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# Multi-dimensional Poisson transform and applications

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**Abstract.** In this paper, we consider several new types of multi-dimensional Poisson transforms for the functions with values in complex Banach spaces. We also prove some new properties of multidimensional vector-valued Laplace transform and analyze certain connections between the solutions of the abstract fractional partial differential equations and the abstract fractional partial difference equations.

## 1. Introduction and preliminaries

Fractional calculus, discrete fractional calculus, fractional differential equations and fractional difference equations are rapidly growing fields of research of many authors; cf. the monographs [3, 9, 10, 13, 14, 22], the doctoral dissertation of E. Bazhlekova [4] and the references quoted therein for more details in this direction. Fractional differential-difference equations have received considerable attention in the last three decades due to their tremendous application potential. The theories of fractional equations on continuous and discrete time domains are well established now and the literature on fractional differential-difference equations rapidly grows. We will only mention here that the fractional differential-difference equations are incredibly important in modeling of various phenomena appearing in mathematical physics, viscoelasticity, optics,

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acoustics, rheology, bioengineering, control theory, electrical and mechanical engineering and so on.

On the other hand, the twodimensional scalar-valued Laplace transform was first considered by D. L. Bernstein [5]-[6] and J. C. Jaeger [12] (1939–1941). The multidimensional scalar-valued Laplace transform has been considered by many authors so far and has numerous delightful applications to the partial differential equations; for more details about the multidimensional Laplace transform and its applications, we refer the reader to the list of references quoted in the recent research article [18].

The vector-valued Poisson transform has been first considered in the pioneering paper [21] by C. Lizama, where the author has also presented some applications of Poisson transform to the abstract fractional difference equations. The main aim of this research article is to extend some structural results from [21] to the higher-dimensional setting. We also further analyze here the multidimensional vector-valued Laplace transform and clarify certain relations between the solutions of the abstract (fractional) partial differential equations and the solutions of the abstract (fractional) difference equations of several variables. Our main results are Theorem 2.3, Theorem 2.7 and Theorem 2.8.

The organization and main ideas of this research article can be simply explained as follows. After fixing the notation and preliminaries used in the paper, we recall the basic definitions about the multidimensional vector-valued Laplace transform and prove some new results in this direction; cf. Proposition 1.1 in Subsection 1.1. Subsection 1.2 recalls the basic definitions about the multi-dimensional fractional calculus; see [18] for more details in this direction. Our main structural results are given in Section 2, where we analyze the multidimensional vector-valued Poisson transforms; the applications to the abstract partial differential-difference equations are given in Subsection 2.1 and the applications to the abstract fractional partial differential-difference equations are given in Subsection 2.2.

**Notation and preliminaries.** In the sequel, we will always assume that $n \in \mathbb{N}$, $(X, \| \cdot \|)$ is a complex Banach space; $\mathbb{N}_n := \{1, ..., n\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(s \in \mathbb{R})$. The finite convolution product $*_0$ of the Lebesgue measurable functions $a(\cdot)$ and $b(\cdot)$ defined on $[0, \infty)$ is given by $(a *_0 b)(t) := \int_0^t a(t-s)b(s)\, ds$, $t \geq 0$. By $\Gamma(\cdot)$ we denote the Euler Gamma function; we set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$ and $g_0(t) := \delta(t)$, the Dirac $\delta$-distribution.

If $u \in L^1_{loc}([0,\infty)^n : X)$, $j \in \mathbb{N}_n$, $\alpha_j > 0$ and $a \in L^1_{loc}([0,\infty)^n)$, then we define

$$J^{\alpha_j}_{t_j} u\bigl(x_1, ..., x_{j-1}, x_j, x_{j+1}, ...x_n\bigr) := \int_0^{x_j} g_{\alpha_j}\bigl(x_j - s\bigr) u\bigl(x_1, ..., x_{j-1}, s, x_{j+1}, ...x_n\bigr)\, ds,$$

and

$$\big(a *_0 u\big)(\mathbf{x}) := \int_0^{x_1} \cdot .... \cdot \int_0^{x_n} a\big(x_1 - s_1, ..., x_n - s_n\big) u\big(s_1, ..., s_n\big)\, ds_1 \cdot ... \cdot ds_n,$$

for any $\mathbf{x} = \big(x_1, ..., x_n\big) \in [0, \infty)^n$.

Further on, if $\alpha > 0$, then the Cesàro sequence $(k^{\alpha}(v))_{v \in \mathbb{N}_0}$ is defined by

$$k^{\alpha}(v) := \frac{\Gamma(v+\alpha)}{\Gamma(\alpha) v!}, \quad v \in \mathbb{N}_0.$$

It is well-known that, for every $\alpha > 0$ and $\beta > 0$, we have $k^{\alpha} *_0 k^{\beta} \equiv k^{\alpha+\beta}$. Define $k^0(0) := 1$ and $k^0(v) := 0$, $v \in \mathbb{N}$; then we have $k^{\alpha} *_0 k^{\beta} \equiv k^{\alpha+\beta}$ for all $\alpha,\ \beta \geq 0$.

If $(u_k)$ is a sequence in $X$, then the Euler forward difference operator $\Delta$ is defined by $\Delta u_k := u_{k+1} - u_k$. The operator $\Delta^m$ is defined inductively; then, for every integer $m \geq 1$, we have:

$$\Delta^m u_k = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} u_{k+j}.$$

If $\mathrm{j} = (j_1, ..., j_n) \in \mathbb{N}_0^n$ and $\mathrm{k} = (k_1, ..., k_n) \in \mathbb{N}_0^n$, then we write $\mathrm{j} \leq \mathrm{k}$ if and only if $j_m \leq k_m$ for all $1 \leq m \leq n$. If the sequences $(a_{\mathrm{k}})_{\mathrm{k} \in \mathbb{N}_0^n}$ and $(b_{\mathrm{k}})_{\mathrm{k} \in \mathbb{N}_0^n}$ are given, then we define $(a *_0 b)(\cdot)$ by

$$(a *_0 b)(\mathrm{k}) := \sum_{\mathrm{j} \in \mathbb{N}_0^n ; \mathrm{j} \leq \mathrm{k}} a_{\mathrm{k}-\mathrm{j}} b_{\mathrm{j}},\ \mathrm{k} \in \mathbb{N}_0^n.$$

It can be simply proved that the convolution product $*_0$ is commutative and associative. Further on, if $a(\cdot)$ is a given sequence in $X$ which depends on the variables $v_1, ..., v_n$, then we define

$$\Delta_{v_i} a\big(v_1, ..., v_i, ..., v_n\big) := a\big(v_1, ..., v_i + 1, ..., v_n\big) - a\big(v_1, ..., v_i, ..., v_n\big).$$

After that, we set $\Delta^2_{v_i v_j} a := \Delta_{v_i} \Delta_{v_j} a$ and $\Delta^2_{v_i v_i} a := \Delta_{v_i} \Delta_{v_i} a$; the terms

$$\Delta^m_{v_{i_1} \dots v_{i_m}} a \text{ and } \Delta^{|\alpha|}_{v_1^{\alpha_1} \cdot ... \cdot v_n^{\alpha_n}} a$$

are defined recursively, as for the partial derivatives of functions ($\alpha_i \in \mathbb{N}_0$; $|\alpha| = \alpha_1 + ... + \alpha_n$). It is worth noting that, for every permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, we have

$$\Delta^{|\alpha|}_{v_1^{\alpha_1} \cdot ... \cdot v_n^{\alpha_n}} a = \Delta^{|\alpha|}_{v_{\sigma(1)}^{\alpha_{\sigma(1)}} \cdot ... \cdot v_{\sigma(n)}^{\alpha_{\sigma(n)}}} a,$$

as easily approved. Many other important results of mathematical analysis, like Green's formula in the plane and Grönwall inequality, have analogues for the difference operators; see [7, pp. 23–25, 43–44] for more details in this direction.

### 1.1. Multidimensional vector-valued Laplace transform

Suppose that $f : [0,+\infty)^n \to X$ is a locally integrable function. Then the multidimensional vector-valued Laplace transform of $f(\cdot)$, denoted by $F(\cdot) = \tilde{f} = \mathcal{L}f$, is defined through

$$F\big(\lambda_1, ..., \lambda_n\big) := \lim_{T\to+\infty} \int_{[0,T]^n} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} f\big(t_1, ..., t_n\big)\, dt_1 \dots dt_n$$
$$:= \int_{[0,+\infty)^n} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} f\big(t_1, ..., t_n\big)\, dt_1 \dots dt_n,$$

if it is well-defined. Our basic assumption henceforth will be:

(GR) $f(\cdot)$ is Lebesgue measurable and there exist real constants $\omega_1 \in \mathbb{R}$, ..., $\omega_n \in \mathbb{R}$, $\eta_1 \in (-1,+\infty)$, ..., $\eta_n \in (-1,+\infty)$ and $\zeta_1 \in (-1,+\infty)$, ..., $\zeta_n \in (-1,+\infty)$ such that

$$\Big\|f\big(t_1, ..., t_n\big)\Big\| \leq M\Big(t_1^{\eta_1} + t_1^{\zeta_1}\Big) \cdot ... \cdot \Big(t_n^{\eta_n} + t_n^{\zeta_n}\Big) \exp\big(\omega_1 t_1 + ... + \omega_n t_n\big),$$
$$\text{for a.e. } t_1 \geq 0, ..., \ t_n \geq 0. \tag{1.1}$$

In this case, the Fubini's theorem implies that the function $F(\lambda_1, ..., \lambda_n)$ is well-defined for $\Re\lambda_1 > \omega_1$, ..., $\Re\lambda_n > \omega_n$ and the Lebesgue dominated convergence theorem implies that $F(\cdot)$ is analytic in this region of $\mathbb{C}^n$ (see [11] for more details about analytic functions of several complex variables).

The collection of all Lebesgue measurable functions $f(\cdot)$ which satisfies condition (GR) forms a vector space with the usual operations. Furthermore, if $f(\cdot)$ satisfies (GR) with $X = \mathbb{C}$, $g : [0,+\infty)^n \to X$ is Lebesgue measurable and there exist real constants $\omega_{1,g} \in \mathbb{R}$, ..., $\omega_{n,g} \in \mathbb{R}$, $\eta_{1,g} \in (-1,+\infty)$, ..., $\eta_{n,g} \in (-1,+\infty)$ and $\zeta_{1,g} \in (-1,+\infty)$, ..., $\zeta_{n,g} \in (-1,+\infty)$ such that

$$\Big\|g\big(t_1, ..., t_n\big)\Big\| \leq M\Big(t_1^{\eta_{1,g}} + t_1^{\zeta_{1,g}}\Big) \cdot ... \cdot \Big(t_n^{\eta_{n,g}} + t_n^{\zeta_{n,g}}\Big) \exp\big(\omega_{1,g} t_1 + ... + \omega_{n,g} t_n\big),$$
$$\text{for a.e. } t_1 \geq 0, ..., \ t_n \geq 0,$$

then the pointwise product $[fg](\cdot)$ also satisfies (GR), provided that

$$\min\big\{\eta_{j,g} + \eta_j,\ \eta_{j,g} + \zeta_j,\ \zeta_{j,g} + \eta_j,\ \zeta_{j,g} + \zeta_j : 1 \leq j \leq n\big\} > -1. \tag{1.2}$$

In our recent research article [18], we have proved the complex inversion theorem for the multidimensional vector-valued Laplace transform. Now we will state and prove the following statements concerning the multi-dimensional Laplace transform, which will be sufficiently enough for our later purposes (cf. also the statements of [2, Theorem 1.5.1, Proposition 1.6.4], which will not be fully generalized to the multi-dimensional setting here):

**Proposition 1.1.** *(i) Suppose that* $f : [0, +\infty)^n \to X$ *satisfies* (GR). *Then we have*

(1.3)

$$F^{(v_1,...,v_n)}(\lambda_1, ..., \lambda_n) = (-1)^{v_1+...+v_n}\Biggl(\mathcal{L}\Biggl[\frac{\cdot_1^{v_1}}{v_1!} \cdot ... \cdot \frac{\cdot_n^{v_n}}{v_n!} f\bigl(\cdot_1, ..., \cdot_n\bigr)\Biggr]\Biggr)(\lambda_1, ..., \lambda_n),$$

*for* $\Re\lambda_1 > \omega_1,\ ...,\ \Re\lambda_n > \omega_n$ *and* $(v_1, ..., v_n) \in \mathbb{N}_0^n$.

*(ii) Suppose that* $a \in L^1_{loc}([0, +\infty)^n)$ *satisfies* (GR) *with* $X = \mathbb{C}$, *the constants* $\omega_1 \in \mathbb{R},\ ...,\ \omega_n \in \mathbb{R}$ *and the constants* $\eta_1 \in (-1, +\infty),\ ...,\ \eta_n \in (-1, +\infty)$ *and* $\zeta_1 \in (-1, +\infty),\ ...,\ \zeta_n \in (-1, +\infty)$ *replaced therein with the constants* $\eta_{1,a} \in (-1, +\infty),\ ...,\ \eta_{n,a} \in (-1, +\infty)$ *and* $\zeta_{1,a} \in (-1, +\infty),\ ...,\ \zeta_{n,a} \in (-1, +\infty)$. *Suppose, further, that* $f \in L^1_{loc}([0, +\infty)^n : X)$ *satisfies* (GR) *with the same constants* $\omega_1 \in \mathbb{R},\ ...,\ \omega_n \in \mathbb{R}$*; then* $(a *_0 f)(\cdot) \in L^1_{loc}([0, +\infty)^n)$ *satisfies* (GR) *and we have:*

$$\mathcal{L}\bigl(a *_0 f\bigr)(\lambda_1, ..., \lambda_n) = \mathcal{L}a(\lambda_1, ..., \lambda_n) \cdot \mathcal{L}f(\lambda_1, ..., \lambda_n), \tag{1.4}$$

*for* $\Re\lambda_1 > \omega_1,\ ...,\ \Re\lambda_n > \omega_n$.

**Proof.** Keeping in mind the estimate (1.1), the part (i) follows from a simple application of the Lebesgue dominated convergence theorem. It can be simply shown that $(a *_0 f)(\cdot)$ is Lebesgue measurable; furthermore, a simple computation involving the Fubini's theorem and the identity $g_c *_0 g_d = g_{c+d}$ for $c,\ d > 0$ shows that there exist positive real constants $M,\ M_1 > 0$ such that:

$$\begin{aligned}
&\bigl\|(a *_0 f)(x_1, ..., x_n)\bigr\| \leq M^n \exp\bigl(\omega_1 x_1 + ... + \omega_n x_n\bigr) \\
&\quad \cdot \int_0^{x_1} \cdot ... \cdot \int_0^{x_n} \Bigl((x_1 - t_1)^{\eta_1} + (x_1 - t_1)^{\zeta_1}\Bigr) \cdot ... \cdot \Bigl((x_n - t_n)^{\eta_n} + (x_n - t_n)^{\zeta_n}\Bigr) \\
&\quad \times \Bigl(t_1^{\eta_1} + t_1^{\zeta_1}\Bigr) \cdot ... \cdot \Bigl(t_n^{\eta_n} + t_n^{\zeta_n}\Bigr)\, dt_1 \cdot ... \cdot dt_n \\
&\quad \leq M_1^n \Bigl(x_1^{\eta_{1;1}} + x_1^{\zeta_{1;1}}\Bigr) \cdot ... \cdot \Bigl(x_n^{\eta_{n;1}} + x_n^{\zeta_{n;1}}\Bigr) \exp\bigl(\omega_1 x_1 + ... + \omega_n x_n\bigr),
\end{aligned}$$

where $\eta_{j;1} = \min(\eta_{j,a} + \eta_j + 1, \eta_{j,a} + \zeta_j + 1, \zeta_{j,a} + \eta_j + 1, \zeta_{j,a} + \zeta_j + 1)$ and $\zeta_{j;1} = \max(\eta_{j,a} + \eta_j + 1, \eta_{j,a} + \zeta_j + 1, \zeta_{j,a} + \eta_j + 1, \zeta_{j,a} + \zeta_j + 1)$ for $1 \leq j \leq n$. The formula (1.4) can be deduced following the lines of the proofs of [8, Theorem 3.1, Theorem 4.1], where the author has considered the double Laplace transform in the scalar-valued setting. ■

### 1.2. Multi-dimensional fractional calculus

In this subsection, we recall the basic definitions about the multi-dimensional generalized Hilfer fractional derivatives and differences ([18]).

If $\delta(t)$ denotes the Dirac delta distribution, then we set $\int_0^t \delta(t-s)f(s)\,ds \equiv f(t)$. Suppose first that $u : [0,\infty) \to X$ is locally integrable, $\alpha > 0$, $m = \lceil \alpha \rceil$, $a \in L^1_{loc}([0,\infty))$ or $a(t) = \delta(t)$, and $b \in L^1_{loc}([0,\infty))$ or $b(t) = \delta(t)$. Set $v_a(t) := \int_0^t a(t-s)u(s)\,ds$, $t \geq 0$. The following extension of the usual Hilfer fractional derivative $D_t^{\alpha,\beta}u(t)$, when $a(t) = g_{(1-\beta)(m-\alpha)}(t)$ and $b(t) = g_{\beta(m-\alpha)}(t)$ for some $\beta \in [0,1]$, has recently been introduced in [20] (for $\beta = 0$, resp. $\beta = 1$, we get the usual Riemann-Liouville fractional derivative $D_R^\alpha u$ of order $\alpha$, resp., the Caputo fractional derivative $\mathbf{D}_C^\alpha u$ of order $\alpha$):

**Definition 1.2.** The generalized Hilfer $(a, b, \alpha)$-fractional derivative of function $u(\cdot)$, denoted shortly by $D^\alpha_{a,b}u$, is defined for any locally integrable function $u(\cdot)$ such that the function $v_a^{(m-1)}(t)$ is locally absolutely continuous for $t \geq 0$, by

$$D^\alpha_{a,b}u(t) := \Bigl(b *_0 v_a^{(m)}\Bigr)(t), \quad \text{a.e. } t \geq 0. \tag{1.5}$$

We similarly define the above notion for locally integrable functions defined on the finite intervals $[0,T]$ and $[0,T)$, where $T > 0$.

Assume now that $u : \mathbb{N}_0 \to X$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $a : \mathbb{N}_0 \to \mathbb{C}$ and $b : \mathbb{N}_0 \to \mathbb{C}$. The following notion is a discrete version of the notion considered above:

**Definition 1.3.** The generalized Hilfer $(a, b, \alpha)$-fractional derivative of sequence $u(\cdot)$, denoted shortly by $\Delta^\alpha_{a,b}u$, is defined by

$$\Delta^\alpha_{a,b}u(v) := \Bigl(b *_0 \Delta^m\bigl(a *_0 u\bigr)\Bigr)(v), \quad v \in \mathbb{N}_0.$$

If $0 \leq \beta \leq 1$, then the usual Hilfer fractional derivative $\Delta^{\alpha,\beta}u$ of order $\alpha$ and type $\beta$ is defined as the generalized Hilfer $(a, b, \alpha)$-fractional derivative of $u(\cdot)$, with $a(v) = k^{(1-\beta)(m-\alpha)}(v)$ and $b(v) = k^{\beta(1-\alpha)}(v)$; for $\beta = 0$, resp. $\beta = 1$, we get the usual Riemann-Liouville fractional derivative $\Delta^\alpha u$ of order $\alpha$, resp., the Caputo fractional derivative $\Delta^\alpha_C u$ of order $\alpha$. Set

$$D^0_{a,b}u := a *_0 b *_0 u \quad \text{and} \quad \Delta^0_{a,b}u := a *_0 b *_0 u.$$

Assume now that $0 < T_j < +\infty$ and $I_j = [0, T_j)$, $I_j = [0, T_j]$ or $I_j = [0, +\infty)$ for $1 \leq j \leq n$. Set $I := I_1 \times I_2 \times ... \times I_n$. Suppose that $u : I \to X$ is a locally integrable function and, for every $j \in \mathbb{N}_n$, $a_j \in L^1_{loc}(I_j)$ or $a_j(t) = \delta(t)$, and $b_j \in L^1_{loc}(I_j)$ or $b_j(t) = \delta(t)$. Suppose further that $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$. Define $\alpha := (\alpha_1, ..., \alpha_n)$ and

$$\mathbb{D}^\alpha_{\mathbf{a},\mathbf{b}}u\bigl(x_1, ..., x_n\bigr) := \left[D^{\alpha_1}_{a_1,b_1}\left(D^{\alpha_2}_{a_2,b_2}\left(...\left(D^{\alpha_n}_{a_n,b_n}u\bigl(\cdot, ..., \cdot\bigr)\right)...\right)\right)\right]\bigl(x_1, ..., x_n\bigr), \tag{1.6}$$

for a.e. $(x_1, ..., x_n) \in I$, provided that the right hand side of (1.6) is well-defined. Here, we suppose that the variables $x_1,\ x_2, ..., x_{n-1}$ are fixed in the computation of the term $D^{\alpha_n}_{a_n,b_n}u(x_1, ..., x_n)$, ..., as well as that that the variables $x_2,\ x_3, ..., x_n$ are fixed in the computation of the final term on the right hand side of (1.6). We call $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}}u$ the multi-dimensional generalized Hilfer $(\mathbf{a}, \mathbf{b}, \alpha)$-fractional derivative of the function $u(\cdot)$. If for each $j \in \mathbb{N}_n$ we have $D^{\alpha_j}_{a_j,b_j} = D^{\alpha_j}_R$, resp., for each $j \in \mathbb{N}_n$ we have $D^{\alpha_j}_{a_j,b_j} = \mathbf{D}^{\alpha_j}_C$, then the corresponding partial fractional derivative $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}}$ is called the multi-dimensional Riemann-Liouville fractional operator (cf. also [22, pp. 340–342]), resp., the multi-dimensional Caputo fractional operator.

In the discrete setting, we suppose that $u : \mathbb{N}_0^n \to X$, $a_j : \mathbb{N}_0 \to \mathbb{C}$ and $b_j : \mathbb{N}_0 \to \mathbb{C}$ are given sequences $(1 \leq j \leq n)$. We define

(1.7)
$$\Delta^{\alpha}_{\mathbf{a},\mathbf{b}}u\big(v_1, ..., v_n\big) := \left[\Delta^{\alpha_1}_{a_1,b_1}\left(\Delta^{\alpha_2}_{a_2,b_2}\left(...\left(\Delta^{\alpha_n}_{a_n,b_n}u\big(\cdot, ..., \cdot\big)\right)...\right)\right)\right]\big(v_1, ..., v_n\big),$$

for any $(v_1, ..., v_n) \in \mathbb{N}_0^n$; note that the right-hand side of (1.7) is always well-defined. We call $\Delta^{\alpha}_{\mathbf{a},\mathbf{b}}u$ the multi-dimensional generalized Hilfer $(\mathbf{a}, \mathbf{b}, \alpha)$-fractional derivative of the sequence $u(\cdot)$; the multi-dimensional Riemann-Liouville fractional difference operator $\Delta^{\alpha}_R$ and the multi-dimensional Caputo fractional difference operator $\Delta^{\alpha}_C$ are defined similarly as above.

In the sequel, we shall primarily use the fractional partial derivatives of the Riemann-Liouville or Caputo type.

## 2. Multi-dimensional Poisson transforms

In this section, we will continue our recent investigation of Poisson like transforms and explain how the already established results and ideas can be simply transferred to the multi-dimensional setting. Suppose that $u : [0, \infty)^n \to X$ is a given locally integrable function and the value of

$$\begin{aligned}[P(u)](\mathrm{v}) &:= [P(u)]\big(v_1, ..., v_n\big) \\ &:= \int_{[0,\infty)^n} e^{-x_1-...-x_n}\frac{x_1^{v_1}}{v_1!}\cdot ... \cdot\frac{x_n^{v_n}}{v_n!}u\big(x_1, ..., x_n\big)\, dx_1\, dx_2 \ldots dx_n\end{aligned}$$

is well-defined for all $v_1 \in \mathbb{N}_0, ..., v_n \in \mathbb{N}_0$. We call the mapping $u \mapsto P(u)$ the multi-dimensional Poisson transform; in terms of the multidimensional vector-valued Laplace transform, we have

$$[P(u)](v_1,...,v_n)=\Biggl(\mathcal{L}\Biggl[\frac{\cdot_1^{v_1}}{v_1!}\cdot ... \cdot\frac{\cdot_n^{v_n}}{v_n!}u(\cdot_1,...,\cdot_n)\Biggr]\Biggr)(1,...,1),\quad (v_1,...,v_n)\in \mathbb{N}_0^n.$$

We can simply prove that
$\int_{[0,\infty)^n}\|u(x_1,...,x_n)\|\,dx_1\,dx_2\,...\,dx_n<+\infty$ implies $\sum_{\mathrm{v}\in\mathbb{N}_0^n}\|[P(u)](\mathrm{v})\|<+\infty.$

We will analyze the multidimensional vector-valued $Z$-transform of sequences and its applications in our forthcoming paper [19], where we will also extend the statement of [21, Theorem 3.1] to the multi-dimensional setting. Now we will reconsider [21, Theorem 3.4] in the multi-dimensional setting:

**Theorem 2.1.** *Suppose that* $a\in L^1_{loc}([0,+\infty)^n)$ *satisfies* (GR) *with* $X=\mathbb{C}$, *the constants* $\omega_1\in(-\infty,1)$, ..., $\omega_n\in(-\infty,1)$ *and the constants* $\eta_1\in(-1,+\infty)$, ..., $\eta_n\in(-1,+\infty)$ *and* $\zeta_1\in(-1,+\infty)$, ..., $\zeta_n\in(-1,+\infty)$ *replaced therein with the constants* $\eta_{1,a}\in(-1,+\infty)$, ..., $\eta_{n,a}\in(-1,+\infty)$ *and* $\zeta_{1,a}\in(-1,+\infty)$, ..., $\zeta_{n,a}\in(-1,+\infty)$. *Suppose, further, that* $f\in L^1_{loc}([0,+\infty)^n:X)$ *satisfies* (GR) *with the same constants* $\omega_1\in(-\infty,1)$, ..., $\omega_n\in(-\infty,1)$. *Then* $[P(a*_0 f)](v_1,...,v_n)$, $[P(a)](v_1,...,v_n)$ *and* $[P(f)](v_1,...,v_n)$ *exist for any* $(v_1,...,v_n)\in\mathbb{N}_0^n$; *furthermore, we have*

(2.1)
$$\bigl[P(a*_0 f)\bigr](v_1,...,v_n)=[P(a)](v_1,...,v_n)\cdot[P(f)](v_1,...,v_n),\quad (v_1,...,v_n)\in\mathbb{N}_0^n.$$

**Proof.** By Proposition 1.1(ii), we have that $(a*_0 f)(\cdot)\in L^1_{loc}([0,+\infty)^n)$ satisfies (GR) and (1.4) holds. It is clear that $[P(a*_0 f)](v_1,...,v_n)$, $[P(a)](v_1,...,v_n)$ and $[P(f)](v_1,...,v_n)$ exist for any $(v_1,...,v_n)\in\mathbb{N}_0^n$. Set now $G:=\mathcal{L}(a*_0 f)$. Then we have:

$$\bigl[P(a*_0 f)\bigr](v_1,...,v_n)=(-1)^{v_1+....+v_n}\frac{1}{v_1!}\cdot ... \cdot\frac{1}{v_n!}G^{(v_1,...,v_n)}(1,...,1)$$
$$=\frac{(-1)^{v_1+....+v_n}}{v_1!\cdot ... \cdot v_n!}\Biggl[[\mathcal{L}a]^{(v_1,...,v_n)}(\lambda_1,...,\lambda_n)\cdot[\mathcal{L}f]^{(v_1,...,v_n)}(\lambda_1,...,\lambda_n)\Biggr]_{(\lambda_1,...,\lambda_n)=(1,...,1)}$$
$$=\frac{(-1)^{v_1+...+v_n}}{v_1!\cdot ... \cdot v_n!}\sum_{j\in\mathbb{N}_0^n;j\leq \mathrm{v}}\binom{v_1}{j_1}\cdot ... \cdot\binom{v_n}{j_n}$$
$$\times[\mathcal{L}a]^{(v_1-j_1,...,v_n-j_n)}(1,...,1)\cdot[\mathcal{L}f]^{(j_1,...,j_n)}(1,...,1)$$
$$=\frac{1}{v_1!\cdot ... \cdot v_n!}\sum_{\mathrm{j}\in\mathbb{N}_0^n;\mathrm{j}\leq\mathrm{v}}\binom{v_1}{j_1}\cdot ... \cdot\binom{v_n}{j_n}[P(a)](\mathrm{v}-\mathrm{j})\cdot[P(f)](\mathrm{j})$$
$$\times(v_1-j_1)!\cdot ... \cdot(v_n-j_n)!\cdot j_1!\cdot .... \cdot j_n![P(a)](\mathrm{v}-\mathrm{j})\cdot[P(f)](\mathrm{j})$$
$$=\sum_{\mathrm{j}\in\mathbb{N}_0^n;\mathrm{j}\leq\mathrm{v}}[P(a)](\mathrm{v}-\mathrm{j})\cdot[P(f)](\mathrm{j}),\quad (v_1,...,v_n)\in\mathbb{N}_0^n,$$

where we have used Proposition 1.1(i)-(ii) and the Leibniz rule. This proves (2.1) and completes the proof. ■

*Remark* 2.1. In the formulation of [21, Theorem 3.4], we must additionally assume that the Laplace transform of function $|a|(\cdot)$ exists at the point 1. Only in this way, we can apply [2, Proposition 1.6.4] as an essential ingredient in the proof of [21, Theorem 3.4], which does not work if $abs(a) < 1 < abs(|a|)$; here, we use the same notion and notation as in [2].

### 2.1. Applications to the abstract partial differential equations

In this subsection, we will present some applications of multidimensional vector-valued Poisson transform to the abstract partial differential equations.

Let us formally set $(x^v/v!) := 0$, if $-v \in \mathbb{N}$. We start with the following illustrative example:

**Example 2.2.** Let us consider the partial differential operator $u_{x_1x_1x_2}(\cdot,\cdot)$, in the dimension $n = 2$, and let us assume that the partial derivatives $u_{x_1x_1x_2}(\cdot,\cdot)$, $u_{x_1x_1}(\cdot,\cdot)$, $u_{x_1}(\cdot,\cdot)$ and the function $u(\cdot,\cdot)$ are continuous on $[0,\infty)^2$, as well as that

$$u_{x_1x_1x_2}(x_1,x_2) = g(x_1,x_2), \quad (x_1,x_2) \in [0,\infty)^2.$$

Let $v_1 \in \mathbb{N}_0$ and $v_2 \in \mathbb{N}_0$ be fixed. Applying the Fubini's theorem and the partial integration with respect to the variable $x_2$, we get:

$$\int_{[0,\infty)^2} e^{-x_1-x_2} \frac{x_1^{v_1+2}}{(v_1+2)!} \frac{x_2^{v_2+1}}{(v_2+1)!} u_{x_1x_1x_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{[0,\infty)^2} e^{-x_1-x_2} \frac{x_1^{v_1+2}}{(v_1+2)!} \left[ \frac{x_2^{v_2+1}}{(v_2+1)!} - \frac{x_2^{v_2}}{v_2!} \right] u_{x_1x_1}(x_1,x_2)\, dx_1\, dx_2, \tag{2.2}$$

provided that

$$\lim_{x_2\to+\infty} e^{-x_2} \frac{x_2^{v_2+1}}{(v_2+1)!} u_{x_1x_1}(x_1,x_2) = 0, \quad x_1 \geq 0 \tag{2.3}$$

and the both double integrals in (2.2) converges absolutely. Applying the Fubini's theorem and the partial integration two times more, with respect to the variable $x_1$, we get:

$$\int_{[0,\infty)^2} e^{-x_1-x_2} \frac{x_1^{v_1+2}}{(v_1+2)!} \frac{x_2^{v_2+1}}{(v_2+1)!} u_{x_1x_1x_2}(x_1,x_2)\, dx_1\, dx_2$$

(2.4)

$$= \int_{[0,\infty)^2} e^{-x_1-x_2} \left[ \frac{x_1^{v_1+2}}{(v_1+2)!} - 2\frac{x_1^{v_1+1}}{(v_1+1)!} + \frac{x_1^{v_1}}{v_1!} \right] \left[ \frac{x_2^{v_2+1}}{(v_2+1)!} - \frac{x_2^{v_2}}{v_2!} \right] u(x_1,x_2)\, dx_1\, dx_2,$$

i.e.,

$$\left[\Delta^3_{x_1^2 x_2} P(u)\right](v_1, v_2) = \int_{[0,\infty)^2} e^{-x_1-x_2} \frac{x_1^{v_1+2}}{(v_1+2)!} \frac{x_2^{v_2+1}}{(v_2+1)!} g(x_1, x_2)\, dx_1\, dx_2,$$

provided that (2.3) holds, as well as

$$\lim_{x_1 \to +\infty} e^{-x_1} \frac{x_1^{v_1+2}}{(v_1+2)!} u_{x_1}(x_1, x_2) = 0, \quad x_2 \geq 0,$$

$$\lim_{x_1 \to +\infty} e^{-x_1} \left[\frac{x_1^{v_1+2}}{(v_1+2)!} - \frac{x_1^{v_1+1}}{(v_1+1)!}\right] u(x_1, x_2) = 0, \quad x_2 \geq 0,$$

the second integral in (2.4) converges absolutely and the integral

$$\int_{[0,\infty)^2} e^{-x_1-x_2} \left[\frac{x_1^{v_1+2}}{(v_1+2)!} - \frac{x_1^{v_1+1}}{(v_1+1)!}\right] \left[\frac{x_2^{v_2+1}}{(v_2+1)!} - \frac{x_2^{v_2}}{v_2!}\right] u_{x_1}(x_1, x_2)\, dx_1\, dx_2$$

converges absolutely.

In general case, one can use the Fubini's theorem and the partial integration in order to see that the following result holds true:

**Theorem 2.3.** *If $D$ is a non-empty subset of $\mathbb{N}_0^n$, $A_\alpha$ is a closed linear operator on $X$ for all $\alpha \in D$, and*

(2.5)
$$\sum_{\alpha \in D} A_\alpha \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}}(x_1, ..., x_n) = \sum_{\alpha \in D} A_\alpha u^{(\alpha)}(x_1, ..., x_n) = g(x_1, ..., x_n),$$

*for any $(x_1, ..., x_n) \in [0, \infty)^n$, then we have*

$$\sum_{\alpha \in D} A_\alpha \left[\Delta^{|\alpha|}_{v_1^{\alpha_1} \cdot \ldots \cdot v_n^{\alpha_n}} P(u)\right](v_1, ..., v_n)$$
$$= \int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+\alpha_1}}{(v_1+\alpha_1)!} \cdot \ldots \cdot \frac{x_n^{v_n+\alpha_n}}{(v_n+\alpha_n)!} g(x_1, ..., x_n)\, dx_1\, dx_2 \ldots dx_n,$$

*for any $(v_1, ..., v_n) \in \mathbb{N}_0^n$, provided that $u : [0, \infty)^n \to X$ is continuous, the integral on the right hand side of the above equality absolutely converges and, for every $v \in \mathbb{N}$, $(v_1, ..., v_n) \in \mathbb{N}_0^n$ and for every multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in D$, we have:*

*0. The integral*

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}A_\alpha u^{(\alpha)}\big(x_1,\ldots,x_n\big)\,dx_1\,dx_2\ldots dx_n$$

*is covergent and the integral*

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}u^{(\alpha)}\big(x_1,\ldots,x_n\big)dx_1\,dx_2\ldots dx_n$$

*is convergent;*

*1. For every multi-index $(\alpha_1,\ldots,\alpha_{n-1},\gamma_n)$, where $0\le\gamma_n\le\alpha_n$, the mapping $\frac{\partial^{\alpha_1+\cdots+\alpha_{n-1}+\gamma_n}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_{n-1}^{\alpha_{n-1}}\partial x_n^{\gamma_n}}$ is continuous on $[0,\infty)^n$,*

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}\left\|u^{(\alpha_1,\ldots,\alpha_{n-1},\gamma_n)}\big(x_1,\ldots,x_n\big)\right\|dx_1\,dx_2\ldots dx_n<+\infty$$

*and, for every multi-index $(\alpha_1,\ldots,\alpha_{n-1},\beta_n)$, where $0\le\beta_n<\alpha_n$, we have*

$$\lim_{x_n\to+\infty} e^{-x_n}\frac{x_n^v}{v!}\frac{\partial^{\alpha_1+\cdots+\alpha_{n-1}+\beta_n}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_{n-1}^{\alpha_{n-1}}\partial x_n^{\beta_n}}\big(x_1,\ldots,x_n\big)=0,\quad x_1\ge0,\ x_2\ge0,\ldots,\ x_{n-1}\ge0;$$

*2. For every multi-index $(\alpha_1,\ldots,\gamma_{n-1})$, where $0\le\gamma_{n-1}\le\alpha_{n-1}$, the mapping $\frac{\partial^{\alpha_1+\cdots+\gamma_{n-1}}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_{n-1}^{\gamma_{n-1}}}$ is continuous on $[0,\infty)^n$,*

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}\left\|u^{(\alpha_1,\ldots,\gamma_{n-1})}\big(x_1,\ldots,x_n\big)\right\|dx_1\,dx_2\ldots dx_n<+\infty$$

*and, for every multi-index $(\alpha_1,\ldots,\beta_{n-1})$, where $0\le\beta_{n-1}<\alpha_{n-1}$, we have*

$$\lim_{x_{n-1}\to+\infty} e^{-x_{n-1}}\frac{x_{n-1}^v}{v!}\frac{\partial^{\alpha_1+\cdots+\beta_{n-1}}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_{n-1}^{\beta_{n-1}}}\big(x_1,\ldots,x_n\big)=0,$$
$$x_1\ge0,\ x_2\ge0,\ldots,\ x_{n-2}\ge0,\ x_n\ge0;$$

*...;*

*n. For every integer $\gamma_1\in[0,\alpha_1]$, the mapping $(\partial^{\gamma_1}u/\partial x_1^{\gamma_1})$ is continuous on $[0,\infty)^n$,*

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}\left\|\frac{\partial^{\gamma_1}u}{\partial x_1^{\gamma_1}}\big(x_1,\ldots,x_n\big)\right\|dx_1\,dx_2\ldots dx_n<+\infty$$

*and, for every integer $\beta_1\in[0,\alpha_1)$, we have*

$$\lim_{x_1\to+\infty} e^{-x_1}\frac{x_1^v}{v!}\frac{\partial^{\beta_1}u}{\partial x_1^{\beta_1}}\big(x_1,\ldots,x_n\big)=0,\quad x_2\ge0,\ x_3\ge0,\ldots,\ x_n\ge0.$$

Observe only that the prescribed assumptions imply, due to the assumption [0.], the closedness of the operators $A_\alpha$ and [14, Theorem 1.2.3], that

$$\sum_{\alpha\in D} A_\alpha \int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+\alpha_1}}{(v_1+\alpha_1)!}\cdot\ldots\cdot\frac{x_n^{v_n+\alpha_n}}{(v_n+\alpha_n)!}u^{(\alpha)}\bigl(x_1,...,x_n\bigr)\,dx_1\,dx_2\ldots dx_n$$
$$=\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+\alpha_1}}{(v_1+\alpha_1)!}\cdot\ldots\cdot\frac{x_n^{v_n+\alpha_n}}{(v_n+\alpha_n)!}g\bigl(x_1,...,x_n\bigr)\,dx_1\,dx_2\ldots dx_n$$

for any $(x_1,...,x_n)\in[0,\infty)^n$, so that the required conclusion follows similarly as in Example 2.2, by means of the assumptions [1.]-[n.].

*Remark* 2.2. It is worth noting that we consider the equation (2.5) without initial conditions. If, for every multi-index $\alpha=(\alpha_1,...,\alpha_n)\in D$, we impose the initial values

$$u_n\bigl(x_1,...x_{n-1}\bigr)=\frac{\partial^{\alpha_1+\ldots+\alpha_{n-1}+\beta_n}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_{n-1}^{\alpha_{n-1}}\partial x_n^{\beta_n}}\bigl(x_1,...,x_{n-1},0\bigr),\ x_1\ge 0,\ x_2\ge 0,...,\ x_{n-1}\ge 0,$$

in [1.], ...,

$$u_1\bigl(x_2,...,x_n\bigr)=\frac{\partial^{\beta_1}u}{\partial x_1^{\beta_1}}\bigl(0,x_2,...,x_n\bigr),\quad x_2\ge 0,\ x_3\ge 0,...,\ x_n\ge 0,$$

in [n.], then the value of

$$\int_{[0,\infty)^n} e^{-x_1-\ldots-x_n}\frac{x_1^{v_1}}{v_1!}\cdot\ldots\cdot\frac{x_n^{v_n}}{v_n!}g\bigl(x_1,...,x_n\bigr)\,dx_1\,dx_2\ldots dx_n,\quad \bigl(v_1,...,v_n\bigr)\in\mathbb{N}_0^n,$$

can be computed in a similar manner. Details can be left to the interested readers.

We will illustrate Theorem 2.3 with two well-known examples:

**Example 2.4.** If $t=x_1$, $x=x_2$, $v_1=i$, $v_2=j$, $a_{i,j}=[Pu](i,j)$ and

$$u_t=u_{xx},\quad \text{resp.},\quad u_{tt}=u_{xx},$$

then

$$a_{i,j+2}=2a_{i+1,j+1}-a_{i+1,j},\quad\text{resp.},\quad -2a_{i+1,j+2}+a_{i,j+2}=-2a_{i+2,j+1}+a_{i+2,j},\tag{2.6}$$

for any $i,\ j\in\mathbb{N}_0$, provided that the requirements of Theorem 2.3 hold. Concerning the uniqueness of solutions of differences equations in (2.6), we will only note here that the first of these equations is uniquely solvable for $(i,j)\in\mathbb{N}_0^2$, provided that the initial values $a_{i,0}$ and $a_{0,j}$ are given for all $i,\ j\in\mathbb{N}_0$, as well as that the second of these equations is uniquely solvable for $(i,j)\in\mathbb{N}_0^2$, provided that the initial values $a_{i,0}$, $a_{i,1}$, $a_{0,j}$ and $a_{1,j}$ are given for all $i,\ j\in\mathbb{N}_0$.

**Example 2.5.** In many published research articles by now, the authors have analyzed the well-posedness and qualitative properties of solutions to the following abstract (degenerate) higher-order differential equation

$$A_n u^{(n)}(t) + A_{n-1} u^{(n-1)}(t) + ... + A_0 u(t) = f(t), \quad t \geq 0,$$

where $A_j$ are differential operators with constant coefficients on the space $X = L^p(\mathbb{R}^n)$, where $1 \leq p \leq +\infty$; cf. [14], [23] and references cited therein for more details in this direction. If we set $t = x_1$ and denote the variables in $L^p(\mathbb{R}^n)$ by $x_2, ..., x_{n+1}$, we can provide a great number of applications of Theorem 2.3 to the abstract partial differential equations with constant coefficients, by applying also certain changes of variables with respect to the variables $x_2, ..., x_{n+1}$.

In order to avoid any form of repeating and plagiarism, we will only emphasize here the following important issues about the multi-dimensional Poisson like transforms:

(i) It is worth noting that Theorem 2.3, Example 2.2 and Example 2.4 can be simply reformulated for the Poisson like transform

$$\begin{aligned}&\bigl[P_{a_1,...,a_n}(u)\bigr](\mathrm{v}) := \bigl[P_{a_1,...,a_n}(u)\bigr]\bigl(v_1, ..., v_n\bigr)\\ &:= \int_{[0,\infty)^n} e^{-a_1x_1-...-a_nx_n} \frac{x_1^{v_1}}{v_1!} \cdot ... \cdot \frac{x_n^{v_n}}{v_n!} u\bigl(x_1, ..., x_n\bigr)\, dx_1\, dx_2 \ldots dx_n,\end{aligned}$$

where $a_j > 0$ and $v_j \in \mathbb{N}_0$ for $1 \leq j \leq n$. In such a way, we can extend [16, Theorem 4] to the higher-dimensional setting. The statement of [16, Theorem 5] can be also simply transferred to the higher-dimensional setting by the use of the Weyl convolution product $(a \circ b)(\cdot)$; cf. [16] for the notion.

(ii) Following our consideration from [16], where we have considered the Poisson like transforms for not exponentially bounded functions, we can also put forward to consideration the following multi-dimensional transform:

$$\begin{aligned}\bigl[P_{\mathbf{a},\mathbf{b},\omega,\mathbf{j}}(u)\bigr](\mathrm{v}) &:= \bigl[P_{\mathbf{a},\mathbf{b},\omega,\mathbf{j}}(u)\bigr]\bigl(v_1, ..., v_n\bigr)\\ &:= \int_{[0,\infty)^n} e^{-b_1(a_1x_1)^{j_1}-...-b_n(a_nx_n)^{j_n}}\\ &\times \frac{(\omega_1x_1)^{v_1}}{v_1!} \cdot ... \cdot \frac{(\omega_nx_n)^{v_n}}{v_n!} u\bigl(x_1, ..., x_n\bigr)\, dx_1\, dx_2 \ldots dx_n,\end{aligned}$$

where $a_s \in \mathbb{R}$, $b_s \in \mathbb{R} \setminus \{0\}$, $\omega_s \in \mathbb{R} \setminus \{0\}$, $j_s \in \mathbb{N}$ and $v_s \in \mathbb{N}_0$ for $1 \leq s \leq n$. Applying the partial integration, we can simply show that for each multi-index $(\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we have

$$\Big[P_{\mathbf{a},\mathbf{b},\omega,\mathbf{j}}\Big(u^{(\alpha_1,...,\alpha_{n-1},\alpha_n)}\Big)\Big]\big(v_1+\alpha_1, ..., v_{n-1}+\alpha_{n-1}, v_n+\alpha_n\big)$$
$$= -\omega_n\Big[P_{\mathbf{a},\mathbf{b},\omega,\mathbf{j}}\Big(u^{(\alpha_1,...,\alpha_{n-1},\alpha_n-1)}\Big)\Big]\big(v_1+\alpha_1, ..., v_{n-1}+\alpha_{n-1}, v_n+\alpha_n-1\big)$$
$$+ j_n a_n^{j_n} b_n \frac{(v_n+\alpha_n+j_n-1)!}{(v_n+\alpha_n)!}\Big[P_{\mathbf{a},\mathbf{b},\omega,\mathbf{j}}\Big(u^{(\alpha_1,...,\alpha_{n-1},\alpha_n-1)}\Big)\Big]$$
$$\big(v_1+\alpha_1, ..., v_{n-1}+\alpha_{n-1}, v_n+\alpha_n+j_n-1\big), \quad (v_1, ..., v_n) \in \mathbb{N}_0^n,$$

under certain logical assumptions. Proceeding in this way, we can find a form of the abstract nonautonomous difference equations of several variables which corresponds to the abstract partial differential equation (2.5).

(iii) In [16], we have also examined the following Poisson like transform:

$$v \mapsto y_{a,b,c,j,\omega}(v) := \int_0^{+\infty} e^{-b(ct^{-1}+at)^j} \frac{(\omega t)^{v-\frac{1}{2}}}{\Gamma(v+\frac{1}{2})} u(t)\,dt, \quad v \in \mathbb{Z},$$

where $a \in \mathbb{R}$, $b,\ c,\ \omega \in \mathbb{R}\setminus\{0\}$ and $j \in \mathbb{N}$. The interested readers may try to introduce some multi-dimensional analogues of this transform as well as to reconsider Theorem 2.3 in this framework.

(iv) It is well known that a class of boundary value problems for the partial differential equations depending of variables $x_1$ and $x_2$, where $0 \le x_1 \le T$ and $x_2 \ge 0$, can be solved using the Fourier series and the method of separation of variables. The Poisson transforms can be also defined and analyzed for the functions defined on the closed rectangles; we can also prove an analogue of Theorem 2.3 in this framework.

### 2.2. Applications to the abstract fractional partial differential equations

In this subsection, we will present some applications of the multidimensional vector-valued Poisson transform to the abstract fractional partial differential equations.

We start with the following illustrative example:

**Example 2.6.** Suppose that $n \ge 2$, $u_j : [0,+\infty) \to \mathbb{C}$ is a locally integrable function $(1 \le j \le n-1)$ and $u_n : [0,+\infty) \to X$ is a locally integrable function. Set $u(x_1, ..., x_n) := u_1(x_1) \cdot ... \cdot u_n(x_n)$, $x_1 \ge 0, ..., x_n \ge 0$. Then an elementary application of the Fubini's theorem shows that

(2.7)
$$\Big[P(u)\Big](v_1,....,v_n)=\Big[P(u_1)\Big](v_1)\cdot ...\cdot\Big[P(u_n)\Big](v_n),\quad (v_1,....,v_n)\in\mathbb{N}_0^n,$$

provided that the integrals which define all terms in (2.7) converge absolutely; here, we do not make any terminological difference between the multi-dimensional Poisson transform with respect to the variables $v_1,...,v_n$ and the one-dimensional Poisson transform with respect to the variable $v_j$ $(1\leq j\leq n)$.

Suppose that $f:[0,+\infty)\rightarrow X$ is a locally integrable function, $\alpha>0$ and $m=\lceil\alpha\rceil$. Then a careful inspection of the proof of [1, Theorem 5.5] shows that the following equality holds true:

$$\Big[P\big(D_t^{\alpha}f\big)\Big](v+m)=\Delta^{\alpha}\big[P(f)\big](v),\ v\in\mathbb{N}_0,$$

provided that $h=g_{m-\alpha}*_0 f\in C^{m-1}([0,+\infty):X)$, $h^{(m-1)}(\cdot)$ is locally absolutely continuous on $[0,+\infty)$ and there exist real numbers $M>0$ and $\omega\in(0,1)$ such that $\|h^{(m)}(t)\|\leq Me^{\omega t}$ for a.e. $t\geq 0$. In [16], we have proved the following formula:

$$\begin{aligned}\Big[P\big(\mathbf{D}_C^{\alpha}f\big)\Big](v+m)&=\Delta^{\alpha}\big[P(f)\big](v)\\&+\frac{(-1)^{v+m+1}}{(v+m)!}\sum_{k=0}^{m-1}(\alpha-1-k)\cdot ...\cdot(\alpha-k-v-m)f^{(k)}(0),\ v\in\mathbb{N}_0,\end{aligned}$$

under certain reasonable assumptions. Using the last two formulae and (2.7), we can simply compute the multi-dimensional Poisson transform of the fractional partial derivatives $\mathbb{D}^{\alpha}u$ forms from the compositions of the Riemann-Liouville fractional derivatives of functions $u_j(\cdot)$ for $j\in J_1$ and the Caputo fractional derivatives of functions $u_j(\cdot)$ for $j\in J_2$, where $\mathbb{N}_n=J_1\cup J_2$. For example, if $J_2=\emptyset$, $\alpha_1\geq 0,...,\alpha_n\geq 0$, $\alpha=(\alpha_1,...,\alpha_n)$ and $m_j=\lceil\alpha_j\rceil$ for $1\leq j\leq n$, then we have:

$$\begin{aligned}&\Bigg[P\Big(D_{R,x_1}^{\alpha_1}\cdot ...\cdot D_{R,x_n}^{\alpha_n}u\Big)\Bigg]\big(v_1+m_1,...,v_n+m_n\big)\\&=\Bigg[P\Big(D_R^{\alpha_1}u_1\Big)\Bigg]\big(v_1+m_1\big)\cdot ...\cdot\Bigg[P\Big(D_R^{\alpha_n}u_n\Big)\Bigg]\big(v_n+m_n\big)\\&=\Delta^{\alpha_1}\Big[P\big(u_1\big)\Big]\big(v_1\big)\cdot ...\cdot\Delta^{\alpha_n}\Big[P\big(u_n\big)\Big]\big(v_n\big)\\&=\Big[\Delta^{\alpha}P(u)\Big]\big(v_1,...,v_n\big),\quad\big(v_1,...,v_n\big)\in\mathbb{N}_0^n,\end{aligned}$$

under the following conditions:

(i) The integrals which define the terms $P(D^{\alpha_1}_{R,x_1} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u)$, $P(D^{\alpha_1}_R u_1), ...,$ $P(D^{\alpha_n}_R u_n)$, $P(u)$, $P(u_1), ...$ and $P(u_n)$ converge absolutely;

(ii) The functions $h_1 = g_{m_1-\alpha_1} *_0 u_1(\cdot), ..., h_n = g_{m_n-\alpha_n} *_0 u_n(\cdot)$ are $(m-1)$-times continuously differentiable on $[0,+\infty)$, the functions $h_1^{(m_1-1)}(\cdot), ...,$ $h_n^{(m_n-1)}(\cdot)$ are locally absolutely continuous on $[0,+\infty)$ and there exist real numbers $M > 0$ and $\omega \in (0,1)$ such that $\|h_j^{(m_j)}(t)\| \leq Me^{\omega t}$ for a.e. $t > 0$ $(1 \leq j \leq n)$.

We continue by stating the following general result (for simplicity, we denote henceforth $D^{\alpha_1}_{R,x_1} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u = u^{(\alpha)}$):

**Theorem 2.7.** *Suppose that $u : [0,\infty)^n \to X$ is a locally integrable function, the term $u^{(\alpha)}$ is well-defined and the following conditions hold true:*

*(i) The integral which defines the term $[P(u^{(\alpha)})](v_1+m_1, ..., v_n+m_n)$ converges absolutely for all $(v_1, ...v_n) \in \mathbb{N}_0^n$.*

*(ii) The function $h_1 = J^{m_1-\alpha_1}_{t_1} D^{\alpha_2}_{R,x_2} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u\big(x_1, x_2, ..., x_n\big) \in C^{m_1-1}([0,+\infty) : X)$, $h_1^{(m_1-1)}(\cdot)$ is locally absolutely continuous on $[0,+\infty)$ and there exist real numbers $M > 0$ and $\omega \in (0,1)$ such that $\|h_1^{(m_1)}(t)\| \leq Me^{\omega t}$ for a.e. $t \geq 0$.*

*(iii) The integral*

$$\int_{[0,+\infty)^n} e^{-x_1-x_2-...-x_n} \frac{x_1^{v_1}}{v_1!} \frac{x_2^{v_2+m_2}}{(v_2+m_2)!} \cdot ... \cdot \frac{x_n^{v_n+m_n}}{(v_n+m_n)!} \times D^{\alpha_2}_{R,x_2} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u\big(x_1, x_2, ..., x_n\big)\, dx_1\, dx_2 ... dx_n$$

*converges absolutely for all $(v_1, ...v_n) \in \mathbb{N}_0^n$.*

*(iv) For every $j \in \{2, ..., n-1\}$, the integral*

$$\int_{[0,+\infty)^n} e^{-x_1-x_2-...-x_n} \frac{x_1^{v_1}}{v_1!} \cdot ... \cdot \frac{x_j^{v_j}}{v_j!} \frac{x_{j+1}^{v_{j+1}+m_{j+1}}}{(v_{j+1}+m_{j+1})!} \cdot ... \cdot \frac{x_n^{v_n+m_n}}{(v_n+m_n)!} \times D^{\alpha_j}_{R,x_j} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u\big(x_1, x_2, ..., x_n\big)\, dx_1\, dx_2 ... dx_n$$

*converges absolutely for all $(v_1, ...v_n) \in \mathbb{N}_0^n$ and the term which defines the term $[P(u)](v_1, ..., v_n)$ converges absolutely for all $(v_1, ...v_n) \in \mathbb{N}_0^n$.*

*(v) For every $j \in \{3, ..., n\}$, we have*

$$h_j = J^{m_{j-1}-\alpha_{j-1}}_{t_{j-1}} D^{\alpha_j}_{R,x_j} \cdot ... \cdot D^{\alpha_n}_{R,x_n} u\big(x_1, x_2, ..., x_n\big) \in C^{m_{j-1}-1}([0,+\infty) : X),$$

*$h_j^{(m_{j-1}-1)}(\cdot)$ is locally absolutely continuous on $[0,+\infty)$ and there exist real numbers $M>0$ and $\omega \in (0,1)$ such that $\|h_j^{(m_{j-1})}(t)\| \leq Me^{\omega t}$ for a.e. $t \geq 0$.*

*Then we have*

(2.8)
$$\Big[P\Big(u^{(\alpha)}\Big)\Big]\big(v_1+m_1,...,v_n+m_n\big)=\Big[\Delta^{\alpha}P(u)\Big]\big(v_1,...,v_n\big),\ \big(v_1,...,v_n\big)\in \mathbb{N}_0^n.$$

**Proof.** Keeping in mind condition (i), we can apply the Fubini's theorem in order to see that

$$\Big[P\Big(u^{(\alpha)}\Big)\Big]\big(v_1+m_1,...,v_n+m_n\big)=\int_{[0,+\infty)^{n-1}}e^{-x_2-...-x_n}\frac{x_2^{v_2+m_2}}{(v_2+m_2)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}$$
$$\times\left[\int_0^{+\infty}e^{-x_1}\frac{x_1^{v_1+m_1}}{(v_1+m_1)!}D_{R,x_1}^{\alpha_1}\cdot...\cdot D_{R,x_n}^{\alpha_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\right]dx_2\,...\,dx_n.$$

Due to (ii)-(iii), we can apply [1, Theorem 5.5] and the Fubini's theorem to obtain that

$$\begin{aligned}
&\Big[P\Big(u^{(\alpha)}\Big)\Big]\big(v_1+m_1,...,v_n+m_n\big)\\
&=\int_{[0,+\infty)^{n-1}}e^{-x_2-...-x_n}\frac{x_2^{v_2+m_2}}{(v_2+m_2)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}\\
&\times\left[\Delta_{x_1}^{\alpha_1}\int_0^{+\infty}e^{-x_1}\frac{x_1^{v_1}}{v_1!}D_{R,x_2}^{\alpha_2}\cdot...\cdot D_{R,x_n}^{\alpha_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\right]dx_2\,...\,dx_n\\
&=\Delta_{x_1}^{\alpha_1}\int_{[0,+\infty)^{n-1}}e^{-x_2-...-x_n}\frac{x_2^{v_2+m_2}}{(v_2+m_2)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}\\
&\times\left[\int_0^{+\infty}e^{-x_1}\frac{x_1^{v_1}}{v_1!}D_{R,x_2}^{\alpha_2}\cdot...\cdot D_{R,x_n}^{\alpha_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\right]dx_2\,...\,dx_n\\
&=\Delta_{x_1}^{\alpha_1}\int_{[0,+\infty)^{n}}e^{-x_1-x_2-...-x_n}\frac{x_1^{v_1}}{v_1!}\frac{x_2^{v_2+m_2}}{(v_2+m_2)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}\\
&\times D_{R,x_2}^{\alpha_2}\cdot...\cdot D_{R,x_n}^{\alpha_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\,dx_2\,...\,dx_n.
\end{aligned}$$

Keeping in mind the remaining assumptions and repeating the above procedure, we get:

$$\begin{aligned}
&\Big[P\Big(u^{(\alpha)}\Big)\Big]\big(v_1+m_1,...,v_n+m_n\big)\\
&=\Delta^{\alpha_1}_{x_1}\int_{[0,+\infty)^n} e^{-x_1-x_2-...-x_n}\frac{x_1^{v_1}}{v_1!}\frac{x_2^{v_2+m_2}}{(v_2+m_2)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}\\
&\times D^{\alpha_2}_{R,x_2}\cdot...\cdot D^{\alpha_n}_{R,x_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\,dx_2\,...\,dx_n\\
&=\Delta^{\alpha_1}_{x_1}\Delta^{\alpha_2}_{x_2}\int_{[0,+\infty)^n} e^{-x_1-x_2-...-x_n}\frac{x_1^{v_1}}{v_1!}\frac{x_2^{v_2}}{v_2!}\frac{x_3^{v_3+m_3}}{(v_3+m_3)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}\\
&\times D^{\alpha_2}_{R,x_2}\cdot...\cdot D^{\alpha_n}_{R,x_n}u\big(x_1,x_2,...,x_n\big)\,dx_1\,dx_2\,...\,dx_n\\
&=\Delta^{\alpha_1}_{x_1}\Delta^{\alpha_2}_{x_2}\cdot...\cdot\Delta^{\alpha_n}_{x_n}\\
&\times\int_{[0,+\infty)^n} e^{-x_1-x_2-...-x_n}\frac{x_1^{v_1}}{v_1!}\cdot...\cdot\frac{x_n^{v_n}}{v_n!}u\big(x_1,x_2,...,x_n\big)\,dx_1\,dx_2\,...\,dx_n,
\end{aligned}$$

which completes the proof. ■

Now we will state and prove the following analogue of Theorem 2.3 for the fractional partial derivatives of the Riemann-Liouville type:

**Theorem 2.8.** *Suppose that $D$ is a non-empty subset of $[0,+\infty)^n$ and $A_\alpha$ is a closed linear operator on $X$ for all $\alpha\in D$; if $\alpha=(\alpha_1,...,\alpha_n)\in D$, then we set $m_j=\lceil\alpha_j\rceil$ for $1\le j\le n$. Suppose further that* (2.5) *holds for a.e. $(x_1,...,x_n)\in[0,\infty)^n$. Then we have*

$$\begin{aligned}
&\sum_{\alpha\in D}A_\alpha\Big[\Delta^\alpha P(u)\Big]\big(v_1,...,v_n\big)\\
&=\int_{[0,\infty)^n} e^{-x_1-...-x_n}\frac{x_1^{v_1+m_1}}{(v_1+m_1)!}\cdot...\cdot\frac{x_n^{v_n+m_n}}{(v_n+m_n)!}g\big(x_1,...,x_n\big)\,dx_1\,dx_2\,...\,dx_n,
\end{aligned}$$

*for any $(v_1,...,v_n)\in\mathbb{N}_0^n$, provided that the following conditions hold true:*

*(i) Condition* [0.] *from the formulation of* Theorem 2.3 *holds;*

*(ii) For every multi-index $\alpha=(\alpha_1,...,\alpha_n)\in D$,* (2.8) *holds true.*

**Proof.** Using conditions (i)-(ii) and [14, Theorem 1.2.3], the required statement simply follows from the next computation:

$$
\begin{aligned}
&= \int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+m_1}}{(v_1+m_1)!} \cdot \ldots \cdot \frac{x_n^{v_n+m_n}}{(v_n+m_n)!} g\big(x_1, \ldots, x_n\big)\, dx_1\, dx_2 \ldots dx_n \\
&= \int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+m_1}}{(v_1+m_1)!} \cdot \ldots \cdot \frac{x_n^{v_n+m_n}}{(v_n+m_n)!} \sum_{\alpha\in D} A_\alpha u^{(\alpha)}\big(x_1, \ldots, x_n\big)\, dx_1\, dx_2 \ldots dx_n \\
&= \sum_{\alpha\in D} A_\alpha \int_{[0,\infty)^n} e^{-x_1-\ldots-x_n} \frac{x_1^{v_1+m_1}}{(v_1+m_1)!} \cdot \ldots \cdot \frac{x_n^{v_n+m_n}}{(v_n+m_n)!} u^{(\alpha)}\big(x_1, \ldots, x_n\big)\, dx_1\, dx_2 \ldots dx_n \\
&= \sum_{\alpha\in D} A_\alpha \Big[P\Big(u^{(\alpha)}\Big)\Big]\big(v_1+m_1, \ldots, v_n+m_n\big) \\
&= \sum_{\alpha\in D} A_\alpha \Big[\Delta^\alpha P(u)\Big]\big(v_1, \ldots, v_n\big),\ \big(v_1, \ldots, v_n\big) \in \mathbb{N}_0^n.
\end{aligned}
$$

■

It is worth noting that Theorem 2.8 can be successfully applied to the fractional partial equations considered in [18, Subsection 5.1, Subsection 5.2]; cf. also [22, Chapter 5].

## 3. Conclusions and final remarks

In this paper, we have examined the multi-dimensional Poisson transforms and some connections between the solutions of the abstract (fractional) differential equations and the abstract (fractional) difference equations depending on several variables. For the proofs of some structural results, we have used the basic properties of the multidimensional vector-valued Laplace transform.

We close the paper with the observation that the existence and uniqueness of almost periodic type solutions for various classes of the abstract integro-differential-difference equations depending on several variables will be analyzed in the forthcoming research monograph [15].

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