re^{-\alpha})<n\le k(r)} \frac{w_n}{r^u},$$

because $Log(r) - Log(c_n) \le \alpha\ \forall n \in [k(re^{-\alpha}), k(r)] \cap \mathbb{N}$. Consequently,

$$\sigma(f,r) \le \alpha \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{r^u} + \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(re^{-\alpha}) - Log(c_n)\big)}{r^u}$$

$$+\alpha \sum_{k(re^{-\alpha})<n\le k(r)} \frac{w_n}{r^u}$$

therefore

$$\sigma(f,r) \le \alpha \sum_{n=0}^{k(r)} \frac{w_n}{r^u} + \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(Log(re^{-\alpha}) - Log(c_n)\big)}{r^u}$$

hence

$$\sigma(f,r) \le e^{-u\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(Log(re^{-\alpha}) - Log(c_n))}{(re^{-\alpha})^u} + \alpha \sum_{0\le n\le k(r)} \frac{w_n}{r^u}.$$

Thus we have

$$\sigma(f,r) \leq e^{-u\alpha}\sigma(f, re^{-\alpha}) + \alpha\psi(f,r). \tag{1}$$

Suppose first that $\sigma(f) < +\infty$. We check that we can pass to superior limits on both sides, so we obtain $\sigma(f) \leq e^{-u\alpha}\sigma(f) + \alpha\psi(f)$ therefore $\sigma(f)\dfrac{(1-e^{-u\alpha})}{\alpha} \leq \psi(f)$. That holds for every $\alpha > 0$, hence by de l'Hopital's theorem, we can derive

$$\psi(f) \geq \rho(f)\sigma(f). \tag{2}$$

Now by (1), we have

$$\sigma(f,r)(1-e^{-u\alpha}) \leq \alpha\psi((r,f),$$

hence passing to inferior limits on both sides, we deduce

$$\frac{\widetilde{\sigma}(f) - e^{-u\alpha}\sigma(f)}{\alpha} \leq \widetilde{\psi}(f)$$

hence

$$\frac{u(\widetilde{\sigma}(f) - e^{-u\alpha}\sigma(f))}{u\alpha} \leq \widetilde{\psi}(f)$$

therefore when $\alpha u = 1$, we obtain

$$\rho(f)(\widetilde{\sigma}(f) - \frac{\sigma(f)}{e}) \leq \widetilde{\psi}(f). \tag{3}$$

We will now show the inequality

$$\psi(f) \leq \rho(f)(e\sigma(f) - \widetilde{\sigma}(f)).$$

Let us fix $\alpha > 0$. We can write

$$\sigma(f,r) = \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(Log(r) - Log(re^{-\alpha}))}{r^u}$$

$$+ \sum_{j=0}^{k(re^{-\alpha})} \frac{w_j(Log(re^{-\alpha}) - Log(c_n))}{r^u} + \sum_{k(re^{-\alpha})<j\leq k(r)} \frac{w_j(Log(r) - Log(c_j))}{r^u}$$

hence

$$\sigma(f,r) \geq \alpha \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{r^u} + \sum_{j=0}^{k(re^{-\alpha})} \frac{w_j(Log(re^{-\alpha}) - Log(c_n))}{r^u}$$

hence

(4) $$\sigma(f,r) \geq \alpha e^{-u\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{(re^{-\alpha})^u} + e^{-u\alpha} \sum_{j=0}^{k(re^{-\alpha})} \frac{w_n(Log(re^{-\alpha}) - Log(c_n))}{(re^{-\alpha})^u}$$

and hence

(5) $$\sigma(f,r) \geq \alpha e^{-u\alpha}\psi(f,re^{-\alpha}) + e^{-u\alpha}\sigma(f,re^{-\alpha}).$$

Therefore, we can deduce

$$\alpha e^{-u\alpha}\psi(f) \leq \limsup_{r\to+\infty} \Big(\sigma(f,r) - e^{-u\alpha}\sigma(f,re^{-\alpha}))\Big)$$

and therefore

(6) $$\alpha e^{-u\alpha}\psi(f) \leq \sigma(f) - e^{-u\alpha}\widetilde{\sigma}(f)).$$

That holds for every $\alpha > 0$ and hence, when $u\alpha = 1$, by (6) we obtain

(7) $$\psi(f) \leq \rho(f)\big(e\sigma(f) - \widetilde{\sigma}(f)\big)$$

which is the left hand inequality of the general conclusion.

Particularly, we notice that when $\sigma(f) < +\infty$, then $\psi(f) < +\infty$. Now, on (4) we can also take the inferior limit on both sides and we deduce

$$\widetilde{\sigma}(f) \geq \alpha e^{-u\alpha}\widetilde{\psi}(f) + e^{-u\alpha}\widetilde{\sigma}(f)$$

therefore

$$\alpha e^{-u\alpha}\widetilde{\psi}(f) \leq \widetilde{\sigma}(f)(1 - e^{-u\alpha}).$$

Then when $u\alpha$ tends to 0 we have

(8) $$\widetilde{\psi}(f) \leq \rho(f)\widetilde{\sigma}(f).$$

Now, suppose that $\sigma(f) = +\infty$. We can find an increasing sequence $r_n$ of limit $+\infty$ such that
$\sigma(f,r_n) = \sup\{\sigma(f,r) \mid r \leq r_n\},\ n \in \mathbb{N}$. Consider (1) when $\min(\alpha, u\alpha) > 1$. Then $\sigma(f,r_ne^{-\alpha}) < \sigma(f,r)_n$, hence of course

$$(1 - e^{-u\alpha})\sigma(f,r_ne^{-\alpha}) \leq (1 - e^{-u\alpha})\sigma(f,r_n)$$

and hence $\sigma(f,r_n) - e^{-u\alpha}\sigma(f,r_ne^{-\alpha}) \geq (1 - e^{-u\alpha})\sigma(f,r_n)$, therefore $(1 - e^{-u\alpha})\sigma(f,r_n) \leq \psi(f,r_n)$, which proves that $\psi(f) = +\infty$.

Thus, $\sigma(f) < +\infty$ is equivalent to $\psi(f) < +\infty$. Consequently, Relations (2), (3), (5), (7), (8) still apply and hence hold as soon as $\sigma(f) < +\infty$ or $\psi(f) < +\infty$.

Now suppose that $\widetilde{\psi}(f) = \psi(f)$. Then we have

$$\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\widetilde{\sigma}(f)$$

therefore $\sigma(f) = \widetilde{\sigma}(f)$, since $\rho(f) > 0$.

Conversely, suppose that $\sigma(f) = \widetilde{\sigma}(f)$. Then by (6) we have

$$\psi(f) \leq \sigma(f)\Big(\frac{e^{u\alpha} - 1}{\alpha}\Big).$$

That holds for every $\alpha > 0$ and then, $\psi(f) \leq u\sigma(f)$, i.e. $\psi(f) \leq \rho(f)\sigma(f)$, hence by (2) we have, $\psi(f) = \rho(f)\sigma(f)$.

But now, by (1), we see that

$$\alpha\psi(f,r) \geq \sigma(f,r) - e^{-u\alpha}\sigma(f, re^{-\alpha})$$

hence, passing to the inferior limit,

$$\alpha\widetilde{\psi}(f) \geq \sigma(f)(1 - e^{-u\alpha})\ \forall\alpha > 0$$

therefore $\widetilde{\psi}(f) \geq \rho(f)\sigma(f)$. But we just showed that $\psi(f) = \rho(f)\sigma(f)$, hence $\widetilde{\psi}(f) = \psi(f)$.

**Proof of Theorem 3** Let $s = \rho(f) \geq t = \rho(g)$. Then $\rho(fg) = s$, hence

$$\sigma(fg) = \limsup_{r\to+\infty} \frac{Log(|fg|(r))}{r^s} = \limsup_{r\to+\infty} \frac{Log(|f|(r).|g|(r))}{r^s}.$$

Now, if $s > t$, then

$$\sigma(fg) = \limsup_{r\to+\infty} \Big(\frac{Log(|f|(r))}{r^s} + \frac{Log(|g|(r))}{r^s}\Big)$$

$$\leq \limsup_{r\to+\infty} \frac{Log(|f|(r))}{r^s} + \limsup_{r\to+\infty} \frac{Log(|g|(r))}{r^t} = \sigma(f) + \sigma(g).$$

Then we notice that when $t < s$, we have

$$\limsup_{r\to+\infty} \frac{Log(|fg|(r))}{r^s} = \limsup_{r\to+\infty} \frac{Log(|f|(r))}{r^s} = \sigma(f)$$

Particularly, if $f$ is clean we have limits instead of limitsup as long as $f$ is concerned. Consequently, if $t < s$, then $fg$ is clean.

Now, suppose that $f$ and $g$ are clean and that $s = t$. Then

$$\limsup_{r\to+\infty} \frac{Log(|f|(r)) + Log(|g|(r))}{r^t} = \lim_{r\to+\infty} \frac{Log(|f|(r)) + Log(|g|(r))}{r^t}$$

$$= \sigma(f) + \sigma(g).$$

Thus $fg$ is clean. And by Theorem 1 and Theorem 2, we have $\psi(fg) = \rho(fg)\sigma(fg) = \rho(f)(\sigma(f) + \sigma(g)) = \psi(f) + \psi(g)$.

**Remark:** A similar proof applies to complex entire functions.

**Proof of Theorem 4**. By hypothesis, there exist $a > 0$ and $R > 0$ such that $\dfrac{Log(|f|(r))}{r^{\rho(f)}} \geq a\ \forall r \geq R$ hence $Log(Log(|f|(r))) \geq Log(a) + \rho(f)Log(r)\ \forall r \geq R$ therefore

$$\frac{Log(Log(|f|(r)))}{Log(r)} \geq \frac{Log(a)}{Log(r)} + \rho(f)\ \forall r \geq R$$

and hence $\widetilde{\rho}(f) \geq \rho(f)$ i.e. $\widetilde{\rho}(f) = \rho(f)$.

In order to prove Theorem 5, we must recall the following Theorem D which is Theorem 1 in [1] and Corollary D.1 which is Theorem 4 in [7] and derives from Theorem D.

**Theorem D:** *Suppose that* $\mathbb{K}$ *is of characteristic* 0. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental, admitting a primitive* $F$. *If there there exists* $c > 0$ *and* $u > 0$ *such that the number of multiple poles of* $F$, *taking multiplicity into account,* $\phi(r, F)$, *satisfy* $\phi(r, F) \leq cr^u$, *then* $f$ *has no quasi-exceptional value.*

**Corollary D.1:** *Suppose that* $\mathbb{K}$ *is of characteristic* 0. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental, admitting primtiives. If* $Log(t(r, f)) \leq O(Log(r))$, *then* $f$ *has no quasi-exceptional value.*

**Proof of Theorem 5**. Let $f = \dfrac{g}{h}$ admit primitives and be such that $\psi(h) < +\infty$. Then $s(r, h) \leq (\psi(h) + 1)r^{\rho(h)}$ when $r$ is big enough. Consequently, $t(r, f) = s(r, h) \leq (\psi(h) + 1)r^{\rho(h)}$. Therefore by Corollary D.1, $f$ has infinitely many zeros. The same applies to $f - b$ for every $b \in \mathbb{K}$, which ends the proof.

**Proof of Corollary 5.2**. Indeed, a meromorphic function having no residue different from zero admits primitives [6].

**Proof of Corollary 5.3**. Let $f = \dfrac{g}{h}$ with $\sigma(h) < +\infty$. Then $f' = \dfrac{g'h - h'g}{h^2}$ and $\sigma(h^2) = 2\sigma(h)$. Thus one can apply Corollary 3.1 to $f'$.

**Proof of Theorem 6**. By hypothesis, there exists $\lambda > 0$ and $R > 0$ such that

$$\frac{Log(Log(|h|(r)))}{Log(r)} + 2\lambda < \widetilde{\rho}(f)\ \forall r > R$$

and hence there exists $R' > R$ such that

$$Log(Log(|h|(r))) + \lambda Log(r) < Log(Log(|f|(r)))\ \forall r > R'$$

therefore $r^{\lambda} Log(|h|(r)) < Log(|f|(r))$, which proves that
$\lim_{r \to +\infty} \dfrac{Log(|h|(r))}{Log(|f|(r))} = 0$, what ends the proof.

**Proof of Theorem 7**. By hypothesis, we have

$$\lim_{r \to +\infty} \frac{Log(|h|(r)))}{Log(|f|(r))} = 0$$

hence $Log(|h|(r))$ is of the form $(Log(|f|(r))(\theta(r))$, with $\lim_{r \to +\infty} \theta(r) = 0$. Therefore, $|h|(r) < |f|(r))$ when $r$ is big enough and hence $|f + h|(r) = |f|(r)$, therefore $\rho(f + h) = \rho(f)$. Then

$$\limsup_{r \to \infty} \frac{\log(f| + h|(r))}{r^{\rho(f+h)}} = \limsup_{r \to \infty} \frac{\log(f|(r))}{r^{\rho(f)}} = \sigma(f),$$

hence $\sigma(f + h) = \sigma(f)$.

Now, there exists $R > 0$ such that $|f + h|(r) = |f|(r) \forall r > R$. Consequently the number of zeros of $f + h$ in each disk $d(0, r)$ equals the number of zeros of $f$ in $d(0, r)$, for every $r > R$. Consequently, since $\rho(f + h) = \rho(f)$, we have $\psi(f + h) = \psi(f)$.

**Proof of Theorem 8**. By hypothesis, we have $|h|(r) < |f|(r)$ when $r$ is big enough and hence

$$|f + h|(r) = |f|(r) \tag{1}$$

therefore $\rho(f + h) = \rho(f)$. Consequently, $\sigma(f + h) = \sigma(f)$.

Moreover, by (1) we notice that when $r$ is big enough, by classical results [6], in each disk $d(0, r)$, the number of zeros of $f + h$ equals this of $f$, therefore $\psi(f + h) = \psi(f)$.

**Proof of Theorem 9**. The statements $\rho(f') = \rho(f)$ and $\sigma(f') = \sigma(f)$ are given in [2] and [3]. Now, suppose that $\sigma(f) = \widetilde{\sigma}(f)$. Set $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. By Theorems 2 and C,

$$e\psi(f) = e\rho(f)\sigma(f) = \limsup_{n\to+\infty} n \sqrt[n]{|a_n|^{\rho(f)}}.$$

But we know that $\frac{1}{n} \leq |n| \leq 1 \ \forall n \in \mathbb{N}$, hence $\lim_{n\to+\infty} \sqrt[n]{|n+1|} = 1$, therefore

$$e\psi(f) = \limsup_{n\to+\infty} n \sqrt[n]{|(n+1)a_{n+1}|^{\rho(f)}} = e\sigma(f')\rho(f') \leq e\psi(f').$$

Similarly, if $\sigma(f') = \widetilde{\sigma}(f')$, then we can see that $\psi(f) \geq \psi(f')$.

In order to prove Theorem 10, we need to recall. the second main Nevanlinna Theorem for p-adic entire functions (for example Theorem C.4.24 in [6]).

**Theorem N:** *Suppose that* $\mathbb{K}$ *is of characteristic* 0*. Let* $f \in \mathcal{A}(\mathbb{K})$ *and let* $a_1, ..., a_q \in \mathbb{K}$*. Then*

$$(q-1)Log(|f|(r)) \leq \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r) - Log(r) + O(1).$$

**Proof of Theorem 10**. We have $\sigma(f, r) = \frac{Log(|f|(r))}{r^{\rho(f)}}$, hence by Theorem N,

$$r^{\rho(f)}(q-1)\sigma(f, r) \leq \sum_{i=1}^{q} \overline{Z}(r, f - a_i) - Z^0(f', r) - Log(r) + O(1).$$

The conclusion is then obvious.

In the proof of Theorem 11, we will use the following basic lemmas:

**Lemma 2** *Let* $f_1, \ f_2$ *be two functions from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$ *such that*

$$\lim_{x\to+\infty} f_1(x) = \lim_{x\to+\infty} f_2(x) = +\infty$$

*and*

$$\limsup_{x\to+\infty} \frac{f_1(x)}{f_2(x)} = b \in \mathbb{R}_+, \ \liminf_{x\to+\infty} \frac{f_1(x)}{f_2(x)} = a > 0.$$

*Then*

$$\lim_{x \to +\infty} \frac{Log(f_1(x))}{Log(f_2(x))} = 1.$$

**Lemma 3:** *Let $\alpha,\ \beta \in \mathbb{R}_+$ and let $g(x) = e^{-x}(\alpha x - \beta)$. Then $g'$ has a unique zero at $1+\dfrac{\beta}{\alpha}$ and $g(1+\dfrac{\beta}{\alpha}) = \alpha e^{-(1+\frac{\beta}{\alpha})}$. Moreover, $g$ is increasing in $[0, 1+\dfrac{\beta}{\alpha}]$ and is decreasing in $[1+\dfrac{\beta}{\alpha}, +\infty[$ and tends to $0$ when $x$ tends to $+\infty$.*

Now let us recall that the definition of a *divisor in* $\mathbb{K}$.

**Defintion:** We call *divisor* in $\mathbb{K}$ a sequence $(a_n, v_n)_{n\in\mathbb{N}}$ where $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{K}$ such that $\lim\limits_{n\to+\infty} |a_n| = +\infty$ and each $v_n$ belongs to $\mathbb{N}*$ [6].

The following Lemma 4 is classical (see for example Corollary B.18.5 of [6]).

**Lemma 4:** *Given a divisor $(a_n, v_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$, there exist functions admitting each $a_n$ as a zero of order $v_n$ and no other zero and two such functions are proportional.*

**Proof of Theorem 11**. We begin the definition of positive increasing sequences $(r_m)_{m\in\mathbb{N}}$, $(\alpha_m)_{m\in\mathbb{N}}$ in $\mathbb{N}^*$, $(\beta_m)_{m\in\mathbb{N}}$, where $r_0 = 1$, $r_{2m} \in |\mathbb{K}|$, $2r_{2m} \le \alpha_m < 2r_{2m} + 1$, We put $q_m = \alpha_m - \alpha_{m-1}$, $\nu_m = Log(r_m)$, $\beta_0 = 0$ and $\beta_m = \beta_{m-1} + q_m(Log(r_{2m}))$.

In $[0, +\infty[$, we define $g_k(\nu) = e^{-\nu}(\alpha_k \nu - \beta_k)$ up to the rank $m$ and suppose that the function $g_k$ satisfies $1 \le g_k(\nu_{2k}) \le 1 + \frac{1}{4k^2}$ and $1 \le g_k(\nu_{2k+2}) \le 1 + \frac{1}{4(k+1)^2}$ $\forall k = 1, ..., m-1$ and $g_{k-1}(\nu_{2k}) = g_k(\nu_{2k})$.

By Lemma 3, $g_k$ is increasing in $[\dfrac{\beta_k}{\alpha_k}, 1+\dfrac{\beta_k}{\alpha_k}]$ from $0$ to a maximum equal to $\alpha_k e^{1+\frac{\beta_k}{\alpha_k}}$ and is decreasing to $0$ when $\nu$ tends to $+\infty$. Hence $g_k$ takes the value $1$ at a unique point $\lambda_{2k}$ in $[\dfrac{\beta_k}{\alpha_k}, 1+\dfrac{\beta_k}{\alpha_k}]$ and at a unique point $\lambda_{2k+2} \in [1+\dfrac{\beta_k}{\alpha_k}, +\infty[$. We then have $g_k(\lambda_{2k}) = e^{-\lambda_{2k}}(\alpha_k\lambda_{2k} - \beta_k)$ and $g_k(\lambda_{2k+2}) = e^{-\lambda_{2k+2}}(\alpha_k\lambda_{2k+2} - \beta_m)$ hence
$\lambda_{2k} = \dfrac{e^{\lambda_{2k}} + \beta_k}{\alpha_k}$ and $\lambda_{2k+2} = \dfrac{e^{\lambda_{2k+2}} + \beta_k}{\alpha_k} > \nu_{2k+1}$ and we can take the value $r_{2k+2} \in |\mathbb{K}|$ close enough to $e^{\lambda_{2k+2}}$ such that, putting $r_{2k+2} = e^{\nu_{2k+2}}$, we then have

$$1 \le g_k(\nu_{2k+2}) = 1 + x_k \le 1 + \frac{1}{4(k+1)^2}$$

and

(1) $$1 \le g_k(\nu_{2k}) = 1 + y_k \le 1 + \frac{1}{4(k)^2}.$$

We notice that $r_{2k+2} > r_{2k+1}$ hence $\nu_{2k+2} > \nu_{2k+1}$. Next the function $g_{k+1}$ is defined in the same way in $[\nu_{2k+2}, \nu_{2k+4}]$ as $g_{k+1}(\nu) = e^{-\nu}(\alpha_{k+1}\nu - \beta_{k+1})$. And we can check that $g_{k+1}(\nu_{2k+2}) = g_k(\nu_{2k+2})$.

Then by Lemma 3, $g_m$ has a maximum at

(2) $$\nu_{2m+1} = 1 + \frac{\beta_m}{\alpha_m},$$

and $g_{m+1}$ has a maximum at $\nu_{2m+3} = 1+\dfrac{\beta_{m+1}}{\alpha_{m+1}}$ and $g_{m+1}(\nu_{2m+3}) = \alpha_{m+1}e^{\frac{\beta_{m+1}}{\alpha_{m+1}}} > 1$, hence $\nu_{2m+3} > \nu_{2m+2}$. Consequently, the sequence $(r_n)_{n\in\mathbb{N}}$ and is strictly increasing. This way, the sequences are now defined for all $m \in \mathbb{N}$. Recall that $q_m = \alpha_m - \alpha_{m-1}$. We put $\Theta_m = \nu_{2m+1} - \nu_{2m}$. Then, $\nu_{2m} = \dfrac{\beta_m + e^{\nu_{2m}}(1 + x_m)}{\alpha_m}$ and hence by (1) and (2) we obtain

(3) $$\Theta_m = 1 - \frac{e^{\nu_{2m}}(1 + y_m)}{\alpha_m} = 1 - \frac{r_{2m}(1 + y_m)}{\alpha_m} = 1 - \frac{r_{2m}(1 + y_m)}{2r_{2m} + \eta_m},$$

where $(\eta_m)_{m\in\mathbb{N}}$ is a positive sequences bounded by 1 and the sequence $(y_m)$, by (1), satisfies $0 \le y_m \le \dfrac{1}{4(m)^2}$. Then

(4) $$\Theta_m \ge \frac{1}{2} - \frac{1}{8(m)^2} > \frac{15}{32}.$$

We can now define by induction the sequences $(r_m)$, $(\nu_m)$, $(g_m)$, $(y_m)$, $(\Theta_m)$ and then $\lim_{n\to+\infty} r_n = +\infty$. Consequently by (3) and (4),

(5) $$\lim_{m\to+\infty} \Theta_m = \frac{1}{2}.$$

We now obtain

$$g_m(\nu_{2m+1}) = e^{-\nu_{2m+1}}(\alpha_m\nu_{2m+1} - \beta_m) = \alpha_m e^{-(1+\frac{\beta_m}{\alpha_m})}$$

and hence, by (2),

$$(6) \qquad g_m(\nu_{2m+1}) = \frac{(2r_{2m} + \eta_m)}{r_{2m+1}} = 2e^{-\Theta_m} + \zeta_m.$$

where $(\zeta_m)_{m\in\mathbb{N}}$ is a positive sequence of limit 0, since $\lim\limits_{m\to+\infty} r_{2m+1} = +\infty$.

We can now define a function $g$ in $[0, +\infty[$ as $g(\nu) = g_m(\nu)$ when $\nu \in [Log(r_{2m}), Log(r_{2m+2})[$.

So, by (5) we have

$$(7) \qquad \lim_{m\to+\infty} g(\nu_{2m+1}) = \frac{2}{\sqrt{e}}.$$

Thus, we can check that

$$(8) \qquad \limsup_{\nu\to+\infty} g(\nu) = \limsup_{m\to+\infty} g(\nu_{2m+1}) < 2.$$

Now, by Lemma 4 we can consider the entire function $f$ admitting $q_m$ zeros on each circle $C(0, r_{2m})$ and no other zero. Let $(a_{j,m})_{(1\le j\le q_m)}$ be the zeros of $f$ on the circle $C(0, r_{2m})$.

Then, when $2m \le r < r_{2m+2}$, the counting functions of zeros of $f$ (counting multiplicity) is of the form

$$Z(f,\ r) = \sum_{k=1}^{m}\sum_{j=1}^{q_k}(a_{k,j}(Log(r) - Log(r_{2k})))$$

$$= \sum_{k=1}^{m} q_k(Log(r) - Log(r_{2k})))$$

and hence, putting $\alpha_m = \sum_{k=1}^m q_k$ and $\beta_m = \sum_{k=1}^m q_k r_{2k}$, the function $g$ appears as the quotient of the counting function of zeros of $f$ (counting multiplicity) by $e^\nu$ when we put $\nu = Log(r)$ . So, we have $\dfrac{Log(|f|(r))}{r} = g(\nu)$ whenever $Log(r) = \nu \in [\nu_{2m}, \nu_{2m+2}[$ and therefore by (1) we can see that

$$\liminf_{r\to+\infty} \frac{Log(|f|(r))}{r} = 1$$

and

$$(9)\qquad \limsup_{r\to+\infty} \frac{Log(|f|(r))}{r} = \lim_{m\to+\infty} |g(\nu_{2m+1})| = \frac{2}{\sqrt{e}}.$$

Moreover, by (9) and Lemma 2, we have

$$\lim_{r\to+\infty} \frac{Log(Log(|f|(r)))}{Log(r)} = 1$$

hence $\rho(f) = 1$.

Further, since $\limsup_{r\to+\infty} \frac{Log(|f|(r))}{r} = \frac{2}{\sqrt{e}}$ and $\liminf_{r\to+\infty} \frac{Log(|f|(r))}{r} = 1$ we can see that $\sigma(f) = \frac{2}{\sqrt{e}}$ and $\widetilde{\sigma}(f) = 1$. Thus, $f$ is not clean though it is regular.

More precisely, by construction, for every $r \in [r_{2m}, r_{2m+2}[$, we have $\psi(f,r) = \frac{2r_{2m}+\eta_m}{r}$ and hence $\psi(f, r_{2m})$ is of the form $2 + y_m$ where $(y_m)_{m\in\mathbb{N}}$ is a sequence of limit 0. Therefore $\psi(f) \geq 2$, while $\rho(f) = 1$. This shows that $f$ does not satisfy the relation $\psi(f) = \rho(f)\sigma(f)$ and hence, this is is not always satisfied when a function $f$ is not clean.

## III. Remarks

**Remark 1:** Of course, by Theorem 2 we know that the function $f$ built in the proof of Theorem 11 satisfies $\psi(f) > \widetilde{\psi}(f)$. But we can directly verify this: on one hand $\psi(f) = 2$ and on the other hand, we can see that $\psi(f, r_{2m+1}) = \frac{\alpha_m}{r_{2m+1}}$ and hence by (5), $\widetilde{\psi}(f) \leq \frac{2}{\sqrt{e}}$.

Next, $f$ must satisfy Theorem 2: $\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)(e\sigma(f) - \widetilde{\sigma}(f))$. Let us check. We have seen that $\psi(f) = 2,\ \rho(f) = 1,\ \sigma(f) = \frac{2}{\sqrt{e}},\ \widetilde{\sigma}(f) = 1$. Then, $\rho(f)(e\sigma(f) - \widetilde{\sigma}(f)) = 2\sqrt{e} - 1 > 2$. That is O'kay.

**Remark 2:** By Corollary 4.1, we see that a clean entire function such that $\sigma(f) > 0$ is regular. The converse is not true, as shows Theorem 11.

In complex analysis, given an entire function $f$, we put

$$M(f,r) = \sup\{|f(z)|_\infty,\quad |z|_\infty = r\}$$

where $| \, . \, |_\infty$ is the archimedean modulus on $\mathbb{C}$. In [4] the authors claimed that if a complex entire function $f$ satisfies

$$\limsup_{r\to+\infty} \frac{Log(Log(M(f,r)))}{Log(r)} = \liminf_{r\to+\infty} \frac{Log(Log(M(f,r)))}{Log(r)},$$

then

$$\limsup_{r\to+\infty} \frac{Log(M(f,r))}{r^\rho} = \liminf_{r\to+\infty} \frac{Log(M(f,r))}{r^\rho},$$

where $\rho = \lim_{r\to+\infty} \dfrac{Log(Log(M(f,r)))}{Log(r)}$. In the field $\mathbb{K}$, we just checked that such a theorem does not hold. Actually, the proof of [4] is put in doubt by the following argument held in Lemma 2 of [4]:

*since*

$$\int_{r_0}^{+\infty} \frac{exp(Log(M(r,f))}{(exp(r^\lambda))^{t-\varepsilon+1}} dr = +\infty,$$

*"then"*

$$\liminf_{r\to+\infty} \frac{exp(Log(M(r,f))}{(exp(r^\lambda))^{t-\varepsilon}} = +\infty.$$

Suppose for example that in $[r_0, +\infty[$, $M(r,f)$ is equivalent to $\dfrac{exp(r\lambda(t-\varepsilon+1))}{r}$.

Then $\dfrac{exp(r\lambda(t-\varepsilon+1))}{exp(r\lambda(t-\varepsilon))}$ is equivalent to $\dfrac{1}{r}$ and hence

$$\int_{r_0}^{+\infty} \frac{exp(Log(M(r,f))}{(exp(r^\lambda))^{t-\varepsilon+1}} dr = +\infty,$$

but

$$\liminf_{r\to+\infty} \frac{exp(Log(M(r,f))}{(exp(r^\lambda))^{t-\varepsilon}} = 0.$$

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