Uniqueness of L-functions sharing finite sets with meromorphic functions having Deficient Poles

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Abstract. In this paper we investigate the uniqueness of L-functions sharing finite sets with meromorphic functions having deficient poles. As a consequence, we have exhibited an extended version of a recent result of A. Banerjee and A. Kundu [1]. The results obtained in this paper improve and extend a recent result due to Khoai-An-Phuong [9] and a result in [17].

1. Introduction. Main results

L-functions in the Selberg class, with the Riemann zeta function as a prototype, are important objects in number theory. In this paper, an L-function always means a non-constant L-function in the Selberg class $\mathcal{S}$, with the normalized condition $a(1) = 1$, which is defined to be a Dirichlet series

$$L(s) = \sum_{i=0}^{\infty} \frac{a(n)}{n^s}$$

satisfying the following hypotheses:

(i) Ramanujan hypothesis: for all positive ϵ, $a(n) \ll n^{\epsilon}$;

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(ii) Analytic continuation: there exists a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;

(iii) Functional equation: there are positive real numbers Q, λ_i, and there exists a positive integer K, and there are complex numbers μ_i,ω with $\Re\mu_i \geq 0$ and $|\omega| = 1$ such that $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) := L(s)Q^s \prod_{i=1}^{K} \Gamma(\lambda_i s + \mu_i)$;

(iv) Euler product hypothesis: $L(s)$ satisfies $L(s) = \prod_p L_p(s)$, where $L_p(s) = \exp\big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\big)$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

On the other hand, an L-function can be analytically continued as a meromorphic function in the complex plane $\mathbb{C}$. Therefore, for the problem of value distribution of L-functions sharing finite sets with meromorphic functions, one of the main tools is the Nevanlinna theory on the value distribution of meromorphic functions.

Let f be a non-constant meromorphic function in $\mathbb{C}$, $a \in \mathbb{C} \cup \{\infty\}$, and k be a nonnegative integer or infinity. We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example [2], [5]): $T(r, f)$, $N(r, f)$, $m(r, f)$, ...

We define

$$\Theta(a, f) = 1 - \lim_{r\to+\infty} \frac{\overline{N}(r, \dfrac{1}{f-a})}{T(r, f)},$$

where r outside possibly a set of finite Lebesgue measure. Clearly

$$0 \leq \Theta(a, f) \leq 1.$$

Denote by $E_f(a)$ the set of all $a-$ points of f where an $a-$ point is counted with its multiplicity, and by $\overline{E}_f(a)$ where an $a-$ point is counted only one time, and by $E_f(a, k)$ the set of all $a-$ points of f where an $a-$ point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

For a non-empty subset $S \subset \mathbb{C} \cup \{\infty\}$, define $E_f(S) = \cup_{a\in S} E_f(a)$, and similarly for $\overline{E}_f(S)$, $E_f(S, k)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{M}(\mathbb{C})$. Two non-constant meromorphic functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity*, (share S CM), if $E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity*, (share S IM), if $\overline{E}_f(S) = \overline{E}_g(S)$, and to *share* S *with weight* k if $E_f(S, k) = E_g(S, k)$.

If the condition $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$) implies $f = g$ for any two non-constant meromorphic (entire) functions f, g of $\mathcal{F}$, then S is called a unique range set for meromorphic (entire) functions of $\mathcal{F}$ counting multiplicity (resp. ignoring multiplicity), and similarly for unique range set for meromorphic (entire) functions of $\mathcal{F}$ with weight k. Clearly $E_f(S) = E_f(S, \infty)$, and $\overline{E}_f(S) =$

$\overline{E}_f(S,0)$. Denote by

$$\mathcal{F}_f\big(\Theta(a,f)\geq t\big)=\big\{f\in\mathcal{M}(\mathbb{C}),t\in\mathbb{R}:\Theta(a,f)\geq t\ \text{ and }\ 1\geq t\big\},$$

$$\mathcal{F}_f\big(\Theta(a,f)\leq t\big)=\big\{f\in\mathcal{M}(\mathbb{C}),t\in\mathbb{R}:\Theta(a,f)\leq t\ \text{ and }\ t\geq 0\big\},$$

and similarly for

$$\mathcal{F}_f\big(\Theta(a,f)>t\big),\ \mathcal{F}_f\big(\Theta(a,f)<t\big).$$

In the last few years, the value distribution and uniqueness of L-functions has been studied extensively. In 2017 Q.-Q. Yuan, X.-M. Li, and H.-X. Yi [17] obtained the following result.

Theorem A. [17] *Let f be a non-constant meromorphic function having finitely many poles, and let L be an L-function. Let $P(z)=z^n+az^m+b$, where m,n are positive integers, satisfying $n>2m+4$, and (m,n)=1, $a,b\in\mathbb{C}$ are non-zero constants. Denote by S the zero set of P. If f and L share S CM, then $f=L$.*

From Theorem A it follows the existence of a class of subsets S with 7 elements, which are zero sets of Yi's polynomials, such that if $E_f(S)=E_L(S)$, then $f=L$, where f is a non-constant meromorphic function having finitely many poles, L is an L-function.

In 2023 H. H. Khoai, V. H. An, and N. D. Phuong [9] by using a class of polynomials, which are not Yi's polynomials, presented a class of subsets $S\subset\mathbb{C}$ with 9 elements such that if $E_f(S)=E_L(S)$, then $f=L$, where L is an L-function and f is a non-constant meromorphic function. They obtained the following result.

Let $n,m\in\mathbb{N}^*$, $a\in\mathbb{C}$, $a\neq 0$.

Consider polynomials $P_K(z)$ of the following form:

$$P_K(z)=(n+m+1)\big(\sum_{i=0}^{m}\tbinom{m}{i}\frac{(-1)^i}{n+m+1-i}z^{n+m+1-i}a^i\big)+1=Q_K(z)+1,$$

where

$$Q_K(z)=(n+m+1)\big(\sum_{i=0}^{m}\tbinom{m}{i}\frac{(-1)^i}{n+m+1-i}z^{n+m+1-i}a^i\big). \tag{1.1}$$

Suppose that

$$Q_K(a)=(n+m+1)(\sum_{i=0}^{m}\tbinom{m}{i}\frac{(-1)^i}{n+m+1-i})a^{n+m+1}\neq -1,-2. \tag{1.2}$$

Then $P_K^{'}(z)=(n+m+1)z^n(z-a)^m$, and $P_K^{'}(z)$ has a zero at 0 of order n, a zero at a of order m. Note that, from the condition (1.2) it follows that $P_K(z)$ has only simple zeros.

Theorem B. *Let f be non-constant meromorphic function, L be a non-constant L-function, $P_K(z)$ be defined as in (1.1) with conditions* (1.2), $S_K = \{z \mid P_K(z) = 0\}$. *If $n \geq 2, m \geq 2, n+m \geq 8$, then the condition $E_f(S_K) = E_L(S_K)$ implies $f = L$.*

A polynomial $P(z)$ is called a *uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, the condition $P(f) = P(g)$ implies $f = g$.

A polynomial $P(z)$ is called a *strong uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, and a non-zero constant c, the condition $P(f) = cP(g)$ implies $f = g$.

Note that $P_K(z)$ be a strong uniqueness polynomial for meromorphic functions if $n \geq 2, m \geq 2, n+m \geq 8$.

Consider polynomials $P(z) \in \mathbb{C}[z]$ of degree q of the form

$$P(z) = a(z-a_1)(z-a_2)\cdots(z-a_q),$$

where the derivative of $P(z)$ has k zeros mutually distinct $d_1, d_2, ..., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively, and then

$$P^{'}(z) = aq(z-d_1)^{q_1}(z-d_2)^{q_2}...(z-d_k)^{q_k}, a \neq 0. \tag{1.3}$$

The number k is called the *derivative index* of $P(z)$.

Suppose that:

$$P(d_i) \neq 0 \text{ for } 1 \leq i \leq k. \tag{1.4}$$

Note that, from the condition (1.4) it follows that $P(z)$has only simple zeros.

We recall the following condition introduced by Fujimoto (see [6]):

$$P(d_i) \neq P(d_j) \text{ for } 1 \leq i < j \leq k,\ k \geq 2. \tag{F}$$

In 2023 A. Banerjee and A. Kundu [1] improved Theorem B. They obtained the following result.

Theorem C. *Let f be a non-constant meromorphic function, L be a L-function, $P(z)$ be defined as in (1.3) with conditions (1.4) and (F), and $S = \{z : P(z) = 0\}$. If $\min\{q_1, q_2\} \geq 2$ when $k = 2$ and $q \geq 2k+4$ and $E_f(S, 2) = E_L(S, 2)$, then we have $f = L$.*

Note that, if $E_f(S) = E_L(S)$, then $E_f(S, 2) = E_L(S, 2)$. Then, applying Theorem C with $k = 2$ and $q \geq 8$, we obtain Theorem B.

Regarding Theorem B and Theorem C it is natural to ask the following question which motivates us to write this paper.

Question 1. *What is the smallest cardinality for such a finite set S such that any meromorphic function f and a L-function L satisfying $E_f(S) = E_L(S)$ must be identical?*

In this paper, we apply the arguments used in [9] and [1] to answer to Question 1.

We shall prove the following main theorem.

Theorem 1. *Let f be a non-constant meromorphic function, L be a non-constant L-function. Let $P(z)$ be a strong uniqueness polynomial of the form (1.3) satisfying conditions (1.4) and (F), and $S = \{z : P(z) = 0\}$, and $E_f(S) = E_L(S)$. Assume that $\min\{q_1, q_2\} \geq 2$ when $k = 2$, and one of the following conditions is satisfied:*

1)

$$q \geq 2k + 3, \quad \textit{and} \quad f \in \mathcal{F}_f\big(\Theta(\infty, f) \geq t\big), \ 1 \geq t \geq \frac{1}{2};$$

2)

$$q \geq 2k + 4, \quad \textit{and} \quad f \in \mathcal{F}_f\big(\Theta(\infty, f) \geq t\big)\}, \quad t < \frac{1}{2}.$$

Then $f = L$.

Applications. We discuss some applications of Theorem 1.

Giving specific values for t in Theorem 1, we can get the following interesting cases for $k = 2$ and $\Theta(\infty, f)$:

i) There exist sets S of 7 elements such that any meromorphic function f and a L-function L satisfying $E_f(S) = E_L(S)$ and $\Theta(\infty, f) = 1$ must be identical.

Indeed, take $k = 2$ and $t = 1$ in Theorem 1. Then $\Theta(\infty, f) = 1$ and applying Theorem 1, Part 1, we obtain $q \geq 7$.

ii) There exist sets S of 8 elements such that any meromorphic function f and a L-function L satisfying $E_f(S) = E_L(S)$ must be identical.

Indeed, take $k = 2$ and $t \leq 0$ in Theorem 1. Then $\Theta(\infty, f) \geq 0$ and applying Theorem 1, Part 2, we obtain $q \geq 8$.

Remark. *i) From Theorem 1, Part 2 we obtain Theorem C.*

ii) Theorem 1 improves and generalizes some previous results of Khoai-An-Phuong [9] and a result in [17].

2. Preliminary results

We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example, [5], [2], [15]). We have other forms of two Fundamental Theorems of the Nevanlinna theory:

Another form of the First Fundamental Theorem (see [15], Theorem 1.2, p.8). *Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$ and let $a \in \mathbb{C}$. Then*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1),$$

where $O(1)$ is a bounded quantity when $r \to +\infty$.

Another form of the Second Fundamental Theorem (see[15], Theorem 1.6', p.22). Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C}$. Then

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) - N_0(r, \frac{1}{f'}) + S(r, f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of the function $(f-a_1)...(f-a_q)$, and $S(r, f) = o(T(r, f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.1. *1/* **[2]** *For any non-constant meromorphic function f, we have*

i) $T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f)$;

ii) $S(r, f^{(k)}) = S(r, f)$.

2/ **[18]** *For any nonconstant meromorphic function f,*

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r, f) + S(r, f).$$

Definition. Let f be a non-constant meromorphic function, and k be a positive integer. We denote by $\overline{N}_{(k}(r, f)$ the counting function of the poles of order $\geq k$ of f, where each pole is counted only once. If z is a zero of f, denote by $\nu_f(z)$ its multiplicity. We denote by $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ the counting function of the zeros z of $f^{'}$ satisfying $f(z) \neq 0$, where each zero is counted only once. We denote by $\overline{N}_{k)}(r, \frac{1}{f})$ the counting function of the zeros z of f satisfying $\nu_f(z) \leq k$, where each zero is counted only once. Let be given two non-constant meromorphic functions f and g. For simplicity, denote by $\nu_1(z) = \nu_f(z)$ (resp.

$\nu_2(z) = \nu_g(z)$), if z is a zero of f (resp. g). Let $f^{-1}(0) = g^{-1}(0)$. We denote by $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1)$ the counting function of the common zeros z, satisfying $\nu_1(z) = \nu_2(z) = 1$. Similarly, we define the counting functions: $N(r, \frac{1}{f}; \nu_1 \geq 2)$, $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$ and $N(r, \frac{1}{g}; \nu_2 \geq 2)$.

Lemma 2.2. *Let f, g be two non-constant meromorphic functions such that $E_f(0) = E_g(0)$. Set*

$$F = \frac{1}{f},\ G = \frac{1}{g},\ H = \frac{F''}{F'} - \frac{G''}{G'}.$$

Suppose that $H \not\equiv 0$. Then

$$N(r, H) \leq \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0).$$

Moreover, if a is a common simple zero of f and g, then $H(a) = 0$.

H. Fujimoto ([3], Proposition 7.1) proved the following:

Lemma 2.3. *Let $P(z)$ be a strong uniqueness polynomial of the form (1.3) satisfying conditions (1.4) and (F). Suppose that $q \geq 5$ and there are two non-constant meromorphic function f and g such that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1$$

for two constants $c_0 \neq 0$ and c_1. If $k \geq 3$ or if $k = 2, min\{q_1, q_2\} \geq 2$, then $c_1 = 0$.

Lemma 2.4. [4] *Let $P(z)$ be a polynomial of the form (1.3) satisfying conditions (1.4) and (F). Then $P(z)$ is a uniqueness polynomial if and only if*

$$\sum_{1 \leq i < j \leq k} q_i q_j > \sum_{i=1}^{k} q_i.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$, or when $k = 2$, $\min\{q_1, q_2\} \geq 2$, and $q_1 + q_2 \geq 5$.

Lemma 2.5. [2]. *Let f be an entire function of finite order ρ. If f has no zeros, then $f(z) = e^{h(z)}$, where $h(z)$ is a polynomial of degree less than ρ.*

Lemma 2.6. [14]. *Let L be a non-constant $L-$ function. Then*

i) $T(r, L) = \frac{d_L}{\pi} r \log r + O(r)$, where $d_L = 2\sum_{i=1}^{K} \lambda_i$ is the degree of $L-$ function, and K, λ_i are respectively the positive integer and positive real number in the functional equation of the definition of $L-$ functions;

ii) $N(r, L) = S(r, L)$,

iii) $\rho(L) = 1$.

Lemma 2.7. [**13**]. *Suppose L is a non-constant L-function, there is no generalized Picard exceptional value of L in the complex plane.*

Lemma 2.8. [**9**] *Let f be a non-constant meromorphic function. Then*

$$\overline{N}(r,\frac{1}{f})-\frac{1}{2}\overline{N}_{1)}(r,\frac{1}{f})\leq\frac{1}{2}N(r,\frac{1}{f}).$$

3. Proof of Theorem 1

Lemma 3.1. *We have*

1)

$$(q-1)T(r,L)+S(r,L)\leq qT(r,f)+S(r,f),$$

2)

$$(q-2)T(r,f)+S(r,f)\leq qT(r,L)+S(r,L),\ S(r,f)=S(r,L).$$

Proof. Applying another form of the two Fundamental Theorems and noting that $\overline{N}(r,L)=S(r,L)$, $E_L(S)=E_f(S)$, we obtain

$$(q-1)T(r,L)\leq\overline{N}(r,L)+\sum_{i=1}^{q}\overline{N}(r,\frac{1}{L-a_i})+S(r,L),$$

$$\begin{aligned}(q-1)T(r,L)+S(r,L)&\leq\sum_{i=1}^{q}\overline{N}(r,\frac{1}{f-a_i})+S(r,f)\\&\leq qT(r,f)+S(r,f).\end{aligned}$$

Similarly,

$$(q-1)T(r,f)\leq\overline{N}(r,f)+\sum_{i=1}^{q}\overline{N}(r,\frac{1}{f-a_i})+S(r,f),$$

$$\begin{aligned}(q-1)T(r,f)+S(r,f)&\leq T(r,f)+\sum_{i=1}^{q}\overline{N}(r,\frac{1}{L-a_i})+S(r,L),\\(q-2)T(r,f)+S(r,f)&\leq qT(r,L)+S(r,L).\end{aligned}$$

Combining the above inequalities, we get

$$\frac{q-1}{q}T(r,L)+S(r,L)\leq T(r,f)+S(r,f)\leq\frac{q}{q-2}T(r,L)+S(r,L),$$

$$\frac{q-2}{q}T(r,f)+S(r,f)\le T(r,L)+S(r,L)\le \frac{q}{q-1}T(r,f)+S(r,f).$$

Therefore $S(r,f)=S(r,L)$.
Lemma 3.1 is proved. ■

Set

$$F=\frac{1}{P(f)},\ L=\frac{1}{P(L)},H=\frac{F''}{F'}-\frac{G''}{G'}.$$

From Lemma 3.1 we obtain $S(r,f)=S(r,L)$. Put $S(r)=S(r,f)=S(r,L)$. Then $T(r,P(f))=qT(r,f)+O(1)$ and $T(r,P(L))=qT(r,L)+O(1)$, and hence $S(r,P(f))=S(r)$ and $S(r,P(L))=S(r)$. We prove following.

Lemma 3.2. $H\equiv 0$ *if one of the following conditions is satisfied:*

1) $q\ge 2k+4-2\Theta(\infty,f)$ *and* $\Theta(\infty,f)<1$.

2) $q\ge 2k+3$ *and* $\Theta(\infty,f)=1$.

Proof. Suppose $H\not\equiv 0$.

Claim 1. *We have*

i) $(q-1)T(r,L)\le \overline{N}(r,\frac{1}{P(L)})-N_o(r,\frac{1}{L'})+S(r)$, *where* $N_o(r,\frac{1}{L'})$ *is the counting function of those zeros of* L', *which are not zeros of the function* $(L-a_1)(L-a_2)\cdots(L-a_q)$.

ii) $(q-1)T(r,f)\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{P(f)})-N_o(r,\frac{1}{f'})+S(r)$, *where* $N_o(r,\frac{1}{f'})$ *is the counting function of those zeros of* f', *which are not zeros of the function* $(f-a_1)\cdots(f-a_q)$.

Proof. $i)$ Applying another form of the two Fundamental Theorems to L and the values $a_1,a_2,...,a_q$, and noticing that

$$N(r,L)=S(r,L),\ \sum_{i=1}^{q}\overline{N}(r,\frac{1}{L-a_i})=\overline{N}(r,\frac{1}{P(L)}),$$

we obtain

$$(q-1)T(r,L)\le \overline{N}(r,L)+\sum_{i=1}^{q}\overline{N}(r,\frac{1}{L-a_i})-N_o(r,\frac{1}{L'})+S(r,L),$$

$$(q-1)T(r,L)\le \overline{N}(r,\frac{1}{P(L)})-N_o(r,\frac{1}{L'})+S(r).$$

$ii)$ The inequality for f is proved by a similar argument.

Claim 2. *We have*

$$N(r,H) \le kT(r,L) + kT(r,f) + \overline{N}(r,f) + N_o(r,\frac{1}{f'}) + N_o(r,\frac{1}{L'}) + S(r).$$

Proof.

Noting that H has only simple poles, from Lemma 2.2 we obtain

$$N(r,H) \le \overline{N}_{(2}(r,P(f)) + \overline{N}_{(2}(r,P(L)) + \overline{N}(r,\frac{1}{P'(f)};P(f)\neq 0) + \overline{N}(r,\frac{1}{P'(L)};P(L)\neq 0) + S(r).$$

On the other hand,

$$\overline{N}_{(2}(r,P(L)) = \overline{N}(r,L) = S(r), \overline{N}_{(2}(r,P(f)) = \overline{N}(r,f).$$

Moreover, we have

$$\overline{N}(r,\frac{1}{[P(L)]'};P(L)\neq 0) \le \sum_{i=1}^{k}\overline{N}(r,\frac{1}{L-d_i};(L-a_1)\cdots(L-a_q)\neq 0) + N_o(r,\frac{1}{L'})$$
$$\le \sum_{i=1}^{k}\overline{N}(r,\frac{1}{L-d_i}) + N_o(r,\frac{1}{L'}) \le kT(r,L) + N_o(r,\frac{1}{L'}) + S(r).$$

Thus

$$\overline{N}(r,\frac{1}{[P(L)]'};P(L)\neq 0) \le kT(r,L) + N_o(r,\frac{1}{L'}) + S(r).$$

Similarly,

$$\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0) \le kT(r,f) + N_o(r,\frac{1}{f'}) + S(r).$$

Claim 3. *We have*

i)

$$\overline{N}(r,\frac{1}{P(L)}) + \overline{N}(r,\frac{1}{P(f)}) \le$$
$$(\frac{q}{2}+k)T(r,L) + (\frac{q}{2}+k)T(r,f) + \overline{N}(r,f) + N_o(r,\frac{1}{L'}) + N_o(r,\frac{1}{f'}) + S(r).$$

ii)

$$(q-2k-2)T(r,L) + (q-2k-2)T(r,f) \le 4\overline{N}(r,f) + S(r).$$

Proof. $i)$ Note that from Lemma 2.2, if a is a common simple zero of $P(f)$ and $P(L)$, then $H(a) = 0$. Therefore,

$$\begin{aligned} N_{1)}(r, \frac{1}{P(L)}) = N_{1)}(r, \frac{1}{P(f)}) &\leq N(r, \frac{1}{H}) \leq T(r, H) + S(r, H) \\ &\leq N(r, H) + S(r), \end{aligned}$$

because by Lemma on logarithmic derivatives, $m(r, H) = o(T(r, H))$, and by Lemma 2.1, $S(r, H) \leq S(r)$. Then, applying Lemma 2.8 and Claim 2, we obtain

$$\overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{P(f)}) = \overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{P(f)}) - N_{1)}(r, \frac{1}{P(L)}) +$$

$$N_{1)}(r, \frac{1}{P(L)}) = \overline{N}(r, \frac{1}{P(L)}) - \frac{1}{2}N_{1)}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{P(f)})$$

$$-\frac{1}{2}N_{1)}(r, \frac{1}{P(f)}) + N_{1)}(r, \frac{1}{P(L)}) \leq \frac{1}{2}(N(r, \frac{1}{P(L)}) + N(r, \frac{1}{P(f)})) + N(r, H)$$

$$\leq (\frac{q}{2} + k)T(r, L) + (\frac{q}{2} + k)T(r, f) + \overline{N}(r, f) + N_o(r, \frac{1}{L'}) + N_o(r, \frac{1}{f'}) + S(r).$$

$ii)$ By Claim 1 and Part $i)$ of Claim 3 we obtain

$$(2q-2)T(r, L) + (2q-2)T(r, f) \leq (q+2k)T(r, L) + (q+2k)T(r, f) + 4\overline{N}(r, f) + S(r),$$

$$(q - 2k - 2)T(r, L) + (q - 2k - 2)T(r, f) \leq 4\overline{N}(r, f) + S(r).$$

Claim 3 is proved.

Now we use Lemma 3.1 and Part ii) of Claim 3 to obtain a contradiction, and complete the proof of $H \equiv 0$.

1) $q \geq 2k + 4 - 2\Theta(\infty, f)$ and $\Theta(\infty, f) < 1$.

Consider $\alpha(x)$ on $x \geq 2k + 4 - 2\Theta(\infty, f),\ x \in \mathbb{R}$,

where

$$\alpha(x) = 2x - 4k - 9 + 4\Theta(\infty, f) + \frac{2k + 6 - 4\Theta(\infty, f)}{x}.$$

We have $\alpha'(x) = 2 - \dfrac{2k + 6 - 4\Theta(\infty, f)}{x^2} > 0$ on $x \geq 2k+4-2\Theta(\infty, f),\ x \in \mathbb{R}$.

Combining this and $q \geq 2k + 4 - 2\Theta(\infty, f)$ and $\Theta(\infty, f) < 1$ we obtain

$$\alpha(q) = 2q - 4k - 9 + 4\Theta(\infty, f) + \frac{2k + 6 - 4\Theta(\infty, f)}{q},$$

$$\alpha(2k + 4 - 2\Theta(\infty, f)) = \frac{2k + 6 - 4\Theta(\infty, f)}{2k + 4 - 2\Theta(\infty, f)} - 1 > 0,$$

$$\alpha(q) \geq \alpha(2k+4-2\Theta(\infty, f)) > 0. \tag{3.1}$$

From (3.1) it is given that

$$\alpha(q) \geq \alpha(2k+4-2\Theta(\infty, f)) > \epsilon > 0, \tag{3.2}$$

where ϵ is a small positive number.

Using Part ii) of Claim 3 and Part 1) of Lemma 3.1 we get

$$(q-2k-2)T(r,L) + (q-2k-2)T(r,f) \leq 4\overline{N}(r,f) + S(r),$$

$$(q-2k-2)T(r,L) + (q-2k-2)T(r,f) \leq 4(1-\Theta(\infty, f) + \frac{\epsilon}{4})T(r,f) + S(r),$$

$$(2q-4k-9+4\Theta(\infty, f) + \frac{2k+6-4\Theta(\infty, f)}{q} - \frac{q-1}{q}\epsilon)T(r,L) \leq S(r),$$

which is a contradiction since (3.2).

2) $q \geq 2k+3$ and $\Theta(\infty, f) = 1$.

Consider $\beta(x)$ on $x \geq 2k+3,\ x \in \mathbb{R}$, where

$$\beta(x) = 2x - 4k - 5 + \frac{2k+2}{x}.$$

We have $\beta'(x) = 2 - \frac{2k+2}{x^2} > 0$ on $x \geq 2k+3,\ x \in \mathbb{R}$.

Combining this and $q \geq 2k+3$ we obtain

$$\beta(q) = 2q - 4k - 5 + \frac{2k+2}{q},\ \beta(2k+3) = 1 + \frac{2k+2}{2k+3} > 1,$$

$$\beta(q) \geq \beta(2k+3) > 1. \tag{3.3}$$

By Part ii) of Claim 3 and $\Theta(\infty, f) = 1$ and Lemma 3.1 we get

$\overline{N}(r,f) = S(r),$ and therefore $(q-2k-2)T(r,L) + (q-2k-2)T(r,f) \leq S(r),$

$$(2q-4k-5+\frac{2k+2}{q})T(r,L) \leq S(r). \tag{3.4}$$

which is a contradiction since (3.3).

So $H \equiv 0$. Lemma 3.2 is proved. ∎

Now we return the proof of Theorem 1. We first prove $H \equiv 0$.

1)

$$q \geq 2k+3, \text{ and } f \in \mathcal{F}_f\big(\Theta(\infty, f) \geq t\big),\ 1 \geq t \geq \frac{1}{2}.$$

If $t = 1$, then $\Theta(\infty, f) = 1$, and then $H \equiv 0$ from Part 2) of Lemma 3.2

If $1 > t \geq \frac{1}{2}$ and $1 > \Theta(\infty, f) \geq \frac{1}{2}$, then

$$q \geq 2k+3 \geq 2k+4-2\Theta(\infty, f) > 2k+2.$$

Therefore, $H \equiv 0$ from Part 1) of Lemma 3.2.

2)

$$q \geq 2k+4, \text{ and } f \in \mathcal{F}_f\big(\Theta(\infty, f) \geq t\big)\}, \quad t < \frac{1}{2}.$$

If $\Theta(\infty, f) < \frac{1}{2}$, then $0 \geq -2\Theta(\infty, f) > -1$. Therefore

$$q \geq 2k+4 \geq 2k+4-2\Theta(\infty, f) > 2k+3,$$

which follows that $H \equiv 0$ from Part 1) of Lemma 3.2.

If $\Theta(\infty, f) \geq \frac{1}{2}$, then $H \equiv 0$ from $q \geq 2k+4 > 2k+3$ and 1).

Thus, $H \equiv 0$. Therefore $\frac{1}{P(L)} = \frac{c}{P(f)} + c_1$ for some constants $c \neq 0$ and c_1. By Lemma 2.3 we obtain $c_1 = 0$. Therefore, there is a constant $c \neq 0$ such that $P(f) = cP(L)$. By $P(z)$ is a strong uniqueness polynomial we obtain $f = L$.

Theorem 1 is proved. ■

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