Differentiably nondegenerate Meromorphic mappings on Kähler manifolds weakly sharing hyperplanes

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Abstract. In this paper, we study the uniqueness problem for differentiably nondegenerate meromorphic mappings from a Kähler manifold into $\mathbb{P}^n(\mathbb{C})$ satisfying a condition (C_ρ) and sharing hyperplanes in general position, where the condition $f^{-1}(H) = g^{-1}(H)$ for some hyperplanes H is replaced by a weaker one that $f^{-1}(H) \subseteq g^{-1}(H)$. An improvement on the algebraic dependence problem of differentiably nondegenerate meromorphic mappings is also given. Moreover, in this case, the condition $f^{-1}(H) \subseteq g^{-1}(H)$ is even omitted for some hyperplanes.

1. Introduction

Let M be an m-dimensional complete connected Kähler manifold, whose universal covering is biholomorphic to a ball $\mathbb{B}(R_0) = \{z \in \mathbb{C}^m; \|z\| < R_0\}$ $(0 < R_0 \le +\infty)$. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping from M into $\mathbb{P}^n(\mathbb{C})$. For $\rho \ge 0$, we say that f satisfies the condition (C_ρ) if there exists a nonzero bounded continuous real-valued function h on M such that

$$\rho\Omega_f + dd^c \log h^2 \ge \mathrm{Ric}\,\omega,$$

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where Ω_f is the full-back of the Fubini-Study form Ω on $\mathbb{P}^n(\mathbb{C})$,
$\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is the Kähler form on M, $\mathrm{Ric}\,\omega = dd^c \log\big(\det\big(h_{i\bar{j}}\big)\big)$, $d = \partial + \bar{\partial}$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. The mapping f is said to be differentiably nondegenerate if there is a point $z_0 \in M$ such that the differential mapping $df(z_0)$ is surjective.

In 1981, S. Drouilhet [1] proved that if two differentiably nondegenerate meromorphic mappings f and g from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ have the same inverse image for a hypersurface of degree at least $n+4$ with only normal crossing intersection and $f = g$ on the inverse image of this hypersurface, then $f \equiv g$. Afterward, in 1986, H. Fujimoto [4] proved the following uniqueness theorem for meromorphic mappings on a complete Kähler manifold, which satisfy the (C_ρ) condition, sharing q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position with $q > n + 3 + 2n\rho$. The result of H. Fujimoto implies the result of S. Drouilhet for the case where the hypersurface is the sum of at least $n + 4$ hyperplanes in general position (as divisors). In 2022, K. Zhou and L. Jin [11] considered the case of meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ where the condition $f^{-1}(H_i) = g^{-1}(H_i)$ is replaced by a weaker one that $f^{-1}(H_i) \subseteq g^{-1}(H_i)$ for some hyperplanes H_i. They proved the following.

Theorem A (see [11, Theorem 1.1]). *Let $f, g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be meromorphic mappings. Let $H_1, \ldots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with $f(\mathbb{C}^m) \not\subseteq H_j, g(\mathbb{C}^m) \not\subseteq H_j$ for $1 \le j \le q$ and $\dim f^{-1}(H_i \cap H_j) \le m - 2$ for $1 \le i < j \le q$. Suppose that:*

(a) $f^{-1}(H_j) = g^{-1}(H_j)$ for $1 \le j \le p$, and $f^{-1}(H_j) \subseteq g^{-1}(H_j)$ for $p < j \le q$,

(b) $f \equiv g$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

Then $f = g$ if any one of the following conditions is satisfied:

(i) f or g is nonconstant and $p = 2n + 2, q > 3n + 3 - 2\sqrt{n}$.

(ii) f or g is linearly nondegenerate and $p = 2n + 2, q \ge 2n + 3$.

(iii) f or g is nonconstant and $p = 2n + 1, q \ge 4n + 3$.

(iv) Both f and g are linearly nondegenerate and $p = n+2, q \ge n^3+n^2+n+4$.

Recently, in [8] the first author extended Theorem A to the case of meromorphic mappings on Kähler manifold. He proved the following theorem.

Theorem B (See [8]). *Let M be a complete, connected Kähler manifold whose universal covering is biholomorphic to $\mathbb{B}(R_0) \subseteq \mathbb{C}^m$ $(0 < R_0 \le +\infty)$. Let $f, g : M \to \mathbb{P}^n(\mathbb{C})$ be linearly nondegenerate meromorphic mappings satisfying a condition (C_ρ). Let $H_1, \ldots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with $\dim f^{-1}(H_i \cap H_j) \le m - 2$ for every $1 \le i < j \le q$, such that* *(i)* $f^{-1}(H_i) = g^{-1}(H_i)\ \forall\ 1 \le i \le p,\ f^{-1}(H_i) \subseteq g^{-1}(H_i)\ \forall\ p+1 \le i \le q,$

(ii) $f = g$ *on* $\bigcup_{i=1}^{q} f^{-1}(H_i),$

where $n+2 \le p \le 2n+2$. *Then* $f \equiv g$ *if*

$$q > 2n + 2 + pn\left(\frac{n+1}{p-n-1} - 1\right) + 2\rho\left(\ell_f + \frac{n+1}{p-n-1}\ell_g\right)$$

$$\text{or}\quad q > 2n + 1 + pn\left(\frac{n}{p-n-1} - \frac{n-1}{n}\right) + 2\rho\left(\ell_f + \frac{n}{p-n-1}\ell_g\right).$$

Here, ℓ_f and ℓ_g are positive integers depending on f and g, respectiely, and in generally bounded above by $\frac{n(n+1)}{2}$. For the case where f and g are differentiably nondegenerate we have $\ell_f = \ell_g = n$ and the assumption of Theorem B is fullfiled with $n+2 \le p \le 2n+2$ and $q > 2n+2+n^2(n+2)+2\frac{pn\rho}{p-n-1}$. This estimate of the number q is actually very far from the sharp. Our first aim in this paper is to study the above problem for the case of differentiably nondegenerate meromorphic mappings. We will give an optimal estimate for the number q of hyperplanes involving the theorem. Namely, we will prove the following result.

Theorem 1.1. *Let M be a complete, connected Kähler manifold whose universal covering is biholomorphic to $\mathbb{B}(R_0) \subseteq \mathbb{C}^m$ $(0 < R_0 \le +\infty)$. Let $f, g : M \to \mathbb{P}^n(\mathbb{C})$ be differentiably nondegenerate meromorphic mappings, which satisfy the condition (C_ρ). Let $H_1, \dots, H_q$ be q hyperplane of $\mathbb{P}^n(\mathbb{C})$ in general position and let $n+2 \le p \le n+3 < q$. Assume that:*

(1) $f^{-1}(H_i) = g^{-1}(H_i)$ *for* $1 \le i \le p$, $f^{-1}(H_i) \subseteq g^{-1}(H_i)$ *for* $p+1 \le i \le q$,

(2) $f = g$ *on* $\bigcup_{i=1}^{q} f^{-1}(H_i)$.

Then $f \equiv g$ if any one of the following condition satisfied:

(a) $p = n+2$ *and* $q > 2n+5+4n\rho$.

(b) $p = n+3$ *and* $q > n+3+2n\rho$.

Then, our result implies the above mentioned result of S. Drouilhet and H. Fujimoto for the case of hyperplane targets. In order to prove Theorem 1.1, we will prove a key lemma (see Lemma 3.2 in Section 3), by which we can estimate the growth of the auxiliary functions. With the useful of Lemma 3.2, we may prove the following algebraic dependence theorem of differentiably nondegenerate meromorphic mappings on Kähler manifolds, where the condition "$f^{-1}(H_i) \subseteq g^{-1}(H_i)\ \forall\ p+1 \le i \le q$" as above is omitted.

Theorem 1.2. *Let M be as in Theorem 1.1. Let $f^1,\dots,f^k : M \to \mathbb{P}^n(\mathbb{C})$ be k differentiably nondegenerate meromorphic mappings, which satisfy the condition(C_ρ). Let ℓ, p and q be positive integers with $n+2 \le p \le q$ and $2 \le \ell \le k$. Let $H_1,\dots,H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that*

(1) $(f^u)^{-1}(H_i) = (f^1)^{-1}(H_i)$ for every $1 \le i \le p$ and $2 \le u \le k$,

(2) $f^{i_1}\wedge\cdots\wedge f^{i_\ell} = 0$ on $\bigcup_{1\le i\le q}(f^1)^{-1}(H_i)$ for every $1 \le i_1 < \cdots < i_\ell \le k$.

Then $f^1\wedge\cdots\wedge f^k \equiv 0$ if $q > n+1+\frac{1}{k-\ell+1}\left(1+\frac{p(k-1)}{p-n-1}\right)+2n\rho\left(1+\frac{(k-1)}{(k-\ell+1)(p-n-1)}\right)$.

In this paper, we only study the uniqueness problem of meromorphic mappings. But our method can be applied to study the finiteness problem of differentiably nondegenerate meromorphic mappings under this *weakly sharing hyperplanes condition*. We refer the readers to the works [6, 7] for this subject. However, the computation in that case certainy very complicate, because there are many more parameters involved. Then, that problem is still an interesting open question.

2. Auxiliary results

Let ν be a divisor on $\mathbb{B}(R_0)$, which is usually regarded as a function from $\mathbb{B}(R_0)$ into $\mathbb{Z}$. The support $\operatorname{Supp}\nu$ is defined as the closure of the set $\{z|\nu(z)\ne 0\}$. For a positive integer k (may be $+\infty$), we define $\nu^{[k]}(z)=\min\{\nu(z),k\}$ and

$$n^{[k]}(t,\nu) := \begin{cases} \int_{\operatorname{Supp}\nu\cap B(t)} \nu^{[k]} v_{m-1} & \text{if } m\ge 2, \text{ where } v_{m-1}(z) = \left(dd^c\|z\|^2\right)^{m-1}, \\ \sum_{|z|\le t} \nu^{[k]}(z) & \text{if } m=1. \end{cases}.$$

The truncated counting function to level k of ν is defined by

$$N^{[k]}(r,r_0;\nu) := \int_{r_0}^{r} \frac{n^{[k]}(t,\nu)}{t^{2m-1}}dt \quad (r_0<r<R_0).$$

We omit the character $^{[k]}$ if $k=+\infty$.

Let φ be a non-zero meromorphic function on $\mathbb{B}(R_0)$. We denote by ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ and set $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$. For convenience, we will write $N_\varphi(r,r_0)$ and $N^{[k]}_\varphi(r,r_0)$ for $N\left(r,r_0;\nu^0_\varphi\right)$ and $N^{[k]}\left(r,r_0;\nu^0_\varphi\right)$, respectively.

Let $f:\mathbb{B}(R_0)\longrightarrow\mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Fix a homogeneous coordinates system $(w_0:\cdots:w_n)$ on $\mathbb{P}^n(\mathbb{C})$. We take a reduced representation

$f = (f_0 : \cdots : f_n)$ and set $\|f\| = \Big(|f_0|^2 + \cdots + |f_n|^2\Big)^{1/2}$. The characteristic function of f is defined by

$$T_f(r, r_0) = \int_{\|z\|=r} \log \|f\| \sigma_m - \int_{\|z\|=r_0} \log \|f\| \sigma_m, \quad 0 < r_0 < r < R_0,$$

where $\sigma_m(z) = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1}$. Here and throughout this paper, we assume that the numbers r_0 and R_0 are fixed with $0 < r_0 < R_0$.

If f is differentiably nondegenerate then there exists $\alpha = (\alpha_0, \ldots, \alpha_n) \in (N^m)^{n+1}$ with $\alpha_0 = (0, \ldots, 0)$ and $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})$, $|\alpha_i| = \sum_{j=1}^m \alpha_{ij} = 1$ such that

$$W_\alpha(f_0, \ldots, f_n) = \det \left(\mathcal{D}^{\alpha_j}(f_i)\right)_{0 \le i, j \le n} \not\equiv 0.$$

The tuple $(\alpha_0, \ldots, \alpha_n)$ is call an admissible set of f, and the function $W_\alpha(f_0, \ldots, f_n)$ is called a genelized Wronskian of f.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$, we (throughout this paper) also denote by the same letter H a linear form defining H, i.e., we may write

$$H(x_0, \ldots, x_n) = \sum_{j=0}^n a_{1j} x_j.$$

We set

$$H(f) := a_0 f_0 + \cdots + a_n f_n.$$

Then, the function $H(f)$ depends on the choice of the local reduced representation of f. However, its zero divisor $\nu_{H(f)}$ does not depend on this choice and hence it is well-defined.

Proposition 2.1 (see [4, Proposition 2.12]). *Let $H_1, \ldots, H_q$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let f be a differentiably nondegenerate meromorphic mapping from the ball $\mathbb{B}^m(R_0) \subseteq \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0, \ldots, f_n)$ and let $(\alpha_0, \ldots, \alpha_n)$ be an admissible set of f. Then, for $0 < r_0 < R_0$ and $0 < tn < p < 1$, there exists a positive constant K such that for $r_0 < r < R < R_0$,*

$$\int_{\|z\|=r} \left| z^{\alpha_0 + \cdots + \alpha_n} \frac{W_{\alpha_0, \ldots, \alpha_n}(f_0, \ldots, f_n)}{H_1(f) \ldots H_q(f)} \right|^t \cdot \|f\|^{t(q-n-1)} \sigma_m \le K \left(\frac{R^{2m-1}}{R - r} T_f(R, r_0) \right)^p.$$

Lemma 2.1 (see [9, Theorem 2.1, p.320]). *Let A be a pure $(m-1)$-dimensional analytic subset of $\mathbb{B}^m(R_0)$ $(0 < R_0 \le +\infty)$. Let ℓ and k be integers with $1 \le \ell \le k \le n+1$. Let $f_j : \mathbb{B}^m(R_0) \to \mathbb{P}^n(\mathbb{C}), 1 \le j \le k$, be meromorphic mappings. Assume that $f_{i_1} \wedge \cdots \wedge f_{i_\ell} = 0$ on A for every $1 \le i_1 < \cdots < i_\ell \le k$. Then we have $\nu_{f_1 \wedge \cdots \wedge f_k}(z) \ge k - \ell + 1$ for all $z \in A$.*

3. Proof of Main Results

We firstly prove the following lemmas.

Lemma 3.1. *Let f be a differentiably nondegenerate meromorphic mapping of a ball $\mathbb{B}^m(R_0)$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(m \ge n)$ with a reduced representation $(f_0 : \cdots : f_n)$. Let $H_0, \ldots, H_q$ be q $(\ge n)$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general possition. Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in (\mathbb{N}^m)^{n+1}$ with $|\alpha_0| = 0, |\alpha_i| = 1$ $(1 \le i \le n)$ such that $W := \det(\mathcal{D}^{\alpha_i} f_j; 0 \le i, j \le n) \not\equiv 0$. Then we have*

$$\sum_{i=0}^{q} \nu_{H_i(f)} - \nu_W \le \nu^{[1]}_{\prod_{i=0}^q H_i(f)}.$$

Proof. Consider a fixed point a, which is a regular point of the analytic set $\mathrm{Supp}\nu_{\prod_{i=0}^q H_i(f)}$ and is not in the indeterminacy locus of f. Since $\{H_i\}_{i=1}^q$ is in general position, a is zero of at most n functions $H_i(f)$. We may suppose that

$$\nu_{H_0(f)}(a) \ge \cdots \ge \nu_{H_\ell(f)}(a) \ge 0 = \nu_{H_{\ell+1}(f)}(z_0) = \cdots = \nu_{H_q(f)}(a) \ (\ell \le n-1).$$

We note that $W = C\det(\mathcal{D}^{\alpha_i} H_j(f))_{0 \le i,j \le n}$ with a nonzero constant C. Also, we may assume that

$$\alpha_1 = (1, 0, 0, \ldots, 0), \alpha_2 = (0, 1, 0, \ldots, 0), \ldots, \alpha_n = (0, 0, \ldots, 0, \overset{n-th}{1}, 0 \ldots, 0).$$

Take a local affine coordinates (U, x) around a, where U is a neighborhood of a in $\mathbb{B}^m(R_0)$, $x = (x_1, \ldots, x_m), x(a) = (0, \ldots, 0)$ such that $\mathrm{Supp}\nu_{\prod_{i=0}^q H_i(f)} \cap U = \{x_1 = 0\} \cap U$.

By reducing U if necessary, we may suppose that $\mathrm{Supp}\nu_{\prod_{i=0}^q H_i(f)} \cap U = \{H_i(f) = 0\} \cap U$ $(0 \le i \le \ell)$ and $H_j(f)$ $(\ell + 1 \le j \le q)$ does not vanishes on U. Then, $H_i(f) = x_1^{t_i} g_j$ $(0 \le i \le \ell)$ with some holomorphic function g_j. Hence,

$$\mathcal{D}^{\alpha_i}\left(\frac{H_j(f)}{H_n(f)}\right) = \frac{\partial}{\partial z_i}\left(\frac{H_j(f)}{H_n(f)}\right) = \sum_{s=1}^{m} \frac{\partial x_s}{\partial z_i} \cdot \frac{\partial}{\partial x_s}\left(\frac{H_j(f)}{H_n(f)}\right) \ (0 \le j \le n-1)$$

and

$$\nu_{\frac{\partial}{\partial x_s}\left(\frac{H_j(f)}{H_n(f)}\right)}(a) \ge \begin{cases} t_j - 1 & \text{if } s = 1 \\ t_j & \text{if } s > 1, \end{cases} \ \forall 1 \le j \le \ell.$$

On the other hand, we have

$$W = CH_n(f)^{n+1} \det\left(\frac{\partial}{\partial z_i}\left(\frac{H_j(f)}{H_n(f)}\right)\right)_{i \le i \le n, 0 \le j \le n-1}.$$

This implies that

$$\begin{aligned}\nu_W(a) &\geq \min\left\{\nu_{\det\left(\frac{\partial}{\partial x_{i_s}}\left(\frac{H_j(f)}{H_n(f)}\right);0\leq j,s\leq n-1\right)}(a); 1\leq i_0<\cdots<i_{n-1}\leq m\right\}\\ &\geq \min\sum_{j=0}^{n-1}\nu_{\frac{\partial}{\partial x_{i_j}}\left(\frac{H_j(f)}{H_n(f)}\right)}(a)\\ &\geq t_1+\cdots+t_\ell-1=\sum_{i=0}^{q}\nu_{H_i(f)}(a)-\nu^{[1]}_{\prod_{i=0}^{q}H_i(f)}(a).\end{aligned}$$

Therefore, we have

$$\sum_{i=0}^{q}\nu_{H_i(f)}(a)-\nu^{[1]}_{\prod_{i=0}^{n}H_i(f)}(a)\leq \nu_W(a).$$

The lemma is proved.

Lemma 3.2. *Let $M=\mathbb{B}^m(R_0)$ $(0<R_0\leq+\infty)$ be a complete connected Kähler manifold. Let k be a positive integer and for each $u\in\{1,\dots,k\}$, let f^u be a differentiably nondegenerate meromorphic mapping from M into $\mathbb{P}^n(\mathbb{C})$, which satisfies the condition (C_ρ) and has a reduced representation $f^u=(f_0^u:\cdots:f_n^u)$. Let $\{H_1^u,\dots,H_{q_u}^u\}$ $(1\leq u\leq k)$ be k families of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, where $q_1,\dots,q_k$ are positive integers. Assume that there exists a non zero holomorphic function h on $\mathbb{B}(\mathbb{R}_0)$ such that:*

(a) $|h|\leq C\|f^1\|^{p_1}\cdots\|f^k\|^{p_k}$, where C is a positive constants,

(b) $\nu_h\geq\sum_{u=1}^k\lambda_u\nu^{[1]}_{\prod_{i=1}^{q_u}H_i^u(f^u)}$, where $\lambda_u(1\leq u\leq k)$ are positive constants.

Then there is an index u such that $\lambda_u(q_u-n-1)-p_u\leq 0$, or

$$\sum_{u=1}^{k}(\lambda_u(q_u-n-1)-p_u)\leq 2n\rho\sum_{u=1}^{k}\lambda_u.$$

Proof. Suppose contrarily that $\lambda_u(q_u-n-1)-p_u>0$, for all $u=1,\dots,k$ and

$$\sum_{u=1}^{k}(\lambda_u(q_u-n-1)-p_u)>2n\rho\sum_{u=1}^{k}\lambda_u.$$

Case 1: $R_0 = +\infty$. By the second main theorem in Nevanlinna theory we have

$$\begin{aligned}\sum_{u=1}^{k}\lambda_u(q_u-n-1)T_{f^u}(r,1) &\le \sum_{u=1}^{k}\lambda_u N^{[1]}_{\prod_{i=1}^{q_u}H_i^u(f^u)}(r,1)+o(\sum_{u=1}^{k}T_{f^u}(r,1))\\ &\le N_h(r,1)+o(\sum_{u=1}^{k}T_{f^u}(r,1))\\ &=\sum_{u=1}^{k}p_uT_{f^u}(r,1)+o(\sum_{u=1}^{k}T_{f^u}(r,1)),\end{aligned}$$

for all $r\in(1,+\infty)$ outside a Lebesgue set of finite measure. This is a contradiction.

Case 2: $R_0 < +\infty$. We suppose that $R_0 = 1$. In this case $\rho > 0$. For each u $(1 \le u \le k)$, choose $(\alpha_0^u, \ldots, \alpha_n^u) \in (\mathbb{N}^m)^{n+1}$ with $|\alpha_0^u| = 0, |\alpha_i^u| = 1$ $(1 \le u \le n)$ such that

$$W(f^u) := \det\left(\mathcal{D}^{\alpha_i^u}(f_j^u); 0 \le i,j \le n\right) \not\equiv 0.$$

By Lemma 3.1, we have

$$\nu_h \ge \sum_{u=1}^{k}\lambda_u\nu^{[1]}_{\prod_{i=1}^{q_u}H_i^u(f^u)} \ge \sum_{u=1}^{k}\lambda_u\left(\sum_{i=1}^{q_u}\nu_{H_i^u(f^u)}-\nu_{W(f^u)}\right).$$

We put $w_u(z) := z^{\alpha_0^u+\cdots+\alpha_n^u}\dfrac{W(f^u)}{\prod_{i=1}^{q_u}H_i^u(f^u)}$ for every $1 \le u \le k$, $t := \frac{2\rho}{\sum_{u=1}^{k}(\lambda_u(q_u-n-1)-p_u)} > 0$ and $\phi := |w_1|^{\lambda_1}\cdots|w_k|^{\lambda_k}\cdot|h|$. Then $a = t\log\phi$ is a plurisubharmonic function on $\mathbb{B}^m(1)$ and $(\sum_{u=1}^{k}\lambda_u)nt < 1$. Therefore, we may choose a positive number p such that $0 < (\sum_{u=1}^{k}\lambda_u)nt < p < 1$.

Since f^u satisfies the condition (C_ρ), there is a continuous plurisubharmonic function φ_u on $\mathbb{B}^m(1)$ such that

$$e^{\varphi_u}dV \le \|f^u\|^{2\rho}v_m.$$

Then the function $\varphi = \lambda'_1\varphi_1 + \cdots + \lambda'_k\varphi_k + a$ is a plurisubharmonic on $\mathbb{B}^m(R_0)$, where $\lambda'_u = \frac{(\lambda_u(q_u-n-1)-p_u)t}{2\rho}$. One has $\sum_{u=1}^{k}\lambda'_u = 1$ and then

$$\begin{aligned}e^{\varphi}dV &= e^{\lambda'_1\varphi_1+\cdots+\lambda'_k\varphi_k+t\log\phi}dV\\ &\le C'\cdot e^{t\log\phi}\cdot\prod_{u=1}^{k}\|f^u\|^{2\lambda'_u\rho}v_m = C'\cdot|\phi|^t\cdot\prod_{u=1}^{k}\|f^u\|^{2\lambda'_u\rho}v_m\\ &= C''\cdot\prod_{u=1}^{k}(|w_u|^{\lambda_u t}\|f^u\|^{2\lambda'_u\rho+p_ut})v_m = C''\cdot\prod_{u=1}^{k}(|w_u|\cdot\|f^u\|^{(q_u-n-1)})^{t\lambda_u}v_m,\end{aligned}$$

where C', C'' are positive constants. Put $x_u = \frac{\sum_{i=1}^k \lambda_i}{\lambda_u}$. Then $\sum_{u=1}^k \frac{1}{x_u} = 1$. Integrating both sides of the above inequality over $\mathbb{B}^m(1)$ and applying Hölder inequality, we get

(3.1)

$$\begin{aligned}\int_{\mathbb{B}^m(1)} e^{\varphi} dV &\le C'' \prod_{u=1}^{k} \left(\int_{\mathbb{B}^m(1)} (|w_u| \cdot \|f^u\|^{(q_u - n - 1)})^{\lambda_u t x_u} v_m \right)^{1/x_u} \\ &= C'' \prod_{u=1}^{k} \left(2m \int_0^1 r^{2m-1} \left(\int_{\|z\|=r} (|w_u| \cdot \|f^u\|^{(q_u-n-1)})^{\lambda_u t x_u} \sigma_m \right) dr \right)^{1/x_u}.\end{aligned}$$

Subcase 2.a: We suppose that

$$\limsup_{r \to 1} \frac{\sum_{u=1}^k T_{f^u}(r, r_0)}{\log 1/(1-r)} < \infty.$$

We see that $\lambda_u t x_u n = (\sum_{i=1}^k \lambda_i) n t < p$. By Proposition 2.1, there exists a positive constant K such that, for every $0 < r_0 < r < R < 1$, we have

$$\int_{\|z\|=r} \left(|w_u| \cdot \|f^u\|^{(q_u-n-1)}\right)^{\lambda_u x_u t} \sigma_m \le K \left(\frac{R^{2m-1}}{R-r} T_{f^u}(R, r_0) \right)^p \ (1 \le u \le k).$$

Choosing $R = r + \dfrac{1-r}{e \max_{1 \le u \le k} T_{f^u}(r, r_0)}$, we have $T_{f^u}(R, r_0) \le 2T_{f^u}(r, r_0)$, for all r outside a subset E of $(0, 1]$ with $\int_E \frac{1}{1-r} dr < +\infty$. Then, the above inequality implies that

$$\int_{\|z\|=r} \left(|w_u| \cdot \|f^u\|^{(q_u-n-1)}\right)^{\lambda_u t x_u} \sigma_m \le \frac{K'}{(1-r)^p} \left(\log \frac{1}{1-r} \right)^p \ (1 \le u \le k),$$

for all r outside E, and for some positive constant K'. The inequality (3.1) yields that

$$\int_{\mathbb{B}^m(1)} e^{\varphi} dV \le C'' 2m \int_0^1 r^{2m-1} \frac{K'}{(1-r)^p} \left(\log \frac{1}{1-r} \right)^p dr < +\infty.$$

This contradicts the results of S.T. Yau [10] and L. Karp [5].

Subcase 2.b: We suppose that

$$\limsup_{r \to 1} \frac{\sum_{u=1}^k T_{f^u}(r, r_0)}{\log 1/(1-r)} = \infty.$$

By [3, Proposition 6.2], we have

$$\sum_{u=1}^{k} p_u T_{f^u}(r,r_0) \geq N_h(r,r_0) + S(r) \geq \sum_{u=1}^{k} \lambda_p N^{[1]}_{\prod_{i=1}^{q} H_i^u(f^u)}(r,r_0) + S(r)$$
$$\geq \sum_{u=1}^{k} \lambda_u (q_u - n - 1) T_{f^u}(r,r_0) + O\Big(\log^+ \frac{1}{1-r} + \log^+ \sum_{u=1}^{k} T_{f^u}(r_0,r)\Big),$$

for every r excluding a set E with $\int_E \frac{dr}{1-r} < +\infty$. This is a contradiction.

Hence, the supposition is false. The lemma is proved.

Proof. [Proof of Theorem 1.1] Since the universal covering of M is biholomorphic to $\mathbb{B}(R_0), 0 < R_0 \leq +\infty$, by using the universal covering if necessary, without loss of generality we assume that $M = B(R_0) \subseteq \mathbb{C}^m$. Let $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$ be reduced representations of f and g, respectively. Suppose contrarily that $f \not\equiv g$. Then there exists

$$P := f_i g_j - f_j g_i \not\equiv 0.$$

(a) Choose $\lambda > 1$ is an arbitrary rational number. Since $f = g$ on $\bigcup_{i=1}^{q} f^{-1}(H_i)$, we have

$$\nu_P \geq \nu^{[1]}_{\prod_{i=1}^{q} H_j(f)} = (\nu^{[1]}_{\prod_{i=1}^{q} H_j(f)} - \lambda \nu^{[1]}_{\prod_{i=1}^{p} H_j(f)}) + \lambda \nu^{[1]}_{\prod_{i=1}^{n+3} H_j(g)}.$$

Take a positive integer k so that $k\lambda$ is an integer and consider the holomorphic function $\tilde{P} = P^k \cdot \prod_{j=1}^{p} H_j(f)^{k\lambda}$. It is clear that

$$\nu_{\tilde{P}} \geq k \nu^{[1]}_{\prod_{j=1}^{q} H_j(f)} + k\lambda \nu^{[1]}_{\prod_{j=1}^{n+3} H_j(g)}$$

and $|\tilde{P}| \leq C\|f\|^{(1+(n+2)\lambda)k}\|g\|^k$ for a positive constant C. Applying Lemma 3.2 to the function $\tilde{P}$, we have one of the following:

- $(q - n - 1) \leq 1 + (n+2)\lambda$,
- $(q - n - 1) - (1 + (n+2)\lambda) + \lambda(n + 3 - n - 1) - 1 \leq 2n\rho(1+\lambda)$.

Let $\lambda \to 1$, we get $q \leq 2n + 5 + 4n\rho$. This is a contradiction. Therefore, we must have $f \equiv g$ in this case.

(b) Choose $\beta \in (\frac{1}{2}, 1)$ is an arbitrary rational number. Similarly as above, we have

$$\nu_P \geq \nu^{[1]}_{\prod_{i=1}^{q} H_j(f)} \geq (1-\beta)\nu^{[1]}_{\prod_{i=1}^{q} H_j(f)} + \beta \nu^{[1]}_{\prod_{i=1}^{n+2} H_j(g)}.$$

Take a positive integer ℓ so that $\ell\beta$ is an integer and consider the holomorphic function P^ℓ. One has

$$\nu_{P^\ell} \geq \ell(1-\beta)\nu^{[1]}_{\prod_{j=1}^{q} H_j(f)} + \ell\beta \nu^{[1]}_{\prod_{j=1}^{n+2} H_j(g)}$$

and $|P^\ell| \le C'\|f\|^\ell\|g\|^\ell$ for a positive constant C'. By Lemma 3.2, one of the following must hold:

- $(1-\beta)(q-n-1) \le 1$,
- $(1-\beta)(q-n-1) - 1 + (2\beta - 1) \le 2\rho(n(1-\beta) + n\beta)$.

Let $\beta \to 1/2$, we get $q \le n + 3 + 2n\rho$. This is a contradiction. Therefore, we must have $f \equiv g$ in this case.

The theorem is proved.

Proof. [Proof of Theorem 1.2] Let $f^u = (f_0^u : \cdots : f_n^u)$ be a reduced representation of f^u for each $1 \le u \le k$. Suppose contrarily that $f^1 \wedge f^2 \wedge \cdots \wedge f^k \not\equiv 0$. Then there exists $0 \le i_1 < \cdots < i_k \le n$ such that

$$P = \det(f_{i_j}^u)_{1 \le u, j \le k} \not\equiv 0.$$

For a regular point z of the analytic set $\bigcup_{i=1}^q (f^1)^{-1}(H_i)$ and not in the indeterminacy locus of f^u $(1 \le u \le k)$, we have$f^{i_1}(z) \wedge \cdots \wedge f^{i_\ell}(z) = 0$ for all $1 \le i_1 < \cdots < i_\ell \le k$. Then by Lemma 2.1, z is a zero of P with multiplicity at least $k - \ell + 1$. Hence, we have

$$\begin{aligned}\nu_P &\ge (k-\ell+1)\nu^{[1]}_{\prod_{i=1}^q H_j(f^1)} \\ &= (k-\ell+1)\nu^{[1]}_{\prod_{i=1}^q H_j(f^1)} - (k-1)\lambda\nu^{[1]}_{\prod_{i=1}^p H_j(f^1)} + \lambda\sum_{u=2}^k \nu^{[1]}_{\prod_{i=1}^p H_j(f^1)},\end{aligned}$$

for every positive rational number $\lambda > \frac{1}{p-n-1}$. Let K be a positive integer such that $K\lambda \in \mathbb{Z}$. We consider the holomorphic function $G = P^K \prod_{i=1}^p H_j(f^1)^{K(k-1)}$. It is clear that

$$\nu_G \ge K(k-\ell+1)\nu^{[1]}_{\prod_{i=1}^q H_j(f^1)} + K\lambda\sum_{u=2}^k \nu^{[1]}_{\prod_{i=1}^p H_j(f^1)}$$

and $|G| \le C\|f^1\|^{K+pK(k-1)\lambda}\|f^2\|^K \cdots \|f^k\|^K$ for a positive constant C. By Lemma 3.2, one of the following must hold:

- $(k-\ell+1)(q-n-1) \le 1 + p(k-1)\lambda$,
- $(k-\ell+1)(q-n-1) - 1 - p(k-1)\lambda + (k-1)(\lambda(p-n-1)-1) \le 2n\rho(k-\ell+1+(k-1)\lambda)$.

Let $\lambda \to 1/(p-n-1)$, we get $q \le n+1+\frac{1}{k-\ell+1}\left(1 + \frac{p(k-1)}{p-n-1}\right) + 2n\rho\left(1 + \frac{(k-1)}{(k-\ell+1)(p-n-1)}\right)$. This is a contradiction.

Therefore, we must have $f^1 \wedge f^2 \wedge \cdots \wedge f^k \equiv 0$. The theorem is proved.

Data availibility

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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