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Weighted sharing of sets in wider sense under the ambit of higher indexed polynomials

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Abstract. In this article, with respect to the recently introduced definition of weighted sharing of sets in the wider sense [8], we have comprehensively extended and improved some well known results in the literature. As the definition involves the manipulations of two polynomials, we perceive that the characteristizations of the underlying polynomials become utmost important. This realization urges us to introduce the definition of the index of polynomial. Noting that, in case of weighted sharing of sets the polynomials so far chosen are of lower index, after defining and streamlining the index concept, we explore the influence of using polynomials with higher indices in the context of weighted sharing of sets in wider sense to improve and extend a number of earlier results.

1. Introduction, background

Value distribution theory is a branch of complex analysis that deals with the distribution of values of analytic functions, especially in the context of meromorphic functions. It involves studying the distribution of zeros, poles and essential singularities of these functions. Set sharing, in the context of complex analysis, usually refers to the phenomenon where two different meromorphic

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functions share the same set of values under certain constraints. The detail definition is as follows:

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$. Let us denote by $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicities then the set $\bigcup_{a\in S}\{z: f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ ($\overline{E}_f(S)=\overline{E}_g(S)$) we say that f and g share the set S CM (IM).

If the readers need further information or a detailed explanation about these concepts, we recommend referring to the original sources cited in the text: [3] and [19]. These sources should provide the required background and information to understand the concepts being mentioned here.

Let us define λ_m over the set of natural numbers as follows:

$$\lambda_m=\begin{cases}2 & \text{if } m=1\\ 1 & \text{if } m\geq 2\end{cases}$$

In 1976, in connection to the famous question of Gross [18], Lin-Yi posed the question (see *Question B*, p. 74, [23]) pertains to meromorphic functions and their relationships when sharing two sets.

Question 1.1. *[23] Can one find two finite sets S_j $(j=1,2)$ such that any two non constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical?*

In 2001, the concept of weighted sharing of sets was introduced, which contributed to the uniqueness theory in complex analysis. The specific details and implications of this notion can be found in the paper by Lahiri [20].

Weighted sharing of sets involve studying the shared properties of sets under certain weighted criteria. This concept could have applications in various areas of uniqueness theory vis-a-vis value distribution theory. The definition is as follows:

Definition 1.1. *[20] Let l be a non negative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_l(a;f)$ the set of all a-points of f, where an a-point of multiplicity t is counted t times if $t\leq l$ and $l+1$ times if $t>l$. Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$. We denote by $E_f(S,l)$ the set $\bigcup_{a\in S}E_l(a;f)$. If $E_f(S,l)=E_g(S,l)$, we say f and g share the set S with weight l and denote it by (S,l). We say, $E_f(S)=E_f(S,\infty)$ and $\overline{E}_f(S)=E_f(S,0)$.*

During last few decades several investigations were done to resolve *Question 1.1.* The following theorem is the best result as far.

Theorem A. *[3] Let us suppose that the polynomial $R(z)$ is defined by*

$$R(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2.$$

Let $S = \{z : R(z) = 0\}$ and $n \geq 8$. Suppose f and g be two non constant meromorphic functions satisfying $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$, then $f \equiv g$.

In 1994, regarding three set sharing problem and the uniqueness of two meromorphic functions, a second relevant question was asked by Yi [27].

Question 1.2. *[27] Can one find three finite sets S_j $(j = 1, 2, 3)$ such that any two non constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ must be identical ?*

Numerous studies were conducted to answer *Question 1.2* (see [2], [7], [13], [9], [14]). Remarkably, the notion of weighted sharing significantly influenced subsequent research efforts related to Gross' question (see [3]-[5], [23], [30]). Among those results we would like to mention the following one which improve the results of [24], [31].

Theorem B. *[11] Let S be defined as in* Theorem A *with $ab^{(n-2)} \notin \{0, 1, 2\}$ and $n \geq 5$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S, l) = E_g(S, l)$, $E_f(\{0\}, k) = E_g(\{0\}, k)$ and $E_f(\{b\}, m) = E_g(\{b\}, m)$ then $f \equiv g$ for $(l, k, m) = (3, 2, 0), (2, 3, 1)$.*

2. Definitions

Recently [8] we have introduced a more comprehensive framework than *Definition 1.1* termed as 'weighted sharing of sets in wider sense' for meromorphic functions.

Definition 2.1. *[8] Let f and g be two non-constant meromorphic functions and $P(z)$ and $Q(z)$ be two polynomials of degree n without any multiple zero. Let*

$$S_P = \{z : P(z) = 0\} \quad \textit{and} \quad S_Q = \{z : Q(z) = 0\}.$$

We say that f and g share the sets S_P and S_Q with weight l in the wider sense if $E_f(S_P, l) = E_g(S_Q, l)$ and we denote it by f, g share $(S_P, S_Q; l)$.

We see that the above definition involves two polynomials that play a pivotal role in characterizing the concept of weighted sharing of sets for meromorphic functions in a wider sense. Thus it will be reasonable to pay attention to the structure of the polynomials used in *Definition 2.1.* If $P = Q$, we get the traditional definition of weighted sharing of sets. Next we recall the following definition that is necessary in subsequent stages.

Definition 2.2. *[12] A polynomial*

$$p(z) = a_n z^n + a_{n-1} z^{z-1} + \ldots + a_1 z + a_0$$

is called an initial term gap polynomial (ITGP) *if $a_i = 0$ but $a_j \neq 0$ for at least one j such that $1 \leq j < i < n$ and an initial term non-gap polynomial* (ITNGP) *if there does not exist any such i.*

In view of *Definition 2.2* we are now at a stage to introduce the definition of index for polynomial as follows:

Definition 2.3. *A polynomial $P^{[s]}(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ is said to be initial term non-gap polynomial of index s, $(1 \leq s \leq n)$ ($ITNGP_s$ in short) if the followings are satisfied*:
i) When $s = 1, \ldots, n-1$, then $a_n \neq 0, a_{n-1} \neq 0, \ldots, a_{n-\overline{s-1}} \neq 0$, but $a_{n-s} = 0$;
ii) When $s = n$ then $a_i \neq 0$, for $i = 1, 2, \ldots, n$.
Note that any polynomial of degree n is of index $s \geq 1$.

Definition 2.4. *[10] Let $P(z)$ be a polynomial such that $P'(z)$ has k mutually distinct zeros given by d_1, d_2, ..., d_k with multiplicities q_1, q_2, ..., q_k, respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, ..., k\}$.*

For the standard definitions and notations of the value distribution theory we refer to [19] and for the definitions of $N(r, a; f \mid \geq s)$, $N(r, a; f \mid = s)$ for $s \geq 1$, $\overline{N}_L(r, 1; f)$, $\overline{N}_L(r, 1; g)$, $N_E^{(k}(r, 1; f)$ and $\overline{N}_*(r, a; f, g)$ we refer to [2], [21], [22], [29].

3. Observations, motivations and main results

In 1995, Yi [28] introduced the following polynomial:

$$P_1(z) = z^n + az^{n-m} + b,$$

where n and m are two positive integers such that $(n, m) = 1$, $n > m$, a and b are two non-zero constants such that the algebraic equation $z^n + az^{n-m} + b = 0$

has no multiple zero. Clearly $P_1(z)$ is a polynomial of index λ_m. Till now a lot of research works have been performed on the uniqueness of two meromorphic functions sharing two sets for the case $m = 1$ (see [4], [5], [9], [13], [25]). However, the case $m \geq 2$ was not being prioritized as that leads to the larger cardinality of the set.

In 1998, Frank-Reinders [15] introduced a new polynomial as follows:

$$P_2(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c,$$

where $n \geq 3$ is an integer and $c \neq 0, 1$ is a complex number. Evidently, the above mentioned polynomial is of index 3. In 2017, this polynomial was further generalized by Banerjee-Mallick [12] in the following manner:

$$P_2^*(z) = z^n + az^{n-m} + bz^{n-2m} + c,$$

where n and $m(= 1)$ be two integers such that $n > 2m$ and a, b and c are three non-zero complex numbers such that $P_2^*(z)$ has no multiple zero and the polynomial is of index 3 as well. Note that the polynomial $R(z)$ use in *Theorem A* is a polynomial of index 1. All the polynomials have significant contributions in case two or three shared set problems.

From the matter discussed so far, we have the following observations:

i) In most of the cases of two shared sets problem, the second set is taken as $\{\infty\}$ as mentioned in *Theorem A*. Hence it is interesting to investigate the case when the second set is taken solely from $\mathbb{C}$. This is our first motivation.

ii) Next we recall the Bi-unique range sets problems (see [1], [26]). In Bi-unique range sets problems, a ground set is selected from $\mathbb{C}$ and the derived set is formed by considering the zeros of the derivative of the generating polynomial of the ground set.

The same situation has also been observed in case three set sharing problems i.e. in the same case one set is taken as 0 like *Theorem B*. So its natural to ask the question that whether 0 can be substituted by a non-zero complex number a in both the cases.

It has been found that 0 is present in the derived set of Bi-unique range sets. Same situation has also been observed in case of three sets sharing problems like *Theorem B*.

So we see that typically, the results of Bi-unique range sets and three set sharing have required the presence of 0 in the one set. The second motivation of writing the paper is to investigate whether the complex number 0 can be substituted by any other non-zero complex number a.

iii) It is clear that, previously the investigations were limited to studying polynomials of index at most three. However, in view of *Definition 2.3*, it is natural to be interested in broadening their exploration. This realization motivate us to solely nurture the cases where the index of the polynomials are ≥ 4. We will show that investigations of higher-indices polynomials can lead to new insights and challenges as they often exhibit more complex behaviors than that were available in the literature.

Let us take two polynomials,

$$
\begin{aligned}
P_6(z) &= \frac{z^6}{6} - \frac{(2a+3b)z^5}{5} + \frac{(a^2+6ab+3b^2)z^4}{4} - \frac{(3a^2b+6ab^2+b^3)z^3}{3} \\
&\quad + \frac{(3a^2b^2+2ab^3)z^2}{2} - a^2b^3z - c_6 \\
&= \widehat{P}_6(z) - c_6, \quad c_6 \neq \widehat{P}_6(a), \widehat{P}_6(b) \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_6(z) &= \frac{z^6}{6a^2b^3} - \frac{(2a+3b)z^5}{5a^2b^3} + \frac{(a^2+6ab+3b^2)z^4}{4a^2b^3} - \frac{(3a^2b+6ab^2+b^3)z^3}{3a^2b^3} \\
&\quad + \frac{(3a^2b^2+2ab^3)z^2}{2a^2b^3} - z - d_6 \\
&= \widehat{Q}_6(z) - d_6, \quad d_6 \neq \widehat{Q}_6(a), \widehat{Q}_6(b), \qquad (3.2)
\end{aligned}
$$

$$
(3.3)\ P_6'(z) = (z-a)^2(z-b)^3 = \widehat{P}_6'(z), Q_6'(z) = \frac{1}{a^2b^3}(z-a)^2(z-b)^3 = \widehat{Q}_6'(z),
$$

with the following conditions
(i) $Q_6(b) = Q_6(a)P_6(b)$,
(ii) $c_6Q_6(a) = d_6$.

The following example shows that the conditions for P_6 and Q_6 are satisfied.

Example 3.1. *Take $a = -1$, $b = 1$. From (3.1), (3.2), (3.3), $P_6(z)$ and $Q_6(z)$ changes to:*

$$
P_{B6}(z) = \frac{z^6}{6} - \frac{z^5}{5} - \frac{z^4}{2} + \frac{2z^3}{3} + \frac{z^2}{2} - z + \frac{3}{10}, \;\; P_{B6}(z) = Q_{B6}(z).
$$

With respect to the above polynomials let us state one of the main results of this paper as follows:

Theorem 3.1. *Let $S_{P_6} = \{z \mid P_6(z) = 0\}$ and $S_{Q_6} = \{z \mid Q_6(z) = 0\}$ where $P_6(z)$ is given by (3.1) and (3.2). Suppose that f and g be two non-constant meromorphic functions satisfying $E_{f^{(k)}}(S_{P_6}, 4) = E_{g^{(k)}}(S_{Q_6}, 4)$, $E_{f^{(k)}}(\{a\}, k_1) = E_{g^{(k)}}(\{a\}, k_1)$ and $E_{f^{(k)}}(\{b\}, k_2) = E_{g^{(k)}}(\{b\}, k_2)$, where $k_1k_2 = 2$, k being any non-negative integer and set $f^{(0)} = f$, then one of the following two conclusions*

holds.
(i) *If* $P_6(z) = Q_6(z)$ *then for any two non-constant meromorphic functions* f *and* g, *we get* $f^{(k)} \equiv g^{(k)}$.
(ii) *If* $P_6(z) \neq Q_6(z)$ *then* $f^{(k)}$ *and* $g^{(k)}$ *satisfy the following equation*

$$\frac{(g^{(k)})^5(h^6-\alpha)}{6} - \frac{(2a+3b)(g^{(k)})^4(h^5-\alpha)}{5} + \frac{(a^2+6ab+3b^2)(g^{(k)})3(h^4-\alpha)}{4}$$
$$-\frac{(3a^2b+6ab^2+b^3)(g^{(k)})^2(h^3-\alpha)}{3} - \frac{(3a^2b^2+2ab^3)(g^{(k)})(h^2-\alpha)}{2}$$
$$-a^2b^3(h-\alpha) \equiv 0,$$

where $h = \frac{f^{(k)}}{g^{(k)}}$ *and* $\alpha = \frac{c_6}{a^2b^3d_6}$.

Next let us define another two polynomials as follows:

$$\begin{aligned} P_8(z) &= \frac{z^8}{8} - \frac{5az^7}{7} + \frac{5a^2z^6}{3} - 2a^3z^5 + \frac{5a^4z^4}{4} - \frac{a^5z^3}{3} - c_8 \\ &= \widehat{P}_8(z) - c_8, c_8 \neq 0, \left(-\frac{a^8}{168}\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} Q_8(z) &= \frac{z^8}{8a^5} - \frac{5z^7}{7a^4} + \frac{5z^6}{3a^3} - \frac{2z^5}{a^2} + \frac{5z^4}{4a} - \frac{z^3}{3} - d_8 \\ &= \widehat{Q}_8(z) - d_8, d_8 \neq 0, \left(-\frac{a^3}{168}\right), \end{aligned} \tag{3.5}$$

$$P_8'(z) = z^2(z-a)^5 = \widehat{P}_8'(z), Q_8'(z) = \frac{1}{a^5}z^2(z-a)^5 = \widehat{Q}_8'(z), \tag{3.6}$$

with the condition $c_8Q_8(a) = d_8P_8(a)$.

The following example shows that the conditions for P_8 and Q_8 are satisfied.

Example 3.2. *Take* $a = 1$ *and an arbitrary value of* c_8 *other than* $\left(\frac{-1}{168}\right)$ *such as* $c_8 = \frac{13}{21}$. *From* (3.5), (3.7), (3.6), $P_8(z)$ *and* $Q_8(z)$ *transform to*

$$P_{B8}(z) = \frac{z^8}{8} - \frac{5z^7}{7} + \frac{5z^6}{3} - 2z^5 + \frac{5z^4}{4} - \frac{z^3}{3} - \frac{13}{21},$$

$$P_{B8}(z) = Q_{B8}(z).$$

With respect to the above mentioned polynomials let us state remaining two main results of this paper.

Theorem 3.2. *Let $S_{P_8} = \{z \mid P_8(z) = 0\}$ and $S_{Q_8} = \{z \mid Q_8(z) = 0\}$ where $P_8(z)$ and $Q_8(z)$ are given by (3.4) and (3.5). Suppose that f and g are two non-constant meromorphic functions satisfying $E_{f^{(k)}}(S_{P_8}, 2) = E_{g^{(k)}}(S_{Q_8}, 2)$ and $E_{f^{(k)}}(\{a\}, 0) = E_{g^{(k)}}(\{a\}, 0)$, where k is a non-negative integer and set $f^{(0)} = f$, then one of the following two conclusions holds.*
(i) If $P_8(z) = Q_8(z)$ then for any two non-constant meromorphic functions f and g, we get $f^{(k)} \equiv g^{(k)}$.
(ii) If $P_8(z) \neq Q_8(z)$ then $f^{(k)}$ and $g^{(k)}$ satisfy the following equation

$$\begin{aligned}(3.7)\qquad &\frac{(g^{(k)})^5(h^8-\beta)}{8} - \frac{5a(g^{(k)})^4(h^7-\beta)}{7} + \frac{5a^2(g^{(k)})^3(h^6-\beta)}{3}\\ &-2a^3(g^{(k)})^2(h^5-\beta) + \frac{5a^4(g^{(k)})(h^4-\beta)}{4} - \frac{a^5(h^3-\beta)}{3} \equiv 0,\end{aligned}$$

where $h = \frac{f^{(k)}}{g^{(k)}}$ and $\beta = \frac{c_8}{a^5 d_8}$.

4. Lemmas

Let us define two meromorphic functions F_i and G_i as follows:

$$(4.1)\qquad F_i \equiv \frac{\widehat{P}_i(f^{(k)})}{c_i}, \qquad G_i \equiv \frac{\widehat{Q}_i(g^{(k)})}{d_i},\ i = 6, 8.$$

On the basis of the two functions in (3.1), we now define the following two auxliary functions H_i and Φ_i as follows:

$$(4.2)\qquad H_i \equiv \left[\frac{F_i''}{F_i'} - \frac{2F_i'}{F_i - 1}\right] - \left[\frac{G_i''}{G_i'} - \frac{2G_i'}{G_i - 1}\right]$$

and

$$(4.3)\qquad \Phi_i \equiv \frac{F_i'}{F_i - 1} - \frac{G_i'}{G_i - 1}.$$

The following lemmas will play key roles in proving our results.

Lemma 4.1. *[29] If F, G be two non-constant meromorphic functions such that they share (1,1) and $H \not\equiv 0$ then*

$$N(r, 1; F \mid= 1) = N(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 4.2. *[5] Let f and g be two non-constant meromorphic functions sharing $(1,l)$, where $0 \le l < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(l-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ \le\ &\frac{1}{2}[N(r,1;f)+N(r,1;g)].\end{aligned}$$

Lemma 4.3. *[6] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + \ldots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \ne 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 4.4. *Let f and g be two non-constant meromorphic functions and F_i and G_i be defined by (4.1) such that $E_{f^{(k)}}(S_{P_6},0) = E_{g^{(k)}}(S_{Q_6},0)$ and $E_{f^{(k)}}(\{a\},p) = E_{g^{(k)}}(\{a\},p)$, $E_{f^{(k)}}(\{b\},p) = E_{g^{(k)}}(\{b\},p)$, $0 \le p < \infty$ and $H_6 \not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;H_6) \le\ &\overline{N}(r,a;f^{(k)}\mid\ge p+1)+\overline{N}(r,b;f^{(k)}\mid\ge p+1)+\overline{N}_*(r,1;F_6,G_6)\\ &+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+\overline{N}_0(r,0;f^{(k+1)})\\ &+\overline{N}_0(r,0;g^{(k+1)}),\end{aligned}$$

where $\overline{N}_0(r,0;f^{(k+1)})$ is the reduced counting function of those zeros of $f^{(k)}$ which are not zeros of $(f^{(k)}-a)(f^{(k)}-b)(F_6-1)$ and $\overline{N}_0(r,0;g^{(k+1)})$ is similarly defined.

Proof. Since $E_{f^{(k)}}(S_{P_6},0) = E_{g^{(k)}}(S_{Q_6},0)$, it follows that F and G share $(1,0)$. We can easily verify that possible poles of H occur at (i) a-points of $f^{(k)}$ of multiplicity $\ge p+1$, (ii) b-points of $f^{(k)}$ of multiplicity $\ge p+1$, (iii) poles of $f^{(k)}$ and $g^{(k)}$, (iv) zeros of $f^{(k+1)}$ which are not the zeros of $(f^{(k)}-a)(f^{(k)}-b)(F_6-1)$, (v) zeros of $g^{(k+1)}$ which are not zeros of $(g^{(k)}-a)(g^{(k)}-b)(G_6-1)$. Since H has only simple poles, the lemma follows from above.

Lemma 4.5. *Let $f^{(k)}$ and $g^{(k)}$ be two non-constant meromorphic functions and F_8 and G_8 be defined by (4.1) satisfying $E_{f^{(k)}}(S_8,0) = E_{g^{(k)}}(S_8,0)$, $E_{f^{(k)}}(\{a\},p) = E_{g^{(k)}}(\{a\},p)$, $0 \le p < \infty$ and $H_8 \not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;H_8) \le\ &\overline{N}(r,a;f^{(k)}\mid\ge p+1)+\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})\\ &+\overline{N}_*(r,1;F_8,G_8)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\\ &+\overline{N}_0(r,0;f^{(k+1)})+\overline{N}_0(r,0;g^{(k+1)}),\end{aligned}$$

where $\overline{N}_0(r,0;f^{(k+1)})$ is the reduced counting function of those zeros of $f^{(k)}$ which are not zeros of $(f^{(k)}-a)(f^{(k)}-b)(F_8-1)$ and $\overline{N}_0(r,0;g^{(k+1)})$ is similarly defined.

Proof. We are omitting the proof of the lemma as it can be carried out in the line of the proof of the *Lemma 4.4*.

Lemma 4.6. *Let f and g be two non-constant meromorphic functions and F_6 and G_6 be given by (4.1) satisfying $E_{f^{(k)}}(S_6,l)=E_{g^{(k)}}(S_6,l)$, $E_{f^{(k)}}(\{a\},p)=E_{g^{(k)}}(\{a\},p)$, $E_{f^{(k)}}(\{b\},p)=E_{g^{(k)}}(\{b\},p)$, $0\le p<\infty$ and $\Phi_6\not\equiv 0$. Then*

$$\begin{aligned}&(3p+2)\overline{N}(r,a;f^{(k)}\mid\geq p+1)\\&\leq \overline{N}_*(r,1;F_6,G_6)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,f^{(k)})+S(r,g^{(k)})\end{aligned}$$

and

$$\begin{aligned}&(4p+3)\overline{N}(r,b;f^{(k)}\mid\geq p+1)\\&\leq \overline{N}_*(r,1;F_6,G_6)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,f^{(k)})+S(r,g^{(k)}).\end{aligned}$$

Proof. By the given condition clearly F_6 and G_6 share $(1,l)$. Also we see that,

$$\Phi_6=\frac{(f^{(k)}-a)^2(f^{(k)}-b)^3f^{(k+1)}}{c_6(F_6-1)}-\frac{(g^{(k)}-a)^2(g^{(k)}-b)^3g^{(k+1)}}{d_6(G_6-1)}.$$

Let, z_0 be a a-point $f^{(k)}$ with multiplicity r. Since, $f^{(k)}$ and $g^{(k)}$ shares $(\{a\},p)$ then z_0 is a zero of Φ_6 of multiplicity $2r+r-1{=}3r-1$ if $r\le p$ and a zero of Φ_6 of multiplicity at least $3(p+1)-1{=}3p+2$ if $r>p$. Hence, by the definition of Φ_6 and by simple calculation we can write that,

$$\begin{aligned}&(3p+2)\overline{N}(r,a;f^{(k)}\mid\geq p+1)\\\leq\;&\overline{N}\left(r,0;\Phi_6\right)\\\leq\;&T(r,\Phi_6)\leq N\left(r,\infty;\Phi_6\right)+S\left(r,F_6\right)+S\left(r,G_6\right)\\\leq\;&\overline{N}_*\left(r,1;F_6,G_6\right)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,f^{(k)})+S(r,g^{(k)}).\end{aligned}$$

The other result can be deduced similarly.

Lemma 4.7. *Let f and g be two non-constant meromorphic functions and F_8 and G_8 be given by (4.1) satisfying $E_{f^{(k)}}(S_8,l)=E_{g^{(k)}}(S_8,l)$, $E_{f^{(k)}}(\{a\},p)=E_{g^{(k)}}(\{a\},p)$, $0\le p<\infty$ and $\Phi_8\not\equiv 0$. Then*

$$\begin{aligned}&(6p+5)\overline{N}(r,a;f^{(k)}\mid\geq p+1)\\\leq\;&\overline{N}_*(r,1;F_8,G_8)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,f^{(k)})+S(r,g^{(k)}).\end{aligned}$$

Proof. The lemma can be proved in the line of the proof of *Lemma 4.6.* So we omit the details.

Lemma 4.8. *Let F_6 and G_6 be given by (4.1). If F_6, G_6 share $(1,l)$, where $0 \leq l < \infty$ and k is a non-negative integer. Then*

$$(i)\quad \overline{N}_L(r,1;F_6) \leq \frac{1}{l+1}\left(\overline{N}(r,0;f^{(k)})+\overline{N}(r,\infty;f^{(k)})-N_{\otimes}(r,0;f^{(k+1)})\right) +S(r,f^{(k)})$$

$$(ii)\quad \overline{N}_L(r,1;G_6) \leq \frac{1}{l+1}\left(\overline{N}(r,0;g^{(k)})+\overline{N}(r,\infty;g^{(k)})-N_{\otimes}(r,0;g^{(k+1)})\right) +S(r,g^{(k)}),$$

where $N_{\otimes}(r,0;f^{(k+1)}) = N(r,0;f^{(k+1)} \mid f^{(k)} \neq 0, w_1, w_2, \ldots, w_6)$ and $w_1, w_2, \ldots w_6$ be the roots of the equation $P_6(z) = 0$, $N_{\otimes}(r,0;g^{(k+1)})$ is defined similarly to $N_{\otimes}(r,0;f^{(k+1)})$. Similar results holds for F_8 and G_8.

Proof. We omit the proof since it can be carried out in the line of the proof of *Lemma 2.10* of [3].

Lemma 4.9. *[16] Let $P(z)$ be a polynomial of degree ≥ 5 without multiple zeros, whose first derivative have mutually k distinct zeros, given by $d_1, d_2, ..., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively. Assume that $P(z)$ satisfies the critical injection property and there are two distinct non-constant meromorphic functions f and g such that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

for some constant $c_0 \neq 0$ and c_1. If $k \geq 3$, or if $k = 2$ and $min\{q_1, q_2\} \geq 2$, then $c_1 = 0$.

Lemma 4.10. *[17] Let $P(z)$ be a monic polynomial without multiple zero whose first derivative have mutually k-distinct zeros, given by $d_1, d_2, .., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively. Suppose that $P(z)$ satisfies the critical injection property. Then $P(z)$ will be a UPM if and only if*

$$\sum_{1 \leq l < m \leq k} q_{_l} q_m > \sum_{i=1}^{k} q_{_l}.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$ then also the above inequality holds.

5. Proofs of the Theorems

Proof. [**Proof of Theorem 3.1**] Let F_6 and G_6 be given by (3.1). Since $f^{(k)}$ and $g^{(k)}$ share $E_{f^{(k)}}(S_{P_6},4)=E_{g^{(k)}}(S_{Q_6},4)$, from (4.1) it follows that F_6 and G_6 share $(1,4)$. Suppose $H_6\not\equiv 0$.

If possible $\Phi_6\equiv 0$. By (4.3) we get,

$$(F_6-1)\equiv A\,(G_6-1)\,, \tag{5.1}$$

where A is a constant.

Next, using (5.1) and the definition of H_6 we get, $H_6\equiv 0$, which is a contradiction. Hence $\Phi_6\not\equiv 0$.

Using *Lemma 4.2* for $l=4$, *Lemma 4.4* for $p=0$, *Lemma 4.6* $p=0$, $p=k_1$ and $p=k_2$, *Lemma 4.3* we get from the Second Fundamental Theorem,

$$\begin{aligned}
&7\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
\le\;& \overline{N}(r,a;f^{(k)})+\overline{N}(r,b;f^{(k)})+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,1;F_6)+\overline{N}(r,a;g^{(k)})\\
&+\overline{N}(r,b;g^{(k)})+\overline{N}(r,\infty;g^{(k)})+\overline{N}(r,1;G_6)-N_0(r,0;f^{(k+1)})\\
&-N_0(r,0;g^{(k+1)})+S(r,f^{(k)})+S(r,g^{(k)})\\
\le\;& N(r,1;F_6\mid=1)-\left(4-\frac{1}{2}\right)\overline{N}_*(r,1;F_6,G_6)+3\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
&+2\overline{N}(r,a;f^{(k)})+2\overline{N}(r,b;f^{(k)})+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\\
&-N_0(r,0;f^{(k+1)})-N_0(r,0;g^{(k+1)})+S(r,f^{(k)})+S(r,g^{(k)})\\
\le\;& 2\overline{N}(r,a;f^{(k)})+\overline{N}(r,a;f\mid\ge k_1+1)+2\overline{N}(r,b;f^{(k)})+\overline{N}(r,b;f\mid\ge k_2+1)\\
&+\left(3+\frac{2}{k+1}\right)\{T(r,f^{(k)})+T(r,g^{(k)})\}+-\frac{5}{2}\overline{N}_*(r,1;F_6,G_6)\\
&+S(r,f^{(k)})+S(r,g^{(k)})\\
\le\;& \left(1+\frac{1}{(3k_1+2)}\right)\{\overline{N}_*(r,1;F_6,G_6)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\}\\
&+\left(\frac{2}{3}+\frac{1}{(4k_2+3)}\right)\{\overline{N}_*(r,1;F_6,G_6)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\}\\
&+\left(3+\frac{2}{k+1}\right)\{T(r,f^{(k)})+T(r,g^{(k)})\}-\frac{5}{2}\overline{N}_*(r,1;F_6,G_6)\\
&+S(r,f^{(k)})+S(r,g^{(k)})\\
\le\;& \left\{3+\frac{11}{3(k+1)}+\frac{1}{(k+1)(3k_1+2)}+\frac{1}{(k+1)(4k_2+3)}\right\}\\
&(T(r,f^{(k)})+T(r,g^{(k)}))+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

i.e.,

$$\left(4-\frac{11}{3(k+1)}-\frac{1}{(3k_1+2)(k+1)}-\frac{1}{(4k_2+3)(k+1)}\right)\{T(r,f^{(k)})+T(r,g^{(k)})\}$$
$$\leq \quad S(r,f^{(k)})+S(r,g^{(k)}).$$

which is a contradiction for $k \geq 0$ and $k_1k_2 = 2$.

Hence $H_6 \equiv 0$. Then for two constants $A(\neq 0)$, B we get

$$\frac{1}{F_6-1} \equiv \frac{A}{G_6-1}+B \tag{5.2}$$

and

$$T(r,f^{(k)}) = T(r,g^{(k)})+S(r,g^{(k)}). \tag{5.3}$$

<u>**Case 1:**</u> Let us assume that $B \neq 0$. Then by (5.2) we can have

$$F_6-1 \equiv \frac{G_6-1}{B\left\{(G_6-1)+\frac{A}{B}\right\}}. \tag{5.4}$$

Let us consider

$$\phi(z) = \widehat{Q}_6(z)-d_6\left(1-\frac{A}{B}\right).$$

<u>**Subcase 1.1:**</u> Let us assume that $A \neq B$.

First we assume that, a is a zero of $\phi(z)$ then it will be a zero of multiplicity 3 and other zeros are simple zeros namely α_i where $i = 1, 2, 3$. Then by the Second Fundamental Theorem, (5.3) and (5.4) we have

$$\begin{aligned}
3T(r,g^{(k)}) &\leq \overline{N}(r,a;g^{(k)})+\sum_{i=1}^{3}\overline{N}(r,\alpha_i;g^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,g^{(k)})\\
&\leq \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,g^{(k)})\\
&\leq \frac{2}{k+1}T(r,g^{(k)})+S(r,g^{(k)}),
\end{aligned}$$

which is contradiction for $k \geq 0$.

Again, if b is a zero of $\phi(z)$ the itnis a zero of multiplicity 4 and other zeros are simple say, β_i for $i = 1, 2$. Using the Second Fundamental Theorem, (5.3) and (5.4) we get

$$\begin{aligned}
2T(r,g^{(k)}) &\leq \overline{N}(r,b;g^{(k)})+\sum_{i=1}^{2}\overline{N}(r,\beta_i;g^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,g^{(k)})\\
&\leq \left(\frac{1}{k+1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{6}\right)T(r,g^{(k)})+S(r,g^{(k)}),
\end{aligned}$$

which is contradiction for $k \geq 0$. Hence we can conclude that all the zeros $\phi(z)$ are simple namely γ_i for $i = 1, 2, \ldots, 6$. Using the Second Fundamental Theorem, (5.3) and (5.4) we get

$$\begin{aligned} 5T(r, g^{(k)}) &\leq \sum_{i=1}^{6} \overline{N}(r, \gamma_i; g^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\ &\leq \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) + S(r.g^{(k)}), \end{aligned}$$

which is contradiction for $k \geq 0$.

Subcase 1.2: Let us assume that $A = B$.

Subcase 1.2.1: Let us take $A \neq -1$. Then (4.2) gives us

$$F_6 \equiv \frac{(1 + A)\left(G_6 - \frac{1}{A+1}\right)}{AG_6}.$$

Here we get

$$\overline{N}\left(r, 0; G_6 - \frac{1}{A+1}\right) = \overline{N}\left(r, 0; F_6\right).$$

Let us take

$$\chi(z) = \widehat{Q}_6(z) - d_6\left(\frac{1}{A+1}\right).$$

Let us first suppose that a is a zero of $\chi(z)$ of multiplicity 3 and another three simple zeros are δ_i for $i = 1, 2, 3$. Evidently $\widehat{Q}_6(z)$ has 6 simple zeros. Using the Second Fundamental Theorem and (5.3) we can write

$$\begin{aligned} 9T(r, g^{(k)}) &\leq \overline{N}(r, a; g^{(k)}) + \sum_{i=1}^{3} \overline{N}(r, \delta_i; g^{(k)}) + \overline{N}(r, 0; G_6) + \overline{N}(r, \infty; g^{(k)}) \\ &\quad + S(r, g^{(k)}) \\ &\leq \overline{N}(r, 0; F_6) + \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\ &\leq \left(6 + \frac{2}{k+1}\right) T(r, g^{(k)}) + S(r, g^{(k)}), \end{aligned}$$

which is contradiction as $k \geq 0$.

Next let us assume that b is zero of $\chi(z)$ of multiplicity 4 and another two simple zeros are η_i for $i = 1, 2$. As $f^{(k)}$ and $g^{(k)}$ share $(\{b\}, 0)$ we can say that b is an e.v.P. of both $f^{(k)}$ and $g^{(k)}$. Using the Second Fundamental Theorem and (5.3) we obtain

$$\begin{aligned}8T(r,g^{(k)}) &\leq \overline{N}(r,b;g^{(k)})+\sum_{i=1}^{2}\overline{N}(r,\eta_i;g^{(k)})+\overline{N}(r,0;G_6)+\overline{N}(r,\infty;g^{(k)})\\ &\quad +S(r,g^{(k)})\\ &\leq \left(2+\frac{2}{k+1}\right)T(r,g^{(k)})+S(r,g^{(k)}),\end{aligned}$$

which is a contradiction for $k \geq 0$.

Hence we can say that all the zeros of $\chi(z)$ are simple and by using the above arguments we can get a contradiction.

Subcase 1.2.2: Let $A = -1$ and zeros of $\widehat{P}_6(z)$ are 0 and θ_i for $i = 1, 2, \ldots, 5$. Then from (5.2) we have

$$F_6 G_6 \equiv 1$$

i.e.

$$g^{(k)}\prod_{i=1}^{5}(g^{(k)}-\theta_i) \equiv \frac{c_6 d_6 a^2 b^3}{f^{(k)}\prod_{i=1}^{5}((f^{(k)})-\theta_i)}.$$

If z_0 is a pole of $g^{(k)}$ of order k_1 then z_0 is θ_i point of $f^{(k)}$ for some $i \in \{1,2,3,4,5\}$ of order k_2. Then we can say that $6k_1 = k_2$, this implies that value of k_2 is at least 6. Using the Second Fundamental Theorem and (5.3) we obtain

$$\begin{aligned}4T(r,f^{(k)}) &\leq \sum_{i=1}^{5}\overline{N}(r,\theta_i;f^{(k)})+\overline{N}(r,\infty;f^{(k)})+S(r,f^{(k)})\\ &\leq \left(\frac{5}{6}+\frac{1}{k+1}\right)T(r,f^{(k)})+S(r,g^{(k)}),\end{aligned}$$

which is contradiction to the fact $k \geq 0$.

Case 2: Let $B = 0$. Then we possess

$$(G_6 - 1) \equiv A\left(F_6 - 1\right). \tag{5.5}$$

Clearly, (5.5) possess

$$T(r,f^{^{(k)}}) = T(r,g^{^{(k)}})+S(r,g^{(k)}). \tag{5.6}$$

Let us first consider $A \neq 1$.

Case 2.1: Let $P_6(a) \neq 1$.

Case 2.1.1: Let us first suppose that $A = \left(\frac{c_6 Q_6(a)}{d_6}\right)$. By some simple calculation, from (5.5) we can write

$$Q_6(g^{(k)}) - Q_6(a) = \frac{Ac_6}{d_6}(P_6(f^{(k)}) - 1). \tag{5.7}$$

Let us consider the polynomial $Q_6(z) - Q_6(a)$. As $Q_6(z)$ is a critically injective polynomial (see *Definition* (2.4)), we can say that b is not a zero of $(Q_6(z) - Q_6(a))$ and we can write $Q_6(z) - Q_6(a) = (z-a)^3 W_3(z)$ where $W_3(a) \neq 0$. It is evident that all the zeros of $W_3(z)$ are simple namely $\tilde{\alpha_i}$, $i = 1, 2, 3$. Let us take the polynomial $(P_6(z) - 1)$.

As $P_6(a) \neq 1$, it is clear that a is not a zero of $(P_6(z) - 1)$.

Further if b is a zero of this polynomial then it will be a zero of multiplicity 4 and another two simple zeros are $\tilde{\beta_i}$ for $i = 1, 2$. We have

$$\overline{N}(r, a; g^{(k)}) + \sum_{i=1}^{3} \overline{N}(r, \tilde{\alpha_i}; g^{(k)}) = \overline{N}(r, b; f^{(k)}) + \sum_{i=1}^{2} \overline{N}(r, \tilde{\beta_i}; f^{(k)}).$$

Since $f^{(k)}$ and $g^{(k)}$ share $(\{a\}, 0)$ and $(\{b\}, 0)$, we can say a and b are e.v.P. of $f^{(k)}$ and $g^{(k)}$. Then by the Second Fundamental Theorem, (5.6) and the above equation we get

$$\begin{aligned} 4T(r, g^{(k)}) &\leq \overline{N}(r, a; g^{(k)}) + \overline{N}(r, b; g^{(k)}) + \sum_{i=1}^{3} \overline{N}(r, \tilde{\alpha_i}; g^{(k)}) + \overline{N}(r, \infty; g^{(k)}) \\ &\quad + S(r, g^{(k)}) \\ &\leq \overline{N}(r, b; f^{(k)}) + \sum_{i=1}^{2} \overline{N}(r, \tilde{\beta_i}; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\ &\leq \left(2 + \frac{1}{k+1}\right) T(r, g^{(k)}) + S(r, g^{(k)}), \end{aligned}$$

which is a contradiction for $k \geq 0$.

Now all the zeros of the polynomial are simple and let us denote them $\tilde{\gamma_i}$ for $i = 1, 2, \ldots, 6$. Then we can write

$$\overline{N}(r, a; g^{(k)}) + \sum_{i=1}^{3} \overline{N}(r, \tilde{\alpha_i}; g^{(k)}) = \sum_{i=1}^{6} \overline{N}(r, \tilde{\gamma_i}; f^{(k)}).$$

By the Second Fundamental Theorem and the above equation we get

$$\begin{aligned} 5T(r,f^{(k)}) &\leq \sum_{i=1}^{6}\overline{N}(r,\tilde{\gamma_i};f^{(k)})+\overline{N}(r,\infty;f^{(k)})+S(r,f^{(k)}) \\ &\leq \overline{N}(r,a;g^{(k)})+\sum_{i=1}^{3}\overline{N}(r,\tilde{\alpha_i};g^{(k)})+\overline{N}(r,\infty;f^{(k)})+S(r,f^{(k)}) \\ &\leq \left(3+\frac{1}{k+1}\right)T(r,f^{(k)})+S(r,f^{(k)}), \end{aligned}$$

which is a contradiction for $k \geq 0$.

<u>Case 2.1.2:</u> Now let us assume that $A \neq \left(\frac{c_6 Q_6(a)}{d_6}\right)$. By simple calculation from (5.5) we can say

$$Q_6(g^{(k)})-\frac{Ad_6}{c_6}P_6(b)=\frac{Ad_6}{c_6}\Big(P_6(f^{(k)})-P_6(b)\Big). \tag{5.8}$$

As $P_6(z)$ is a critically injective polynomial, a cannot be a zero of the polynomial $(P_6(z)-P_6(b))$. Hence b is always a zero of of $(P_6(z)-P_6(b))$ of multiplicity 4 and another simple zeros are
$tilde\eta_i$ for $i=1,2$. Here we can write

$$(P_6(f^{(k)})-P_6(b))=(f^{(k)}-b)^4(f^{(k)}-\eta_1)(f^{(k)}-\eta_2). \tag{5.9}$$

Let us consider the polynomial $\Big(Q_6(z)-\frac{Ad_6}{c_6}P_6(b)\Big)$.

If a is a zero of the polynomial $\Big(Q_6(z)-\frac{Ad_6}{c_6}P_6(b)\Big)$, then it will be of multiplicity 3 and other zeros are simple, say $\tilde{\delta_i}$ for $i=1,2,3$. Then we can write from (5.8) and (5.9)

$$\overline{N}(r,a;g^{(k)})+\sum_{i=1}^{3}\overline{N}(r,\tilde{\delta_i};g^{(k)})=\overline{N}(r,b;f^{(k)})+\sum_{i=1}^{2}\overline{N}(r,\eta_i;f^{(k)}).$$

In this case also a and b are e.v.P. of both $f^{(k)}$ and $g^{(k)}$. By the Second Fundamental Theorem, (5.6) and the above equation,

$$\begin{aligned} 4T(r,g^{(k)}) &\leq \overline{N}(r,a;g^{(k)})+\overline{N}(r,b;g^{(k)})+\sum_{i=1}^{3}\overline{N}(r,\tilde{\delta_i};g^{(k)})+\overline{N}(r,\infty;g^{(k)}) \\ &\quad +S(r,g^{(k)}) \\ &\leq \overline{N}(r,b;f^{(k)})+\sum_{i=1}^{2}\overline{N}(r,\tilde{\eta_i};f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,g^{(k)}) \\ &\leq \left(2+\frac{1}{k+1}\right)T(r,g^{(k)})+S(r,g^{(k)}), \end{aligned}$$

which is a contradiction for $k \geq 0$.

Next, if b is a zero of the polynomial $\left(Q_6(z) - \frac{Ad_6}{c_6}P_6(b)\right)$ the we get

$$A = \frac{c_6 Q_6(b)}{d_6 P_6(b)}.$$

As we have from the hypothesis of the *Theorem 3.1* that $Q_6(b) = Q_6(a)P_6(b)$ then we have $A = \frac{c_6 Q_6(a)}{d_6}$ which is contradiction.

Now, if we assume that all the zeros of the polynomial $\left(Q_6(z) - \frac{Ad_6}{c_6}P_6(b)\right)$ are simple, we using the same methodology that is used to solve the situation in *Case 2.1.2*, we will arrive at a contradiction.

Case 2.2: let $P_6(a) = 1$. Then from (5.5) calculating we have

$$P_6(f^{(k)}) - 1 = \frac{c_6}{Ad_6}\left(Q_6(g^{(k)}) - \frac{Ad_6}{c_6}\right).$$

As $P_6(a) = 1$, we can say that a is zero of the polynomial $(P_6(z)) - 1$ of multiplicity 3 and other two simple zeros are denoted by $\tilde{\theta}_i$ for $i = 1, 2$. Hence we can write

$$(f^{(k)} - a)^3(f^{(k)} - \tilde{\theta}_1)(f^{(k)} - \tilde{\theta}_2) = \frac{c_6}{Ad_6}\left(Q_6(g^{(k)}) - \frac{Ad_6}{c_6}\right).$$

If a is a zero of the polynomial $\left(Q_6(g^{(k)}) - \frac{Ad_6}{c_6}\right)$ then we get $c_6 Q_6(a) = d_6$ which is contradiction to hypothesis of *Theorem (3.1)*. Hence the polynomial $\left(Q_6(g^{(k)}) - \frac{Ad_6}{c_6}\right)$ can have multiple zero b of multiplicity 4 or all simple zeros. In both the cases using the same calculations that has been done in *Case 2.1* we will get contradictions. Hence $A = 1$ and We have

$$F_6 = G_6$$

which implies that

$$\frac{(g^{(k)})^5(h^6 - \alpha)}{6} - \frac{(2a + 3b)(g^{(k)})^4(h^5 - \alpha)}{5} + \frac{(a^2 + 6ab + 3b^2)(g^{(k)})3(h^4 - \alpha)}{4}$$
$$-\frac{(3a^2 b + 6ab^2 + b^3)(g^{(k)})^2(h^3 - \alpha)}{3} - \frac{(3a^2 b^2 + 2ab^3)(g^{(k)})(h^2 - \alpha)}{2}$$
$$-a^2 b^3(h - \alpha) \equiv 0,$$

where $h = \frac{f^{(k)}}{g^{(k)}}$ and $\alpha = \frac{c_6}{a^2 b^3 d_6}$.

Next if $P_6(z) = Q_6(z)$ we have from (5.2), *Lemma 4.9* and *Lemma 4.10*, $f^{(k)} \equiv g^{(k)}$.

Proof. [**Proof of Theorem 3.2**] Let F_8 and G_8 be given by (4.1). Since $f^{(k)}$ and $g^{(k)}$ share $E_{f^{(k)}}(S_{P_8},2)=E_{g^{(k)}}(S_{Q_8},2)$, from (4.1) it follows that F_8 and G_8 share $(1,2)$. Suppose $H_8\not\equiv 0$. Hence $\Phi_8\neq 0$.

Using *Lemma 4.2* for $l=2$, *Lemma 4.5* for $p=0$, *Lemma 4.7* $p=0$, *Lemma 4.3* and *Lemma 4.8* for $l=2$ we get from the Second Fundamental Theorem,

$$\begin{aligned}
& 9\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
\leq\;& \overline{N}(r,0;f^{(k)})+\overline{N}(r,a;f^{(k)})+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,1;F_8)+\overline{N}(r,0;g^{(k)})\\
& +\overline{N}(r,a;g^{(k)})+\overline{N}(r,\infty;g^{(k)})+\overline{N}(r,1;G_8)-N_0(r,0;f^{(k+1)})-N_0(r,0;g^{(k+1)})\\
& +S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& N(r,1;F_8\mid=1)+2\overline{N}(r,a;f^{(k)})+6\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
& -\left(2-\frac{1}{2}\right)\overline{N}_*(r,1;F_8,G_8)-N_0(r,0;f^{(k+1)})-N_0(r,0;g^{(k+1)})\\
& +S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& 3\overline{N}(r,a;f^{(k)})+8\{T(r,f^{(k)})+T(r,g^{(k)})\}-\frac{1}{2}\overline{N}_*(r,1;F_8,G_8)\\
& +S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& 8\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
& +\frac{3}{5}\left\{\overline{N}_*(r,1;F_8,G_8)+\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\right\}\\
& -\frac{1}{2}\overline{N}_*(r,1;F_8,G_8)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& \frac{43}{5}\{T(r,f^{(k)})+T(r,g^{(k)})\}+\frac{1}{10}\overline{N}_*(r,1;F_8,G_8)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& \left(\frac{43}{5}+\frac{1}{15}\right)\{T(r,f^{(k)})+T(r,g^{(k)})\}+S(r,f^{(k)})+S(r,g^{(k)}),
\end{aligned}$$

which is a clear contradiction. Hence $H_8\equiv 0$. Then for two constants $A(\neq 0)$, B we get

$$\frac{1}{F_8-1}\equiv\frac{A}{G_8-1}+B \tag{5.10}$$

and

$$T(r,f^{(k)})=T(r,g^{(k)})+S(r,g^{(k)}). \tag{5.11}$$

Using the same arguments that were used to prove $B=0$ in *Theorem 3.1* we get

$$(G_8-1)=A(F_8-1). \tag{5.12}$$

Let us first suppose that $A\neq 1$.

Case 1: Let us take $A \neq -\left(\frac{c_8}{P_8(a)}\right)$. Then from (5.12) we can write that

$$Q_8(g^{(k)}) - \frac{Ad_8}{c_8}P_8(a) = \frac{Ad_8}{c_8}(P_8(f^{(k)}) - P_8(a)). \tag{5.13}$$

For the polynomial $(P_8(z) - P_8(a))$, a is zero of this polynomial of multiplicity 6 and another two simple zeros are denoted by $\hat{\alpha_i}$ for $i = 1, 2$. As $P_8(z)$ is a critically injective polynomial, b can not be a zero of the polynomial $(P_8(z) - P_8(a))$. Hence we have

$$(P_8(f^{(k)}) - P_8(a)) = (f^{(k)} - a)^6(f^{(k)} - \hat{\alpha_1})(f^{(k)} - \hat{\alpha_2}).$$

Next we consider the polynomial $\left(Q_8(z) - \frac{Ad_8}{c_8}P_8(a)\right)$.

Now it is clear that 0 is not a zero of the $\left(Q_8(z) - \frac{Ad_8}{c_8}P_8(a)\right)$ as

$$Q_8(0) - \frac{Ad_8}{c_8}P_8(a) = -d_8\left(1 + \frac{AP_8(a)}{c_8}\right) \neq 0.$$

Again we can see that if a is zero of the polynomial $\left(Q_8(z) - \frac{Ad_8}{c_8}P_8(a)\right)$, from the hypothesis of the *Theorem 3.2* we have

$$Q_8(a) - \frac{Ad_8}{c_8}P_8(a) = 0 \quad \text{i.e.} \quad A = \frac{c_8 Q_8(a)}{d_8 P_8(a)} = 1,$$

which is a contradiction.

Hence all the zeros of the polynomial $\left(Q_8(z) - \frac{Ad_8}{c_8}P_8(a)\right)$ are simple namely $\hat{\beta}_i$ for $i = 1, 2, \ldots 8$. Using the Second Fundamental Theorem, (5.11) and (5.13) we obtain

$$\begin{aligned}
7T(r, g^{(k)}) &\leq \sum_{i=1}^{8} \overline{N}(r, \hat{\beta}_i; g^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\
&\leq \overline{N}(r, a; f^{(k)}) + \sum_{i=1}^{2} \overline{N}(r, \hat{\alpha_i}; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, f^{(k)}) \\
&\leq \left(2 + \frac{1}{k+1}\right) T(r, g^{(k)}) + S(r, g^{(k)}),
\end{aligned}$$

which is a contradiction.

Case 2: Let us suppose that $A = -\left(\frac{c_8}{P_8(a)}\right)$. Clearly, 0 is zero of $\left(Q_8(z) - \frac{Ad_8}{c_8}P_8(a)\right)$ of multiplicity 3 and other zeros are simple say $\hat{\gamma}_i$ for

$i = 1, 2, \ldots 5$. Using the Second Fundamental Theorem, (5.11) and (5.13) we possess

$$\begin{aligned} 5T(r, g^{(k)}) &\leq \overline{N}(r, 0; g^{(k)}) + \sum_{i=1}^{5} \overline{N}(r, \hat{\gamma}_i; g^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\ &\leq \overline{N}(r, a; f^{(k)}) + \sum_{i=1}^{2} \overline{N}(r, \hat{\alpha}_i; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, g^{(k)}) \\ &\leq \left(2 + \frac{1}{k+1}\right) T(r, g^{(k)}) + S(r, g^{(k)}), \end{aligned}$$

which is a contradiction.

As

$$Q_8(a) - \frac{Ad_8}{c_8} P_8(a) = Q_8(a) + d_8 = \widehat{Q}_8(a) \neq 0,$$

in this case a is not a zero of the polynomial $\left(Q_8(z) - \frac{Ad_8}{c_8} P_8(a)\right)$.

Hence all the zeros of the polynomial $\left(Q_8(z) - \frac{Ad_8}{c_8} P_8(a)\right)$ are simple and using the same arguments that were used to handle the situation in *Case 1* we will again arrive a contradiction.

So $A = 1$ and We have

$$F_8 = G_8$$

which implies

$$\begin{aligned} &\frac{(g^{(k)})^5(h^8 - \beta)}{8} - \frac{5a(g^{(k)})^4(h^7 - \beta)}{7} + \frac{5a^2(g^{(k)})^3(h^6 - \beta)}{3} - 2a^3(g^{(k)})^2(h^5 - \beta) \\ &\quad + \frac{5a^4(g^{(k)})(h^4 - \beta)}{4} - \frac{a^5(h^3 - \beta)}{3} \equiv 0, \end{aligned}$$

where $h = \frac{f^{(k)}}{g^{(k)}}$ and $\beta = \frac{c_8}{a^5 d_8}$.

Next if $P_8(z) = Q_8(z)$ we have from (5.10), *Lemma 4.9* and *Lemma 4.10*, $f^{(k)} \equiv g^{(k)}$.

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