J. Math. Math. Sci. (2024) 115–133

Second-order optimality conditions for strict local Pareto minima of constrained nonsmooth multiobjective optimization problems

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(Received Sep. 15, 2024)

Abstract. This paper presents primal and dual Fritz John second-order necessary conditions for strict local Pareto minima of order two of nonsmooth vector optimization problems in terms of the Páles–Zeidan second-order directional derivatives without constraint qualifications. Dual second-order Karush–Kuhn–Tucker necessary and sufficient conditions for strict local Pareto minima of order two are established under a suitable constraint qualification.

1. Introduction

Vector equilibrium problem plays an important role in nonlinear analysis. It provides a unified mathematical model including vector optimization problem, vector variational inequality problem, and some other problems. There are a lot of papers dealing with optimality conditions for solutions of vector equilibrium problems and vector inequalities (see, e. g., [3, 7–9, 11, 12, 21–25] and references therein). In recent years, second-order necessary optimality conditions have been received attention because of their extension beyond first-order necessary conditions. There have been many papers to deal with second-order optimality conditions (see, e. g., [1, 2, 4, 5, 10, 13–15, 19, 20, 30, 31] and references therein).

Key words and phrases: Primal and dual second-order necessary efficiency conditions; Páles–Zeidan second-order directional derivatives; First and second-order tangent vectors
2020 Mathematics Subject Classification: *90C46; 90C29*

Second-order optimality conditions were derived by Ben-Tal [2], Aghezzaf and Hachimi [1], Jiménez and Novo [15] for optimization problems with twice differentiable functions, Ginchev and Ivanov [10] for continuously differentiable scalar problems, Ivanov [14] for scalar problems involving Lipschitz second-order Hadamard differentiable functions. Recently, Constantin [4] established primal and dual second-order necessary conditions for Lipschitz optimization problems with inequality constraints by using second-order directional derivatives. Constantin [6] derived some second-order optimality conditions for strict efficiency under Zingwill constraint qualification.

The purpose of this paper is to develop primal and dual second-order necessary optimality conditions for strict local Pareto minima of multiobjective optimization problems involving inequality and set constraints in terms of the Páles–Zeidan second-order directional derivatives in Banach spaces.

The paper is organized as follows. After some preliminaries, Section 3 is devoted to develop primal second-order necessary conditions for strict local Pareto minima of order two of nonsmooth multiobjective optimization problems involving inequality and set constraints without constraint qualifications in terms of the Páles–Zeidan second-order upper generalized directional derivatives. Section 4 deals with dual second-order Fritz John and Karush-Kukn-Tucker necessary optimality conditions for strict local Pareto minima under a suitable constraint qualification. Section 5 gives dual second-order sufficient optimality conditions for strict local Pareto minima.

2. Preliminaries

Let X be a real Banach space, and let f be a real-valued function defined on X, which is Lipschitz near $\overline{x} \in X$. We recall that the Clarke generalized derivative of f at $\overline{x} \in X$ in a direction $v \in X$ is defined as

$$f^0(\overline{x}; v) := \limsup_{(x,t)\to(\overline{x},0^+)} \frac{f(x+tv)-f(x)}{t}. \tag{2.1}$$

Following [30], the Páles–Zeidan second-order upper(lower) generalized directional derivative of f at $\overline{x}$ in the direction v is defined as

$$f_+^{00}(\overline{x}; v) := \limsup_{t\downarrow 0} \frac{f(\overline{x}+tv)-f(\overline{x})-tf^0(\overline{x}; v)}{t^2/2}, \tag{2.2}$$

$$\Big(resp. \quad f_-^{00}(\overline{x}; v) := \liminf_{t\downarrow 0} \frac{f(\overline{x}+tv)-f(\overline{x})-tf^0(\overline{x}; v)}{t^2/2}\Big). \tag{2.3}$$

Let $f : X \to \mathbb{R}$ be Fréchet differentiable at $\overline{x} \in X$, and $\nabla f(\overline{x})$ be its gradient at $\overline{x}$. The following limit is called second-order directional derivative at $\overline{x}$ in the direction $v \in \mathbb{R}^n$

$$f''(\overline{x}; v) := \lim_{t\downarrow 0} \frac{f(\overline{x}+tv) - f(\overline{x}) - t\nabla f(\overline{x})(v)}{t^2/2}.$$

Note that if f is continuously Fréchet differentiable near $\overline{x}$ with the Fréchet derivative $\nabla f(\overline{x})$, then f is Lipschitz near $\overline{x}$, and $f^0(\overline{x}; v) = \nabla f(\overline{x})v$ $(\forall v \in X)$; If f is continuously Fréchet differentiable near $\overline{x}$ and second-order directionally differentiable at $\overline{x}$ in a direction $v \in X$, then (see [4])

$$f_+^{00}(\overline{x}; v) = f\text{''}(\overline{x}; v).$$

Let C be a nonempty subset of X. Following [5], an element $u \in X$ is called a tangent vector to C at $\overline{x} \in \mathrm{cl}C$ if

$$\lim_{t\downarrow 0} \frac{1}{t} d(\overline{x} + tu; C) = 0, \tag{2.4}$$

where $d(x; C)$ stands for the distance from x to C, $\mathrm{cl}C$ is the closure of C. The set of all tangent vectors to C at $\overline{x}$ is denoted by $T_{\overline{x}}(C)$, and is called the tangent cone to C at $\overline{x}$. Note that $T_{\overline{x}}(C)$ is a nonempty closed cone containing $0 \in X$. Moreover, (2.4) is equivalent to the existence of a function $\beta : (0, +\infty) \to X$ with $\beta(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + t(u + \beta(t)) \in C \quad (\forall t > 0).$$

Following [18], we also have

$$T_{\overline{x}}(C) = \{u \in X \ : \ \exists t_n \to 0^+, \ \exists u_n \to u \ \text{ such that } \ x_n = \overline{x} + t_n u_n \in C, \ \forall n \geq 1\}.$$

Adapting the definition in [5], an element $v \in X$ is said to be a second-order tangent vector to C at $\overline{x}$ if there exists $u \in X$ such that

$$\lim_{t\downarrow 0} \frac{1}{t^2} d(\overline{x} + tu + \frac{t^2}{2} v; C) = 0\,. \tag{2.5}$$

The vector u satisfying (2.5) is said to be associated with v. Denote by $T^2_{\overline{x}}(C)$ the set of all second-order tangent vectors to C at $\overline{x}$. Observe that $v \in T^2_{\overline{x}}(C)$ with the associated vector u is equivalent to the existence of a function $\beta_1 : (0, +\infty) \to X$ with $\beta_1(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + tu + \frac{t^2}{2}(v + \beta_1(t)) \in C \ (\forall t > 0).$$

Moreover, $v \in T^2_{\overline{x}}(C)$ implies that $u \in T_{\overline{x}}(C)$, and $T^2_{\overline{x}}(C)$ is a nonempty cone containing $0 \in X$ (see [5, 18]).

3. Primal Second-Order Necessary Conditions for strict local Pareto minima

Let X be a real Banach space, and let f be a function from X to $\mathbb{R}^m$, and so, $f = (f_1, \ldots, f_m)$. Let $g_1, \ldots, g_n$ be extended-real-valued functions defined on X, and C be a subset of X. We set $J := \{1, \ldots, m\}$, $I := \{1, \ldots, n\}$ and

$$M := \Big\{x \in C : \ g_i(x) \leqslant 0, i \in I\Big\}.$$

Let us consider the following multiobjective optimization problem:

$$(MP) \qquad\qquad \min\{f(x) : \ x \in M\}.$$

For $\overline{x} \in X$, we denote:

$$I(\overline{x}) := \{i \in I : g_i(\overline{x}) = 0\},$$

for $u \in X$:

$$I(\overline{x}, u) := \{i \in I(\overline{x}) : g_i^0(\overline{x}; u) = 0\},$$
$$J(\overline{x}, u) := \{k \in J : f_k^0(\overline{x}; u) = 0\}.$$

Remark that a vector $\overline{x} \in X$ is called a weak local Pareto minimum for (MP) if there exists a neighborhood V of $\overline{x}$ such that there is no $x \in V \cap M$ satisfying

$$f_k(x) < f_k(\overline{x}) \quad \forall k \in J.$$

Adapting the definition in Jiménez [18], a vector $\overline{x} \in M$ is called a strict local Pareto minimum of order two for (MP) if there exist a constant $\alpha > 0$ and a neighborhood V of $\overline{x}$ such that

$$\big(f(x) + \mathbb{R}^m_+\big) \cap B\big(f(\overline{x}), \alpha\|x - \overline{x}\|^2\big) = \emptyset, \quad \forall x \in M \cap V, x \neq \overline{x}, \qquad (*)$$

where $\mathbb{R}^m_+$ is the nonnegative orthant in $\mathbb{R}^m$, $B\big(f(\overline{x}), \alpha\|x-\overline{x}\|^2\big)$ stands for the open ball of radius $\alpha\|x - \overline{x}\|^2$ around $f(\overline{x})$.

Remark 3.1. 1. Condition (*) is equivalent to the following

$$\|f(x) + d - f(\overline{x})\| > \alpha\|x - \overline{x}\|^2 \quad \forall x \in M \cap V, x \neq \overline{x}, \ \forall d \in \mathbb{R}^m_+.$$

2. The definition of strict local Pareto minimizer becomes the usual notion of a strict local minimizer of order two when $X = \mathbb{R}$ and $m = 1$ and then we obtain that

$$(f(x) + \mathbb{R}_+) \cap B(f(\overline{x}), \alpha||x - \overline{x}||^2) = \emptyset \quad \forall x \in M \cap V \setminus \{\overline{x}\},$$

which is equivalent to the following

$$f(x) > f(\overline{x}) + \alpha||x - \overline{x}||^2.$$

3. It should be noted here that if $\overline{x} \in M$ is a strict local minimum of order two for (MP) then it is a weak local Pareto minimum for (MP) (see [18]).

We recall Theorem 3.7 by Jiménez [18] on strict local Pareto minima of order two in terms of the data of Problem (MP), which will be used to prove primal necessary condition for strict local Pareto minima in next section.

Proposition 3.1. [18] *$\overline{x} \in M$ is a strict local Pareto minimum of order two for (MP) if and only if there exist a number $\alpha > 0$, a neighborhood U of $\overline{x}$, and at most m sets $V_i, i \in I' \subset I$, such that $\{V_i, i \in I'\}$ is a covering of $M \cap U \setminus \{\overline{x}\}$, and verifying*

$$f_i(x) > f_i(\overline{x}) + \alpha\|x - \overline{x}\|^2 \quad (\forall x \in M_i \setminus \{\overline{x}\}),$$

where $M_i = (M \cap U \cap V_i) \cup \{\overline{x}\}$.

This section deals with primal second-order necessary conditions for strict local Pareto minima of order two for (MP) in terms of Palés–Zeidan's second-order upper generalized directional derivatives.

Adapting the definition in [13], a vector $u \in X$ is called critical direction at $\overline{x} \in M$ if

$$\begin{cases} f_k^0(\overline{x}; u) \leqslant 0 \quad (\forall k \in J), \\ g_i^0(\overline{x}; u) \leqslant 0 \quad (\forall i \in I(\overline{x})), \\ u \in T_{\overline{x}}(C). \end{cases}$$

We shall begin with a primal second-order necessary condition for strict local Pareto minimum of (MP).

Theorem 3.2. *Let $\overline{x} \in M$ be a strict local Pareto minimum of (MP). Assume that the functions $f_k (k \in J)$ and g_i $(i \in I(\overline{x}))$ are locally Lipschitz at $\overline{x}$, the functions g_i $(i \notin I(\overline{x})$ are continuous at $\overline{x}$. Then, for every critical direction $u \in X$, $u \neq 0$, there is no $v \in T^2_{\overline{x}}(C)$ with the associated vector u satisfying the following system:*

$$f_k^0(\overline{x}; v) + f_{k,+}^{00}(\overline{x}; u) \leq 0 \quad (\forall k \in J(\overline{x}, u)), \tag{3.1}$$

$$g_i^0(\overline{x}; v) + g_{i,+}^{00}(\overline{x}; u) < 0 \quad (\forall i \in I(\overline{x}; u)). \tag{3.2}$$

Remark 3.2. Constantin [6] established primal and dual Fritz John second-order necessary conditions for strict local Pareto minima of Lipschitz multiobjective optimization problems in terms of the Páles–Zeidan second-order directional derivatives with Zingwill constraint qualification. In this paper, we develop second-order primal and dual Fritz John optimality conditions for strict local Pareto minima of Lipschitz multiobjective optimization problems in terms of the Páles–Zeidan second-order directional derivatives without any constraint qualification. The results obtained here are new and significant in the theory of optimality conditions.

Proof. [Proof of Theorem 3.2] Assume the contrary, that there is a critical direction $u_0 \in X$ such that the system (3.1)–(3.2) has a solution $v_0 \in T^2_{\overline{x}}(C)$. Hence, there exists a mapping $\beta(t) : (0, +\infty) \to X$ with $\beta(t) \to 0$, as $t \to 0^+$ such that

$$\overline{x} + tu_0 + \frac{t^2}{2}(v_0 + \beta(t)) \in C \ (\forall t > 0). \tag{3.3}$$

First, let us consider the inequality constraints. We divide three cases.
(a) Let $i \in I(\overline{x}; u_0)$. Then $g_i(\overline{x}) = 0$ and $g_i^0(\overline{x}; u_0) = 0$. Let us show that there exists $\overline{\epsilon}_i > 0$ such that

$$g_i\big(\overline{x} + tu_0 + \frac{t^2}{2}(v_0 + \beta(t))\big) < 0 \ (\forall t \in (0, \overline{\epsilon}_i), \forall i \in I(\overline{x}; u_0)). \tag{3.4}$$

We first show that there exists $\epsilon_i > 0$ such that for every $t \in (0, \epsilon_i)$, $g_i(\overline{x} + tu_0 + \frac{1}{2}t^2 v_0) < 0$. If it were not so, for $\epsilon_i = \frac{1}{n} > 0$ ($n \in \mathbb{N}$), there would exist $t_n \in (0, \frac{1}{n})$ such that $g_i(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) = g_i(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) - g_i(\overline{x}) \geqslant 0$. Consequently,

$$\begin{aligned}
&g_i^0(\overline{x}; v_0) + g_{i,+}^{00}(\overline{x}; u_0) \\
&\geqslant \limsup_{n \to +\infty} \frac{g_i(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) - g_i(\overline{x} + t_n u_0)}{t_n^2/2} \\
&+ \limsup_{n \to +\infty} \frac{g_i(\overline{x} + t_n u_0) - g_i(\overline{x}) - t_n g_i^0(\overline{x}; u_0)}{t_n^2/2} \\
&\geqslant \limsup_{n \to +\infty} \frac{g_i(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) - g_i(\overline{x})}{t_n^2/2} \\
&\geqslant 0,
\end{aligned}$$

which conflicts with (3.2), and so, there exists $\epsilon_i > 0$ such that for every $t \in (0, \epsilon_i)$, $g_i(\overline{x} + tu_0 + \frac{1}{2}t^2 v_0) < 0$. Since $\beta(t) \to 0$ as $t \to 0^+$, in view of the continuity of g_i at $\overline{x}$, there is a number $\overline{\epsilon}_i \in (0, \epsilon_i)$ such that (3.4) holds for every $t \in (0, \overline{\epsilon}_i)$.
(b) Let $j \in I(\overline{x}) \setminus I(\overline{x}; u_0)$. Then $g_j(\overline{x}) = 0$ and $g_j^0(\overline{x}; u_0) < 0$. We show that there exists $\overline{\delta}_j > 0$ such that

$$g_j(\overline{x} + tu_0 + \frac{1}{2}t^2(v_0 + \beta(t)) < 0 \ (\forall t \in (0, \overline{\delta}_j), \forall j \in I(\overline{x}) \setminus I(\overline{x}; u_0)). \tag{3.5}$$

To prove (3.5), we first prove that there exists $\delta_j > 0$ such that

$$g_j(\overline{x} + tu_0 + \frac{1}{2}t^2 v_0) < 0 \ (\forall t \in (0, \delta_j), \forall j \in I(\overline{x}) \setminus I(\overline{x}; u_0)). \tag{3.6}$$

If this were false, for $\delta_j = \frac{1}{n} > 0$ ($n \in \mathbb{N}$), there would exist $t_n \in (0, \frac{1}{n})$ such that $g_j(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) \geqslant 0 = g_j(\overline{x})$. Since g_j is locally Lipschitz at $\overline{x}$, it follows that

$$\begin{aligned} 0 &\leqslant \frac{g_j(\overline{x} + t_n u_0 + \frac{1}{2}t_n^2 v_0) - g_j(\overline{x} + t_n u_0)}{t_n} + \frac{g_j(\overline{x} + t_n u_0) - g_j(\overline{x})}{t_n} \\ &\leqslant \frac{1}{2} L_j t_n ||v_0|| + g_j^0(\overline{x}; u_0), \end{aligned}$$

where L_j is the Lipschitz constant of g_j at $\overline{x}$. By letting $n \to +\infty$, we arrive at $g_j^0(\overline{x}; u_0) \geqslant 0$ ($\forall j \in I(\overline{x}) \setminus I(\overline{x}; u_0)$), a contradiction. Hence, (3.6) holds. Making use of the continuity of g_j at $\overline{x}$, it follows from (3.6) that there exists a number $\overline{\delta}_j \in (0, \delta_j)$ such that (3.5) holds.
(c) Let $r \in I \setminus I(\overline{x})$. Then $g_r(\overline{x}) < 0$. By the continuity of g_r, there exists $\sigma_r > 0$) such that $g_r(\overline{x} + tu_0 + \frac{1}{2}t^2 v_0) < 0$ ($\forall t \in (0, \sigma_r), \forall r \in I \setminus I(\overline{x})$). Since $\beta(t) \to 0$ as $t \to 0^+$, there is $\overline{\sigma}_r \in (0, \sigma_r)$ such that

$$g_r\big(\overline{x} + tu_0 + \frac{1}{2}t^2(v_0 + \beta(t))\big) < 0 \ (\forall t \in [0, \overline{\sigma}_r), \forall r \in I \setminus I(\overline{x})). \tag{3.7}$$

Now, we set $\overline{\alpha} := \min\{\overline{\epsilon}_i, \overline{\delta}_j, \overline{\sigma}_r : i \in I(\overline{x}; u_0), j \in I(\overline{x}) \setminus I(\overline{x}; u_0), r \in I \setminus I(\overline{x})\}$. It follows from (3.4), (3.5) and (3.7) that

$$g_i\big(\overline{x} + tu_0 + \frac{1}{2}t^2(v_0 + \beta(t))\big) < 0 \ (\forall t \in (0, \overline{\alpha}), \forall i \in I). \tag{3.8}$$

From (3.3) and (3.8), we deduce that $\overline{x} + tu_0 + \frac{1}{2}t^2(v_0 + \beta(t))$ ($\forall t \in (0, \overline{\alpha})$) are feasible points of Problem (MP).

Next, we consider the objective functions.

Since $\overline{x}$ is a strict Pareto minimum of f on M, in view of Proposition 3.1, there are $\alpha > 0$, a neighborhood W of $\overline{x}$ and at most m sets $V_k, k \in J' \subset J$ such that $\{V_k : k \in J'\}$ is a coverring of $(M \cap W) \setminus \{\overline{x}\}$, and

$$f_k(x) > f_k(\overline{x}) + \alpha \parallel x - \overline{x} \parallel^2 \ (\forall x \in M_k \setminus \{\overline{x}\})$$

where $M_k := (M \cap W \cap V_k) \cup \{\overline{x}\}$.
For all sequence $t_n \downarrow 0$, $t_n \in (0, \overline{\alpha})$, we have $\overline{x} + t_n u_0 + \frac{t_n^2}{2}(v_0 + \beta(t_n)) \in \bigcup_{k \in J'} M_k$. Hence, there exist an index k and an infinite subsequence $t_n' \downarrow 0$

of $\{t_n\}$, without loss of generality we still denote by t_n, such that $\overline{x} + t_n u_0 + \frac{t_n^2}{2}(v_0 + \beta(t_n)) \neq \overline{x}$ and $\overline{x} + t_n u_0 + \frac{t_n^2}{2}(v_0 + \beta(t_n)) \in M_k, \forall n$.

Therefore,

$$f_k\big(\overline{x} + t_n u_0 + \frac{t_n^2}{2}(v_0 + \beta(t_n))\big) > f_k(\overline{x}) + \alpha \parallel t_n u_0 + \frac{t_n^2}{2}(v_0 + \beta(t_n)) \parallel^2 . \tag{3.9}$$

We now show that $f_k^0(\overline{x}; u_0) = 0$.

We have

$$|f_k\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n))\big) - f_k(\overline{x})| \leq D_k \frac{t_n^2}{2}\|v_0 + \beta(t_n)\|, \tag{3.10}$$

and

$$|f_k\big(\overline{x} + t_n u_0 + \frac{t_n^2}{2}\beta(t_n)\big) - f_k(\overline{x} + t_n u_0)| \leq D_k \frac{t_n^2}{2}\|\beta(t_n)\|, \tag{3.11}$$

where D_k is the Lipschitz constant of f_k at $\overline{x}$.

Therefore,

$$\lim_{n\to\infty} \frac{f_k(\overline{x}) - f\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n))\big)}{t_n} = 0, \tag{3.12}$$

$$\lim_{n\to\infty} \frac{f_k(\overline{x} + t_n u_0) - f\big(\overline{x} + t_n u_0 + \frac{t_n^2}{2}\beta(t_n)\big)}{t_n^2/2} = 0. \tag{3.13}$$

From (3.9) and (3.12), we have:

$$\begin{aligned} f_k^0(\overline{x}; u_0) &\geq \limsup_{n\to\infty} \frac{f_k\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n)) + t_n u_0\big) - f_k\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n))\big)}{t_n} \\ &= \limsup_{n\to\infty} \frac{f_k\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n)) + t_n u_0\big) - f_k(\overline{x})}{t_n} \\ &\quad + \lim_{n\to\infty} \frac{f_k(\overline{x}) - f\big(\overline{x} + \frac{t_n^2}{2}(v_0 + \beta(t_n))\big)}{t_n} \geq 0. \end{aligned}$$

Hence, $f_k^0(\overline{x}; u_0) \geq 0$. On the other hand u_0 is a critical direction, we get

$$f_k^0(\overline{x}; u_0) = 0. \tag{3.14}$$

It follows from (3.9), (3.13) and (3.14) that

$$\begin{aligned}
&f_k^0(\overline{x};v_0)+f_{k,+}^{00}(\overline{x};u_0)\\
&\ge \limsup_{n\to\infty}\frac{f_k\big(\overline{x}+t_nu_0+\frac{t_n^2}{2}\beta(t_n)+\frac{t_n^2}{2}v_0\big)-f_k\big(\overline{x}+t_nu_0+\frac{t_n^2}{2}\beta(t_n)\big)}{t_n^2/2}\\
&+\limsup_{n\to\infty}\frac{f_k(\overline{x}+t_nu_0)-f_k(\overline{x})-t_nf_k^0(\overline{x};u_0)}{t_n^2/2}\\
&\ge 2\limsup_{n\to\infty}\frac{f_k\big(\overline{x}+t_nu_0+\frac{t_n^2}{2}(v_0+\beta(t_n))\big)-f_k(\overline{x})}{t_n^2}\\
&+2\lim_{n\to\infty}\frac{f_k(\overline{x}+t_nu_0)-f_k\big(\overline{x}+t_nu_0+\frac{t_n^2}{2}\beta(t_n)\big)}{t_n^2}\\
&\ge 2\limsup_{n\to\infty}\frac{\alpha\|t_nu_0+\frac{t_n^2}{2}(v_0+\beta(t_n))\|^2}{t_n^2}=2\alpha\|u_0\|^2>0.
\end{aligned}$$

Hence,

$$f_k^0(\overline{x};v_0)+f_{k,+}^{00}(\overline{x};u_0)>0, \tag{3.15}$$

which contradicts (3.1). Consequently, the system (3.1)-(3.2) has no solution $v\in T_{\overline{x}}^2C$. The proof is complete. ■

Example 3.3. Let $X=\mathbb{R}^2$, $C=\mathbb{R}_+^2$, $f_1(x_1,x_2)=x_1^2+x_2^2$, $f_2(x_1,x_2)=|x_1|$, $g_1(x_1,x_2)=x_1^2-x_2$.
We have

$$M=\{x\in C\mid g_1(x_1,x_2)\le 0\}=\{(x_1,x_2)\mid x_1\ge 0,\ x_2\ge x_1^2\}$$

Observe that $\overline{x}=(0,0)$ is a strict local Pareto minimum of order two for (MP). Indeed, for all $x=(x_1,x_2)\in M\setminus\{(0,0)\}$ and $d=(d_1,d_2)\in\mathbb{R}_+^2$, we get

$$\begin{aligned}
\|f(x)+d-f(\overline{x})\|&=\sqrt{(x_1^2+x_2^2+d_1)^2+(|x_1|+d_2)^2}\\
&\ge |x_1^2+x_2^2+d_1|\ge x_1^2+x_2^2>\frac{1}{2}\|x-\overline{x}\|^2.
\end{aligned}$$

It is easy to show that, for all $v=(v_1,v_2)\in\mathbb{R}^2$

$$\begin{aligned}
&f_1'(\overline{x};v)=0,\ f_2'(\overline{x};v)=|v_1|,\ g_1'(\overline{x};v)=-v_2,\ T_{\overline{x}}(C)=T_{\overline{x}}^2(C)=\mathbb{R}_+^2,\\
&f_{1,+}^{00}(\overline{x};v)=2(v_1^2+v_2^2),\ f_{2,+}^{00}(\overline{x};v)=0,\ g_{1,+}^{00}(\overline{x};v)=2v_1^2.
\end{aligned} \tag{3.16}$$

From (3.16), $u=(u_1,u_2)\ne 0$ is a critical direction at $\overline{x}$ of (MP) iff $u_1=0$, $u_2>0$. Then, system (3.1)-(3.2) become

$$0+2(0^2+u_2^2)\le 0,$$

$$|v_1| + 0 \le 0,$$
$$-v_2 + 2u_2^2 \le 0.$$

It is easy to see that there is no $v \in T^2_{\overline{x}}(C)$ satisfying above system.

For $C = X$, we obtain the following consequence.

Corollary 3.1. *Assume that $C = X$ and $\overline{x} \in M$ is a strict local Pareto minimum of (MP). Assume further that the functions $f_k\,(k \in J), g_i\,(i \in I(\overline{x}))$ are locally Lipschitz at $\overline{x}$ and the functions $g_i\,(i \in I \setminus I(\overline{x}))$ are continuous at $\overline{x}$. Then for for every critical direction $u \in X,\ u \neq 0$, there is no $v \in X$ satisfying the following system:*

$$f_k^0(\overline{x}; v) + f_{k,+}^{00}(\overline{x}; u) \le 0 \quad (\forall k \in J(\overline{x}, u)),$$
$$g_i^0(\overline{x}; v) + g_{i,+}^{00}(\overline{x}; u) < 0 \quad (\forall i \in I(\overline{x}; u)).$$

Proof. It can be seen that in the case $C = X$, we have $T^2_{\overline{x}}(C) = X$. Applying Theorem 3.2, we obtain the desired conclusion. ■

4. Dual Second-Order Necessary Conditions for Strict Efficiency

In this section, we shall give dual second-order Fritz John necessary conditions for strict local Pareto minima.

Theorem 4.1. *Let X be a finite-dimensional normed space,* $\dim X = q$, *$C = X$ and $\overline{x} \in M$ a strict local Pareto minimum of order two for problem (MP). Assume that the functions f_k $(k \in J)$ and g_i $(i \in I(\overline{x}))$ are Gâteaux differentiable with the Gâteaux derivative f'_k $(k \in J)$ and $g'_i(\overline{x})(i \in I(\overline{x}))$, respectively and the functions $f_k(k \in J)$ and g_i $(i \in I(\overline{x}))$ are locally Lipschitz at $\overline{x}$, the functions g_i $(i \notin I(\overline{x}))$ are continuous at $\overline{x}$. Suppose also that for every $v \in X$, $f_{k,+}^{00}(\overline{x}; v)$, $g_{i,+}^{00}(\overline{x}; v)$ are finite $\forall k \in J(\overline{x}, v)$, $i \in I(\overline{x}, v)$. Then, for every nonzero critical direction $u \in X$, there exist $\lambda_k \geqslant 0$ $(k \in J(\overline{x}, u)), \mu_i \geqslant 0$ $(i \in I(\overline{x}, u))$, not all zero, such that*

$$\sum_{k \in J(\overline{x}, u)} \lambda_k f'_k(\overline{x}) + \sum_{i \in I(\overline{x}, u)} \mu_i g'_i(\overline{x}) = 0, \tag{4.1}$$

$$\sum_{k \in J(\overline{x}, u)} \lambda_k f_{k,+}^{00}(\overline{x}; u) + \sum_{i \in I(\overline{x}, u)} \mu_i g_{i,+}^{00}(\overline{x}; u) \ge 0. \tag{4.2}$$

Moreover, assume that the above critical direction u satisfies: There exists $v_* \in X$ such that

$$(CQ) \qquad g_i'(\overline{x})(v_*) + g_{i,+}^{00}(\overline{x}; u) < 0 \qquad (\forall i \in I(\overline{x}; u)).$$

Then there exists $k \in J(\overline{x}, u)$ such that $\lambda_k \neq 0$.

Proof. (i) Denote $p = |J(\overline{x}, u)| + |I(\overline{x}, u)|$, consider the matrix A with the rows

$$\{f_k'(\overline{x}) \ : k \in J(\overline{x}, u)\} \text{ and } \{g_i'(\overline{x}) \ : i \in I(\overline{x}, u)\},$$

and the vector b with the components

$$\{f_{k,+}^{00}(\overline{x}; u) \ : k \in J(\overline{x}, u)\} \text{ and } \{g_{i,+}^{00}(\overline{x}, u) \ : i \in I(\overline{x}, u)\}.$$

It is easy to see that $A = (a_{ij})$ has p rows and q columns.

With these notations, taking account of Corollary 3.1, it follows that the linear system $Az + b < 0$ has no solution $v \in X$, which is equivalent to the fact that the linear program

$$\min\{y : Aw + b \leq \hat{y}\},$$

where by $\hat{y} \in \mathbb{R}^p$ is denoted the vector with all the components equal to y, has a non-negative optimal solution. An equivalent form of the last program is

$$\min\{y \ : -Aw + \hat{y} \geq b\} = \min\{\langle c, e\rangle \ : He \geq b\},$$

where $e = (w_1, w_2, \ldots, w_q, y), \ c = (0, 0, \ldots, 0, 1)$ and

$$H = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1q} & 1 \\ -a_{21} & -a_{22} & \cdots & -a_{2q} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{p1} & -a_{p2} & \cdots & -a_{pq} & 1 \end{pmatrix}.$$

By the duality theorem, the dual program

$$\max\{b^T\delta \ : -A^T\delta = 0, \ \sum_{i=1}^{p} \delta_i = 1, \ \delta_i \geq 0, \ \forall\, i = \overline{1, p}\}$$

the system

$$A^T\delta = 0, \ b^T\delta \geq 0, \ \delta \geq 0, \ \delta \neq 0,$$

has a solution. Here, the vector $\delta = (\lambda, \mu)$ has the same dimension as the vector b, λ has the components $\{\lambda_k \ : k \in J(\overline{x}, u)\}$, μ has the components $\{\mu_i \ : i \in I(\overline{x}, u)\}$, A^T is the transpose of the matrix A, and $\delta = (\delta_1, \ldots, \delta_p) \geq 0$ means $\delta_i \geq 0$ for all $i \in \{1, ..., p\}$.

We obtained that there exist $\lambda_k \geq 0,\ k \in J(\overline{x}, u), \mu_i \geq 0,\ i \in I(\overline{x}, u)$, not all equal to zero, such that

$$\sum_{k\in J(\overline{x},u)} \lambda_k f_k'(\overline{x}) + \sum_{i\in I(\overline{x},u)} \mu_i g_i'(\overline{x}) = A^T\delta = 0,$$

$$\sum_{k\in J(\overline{x},u)} \lambda_k f_{k,+}^{00}(\overline{x}; u) + \sum_{i\in I(\overline{x},u)} \mu_i g_{i,+}^{00}(\overline{x}; u) = b^T\delta \geq 0.$$

(ii) We now suppose that the constraint qualification (CQ) holds. From (4.1), choose $v = v_*$, we get

(4.3)
$$\sum_{k\in J(\overline{x},u)} \lambda_k \big(f_k'(\overline{x})(v_*) + f_{k,+}^{00}(\overline{x}; u)\big) + \sum_{i\in I(\overline{x},u)} \mu_i \big((g_i'(\overline{x})(v_*) + g_{i,+}^{00}(\overline{x}; u)\big) \geq 0.$$

If $\lambda_k = 0,\ \forall k \in J(\overline{x}, u)$, we have

$$\sum_{i\in I(\overline{x},u)} \mu_i \big((g_i'(\overline{x})(v_*) + g_{i,+}^{00}(\overline{x}; u)\big) \geq 0. \tag{4.4}$$

On the other hand, there are at least one $\mu_i > 0$ ($i \in I(\overline{x}, u)$, as $\lambda_k \geqslant 0$ ($k \in J(\overline{x}, u)), \mu_i \geqslant 0$ ($i \in I(\overline{x}, u)$) and not all zero. From (CQ), we obtain

$$\sum_{i\in I(\overline{x},u)} \mu_i \big((g_i'(\overline{x})(v_*) + g_{i,+}^{00}(\overline{x}; u)\big) < 0.$$

But this contradicts (4.4). Consequently, exists $k \in J(\overline{x}, u)$ such that $\lambda_k \neq 0$.

■

Example 4.2. Let $X = C = \mathbb{R}^2,\ f_1(x_1, x_2) = x_1^2 + x_2^2,\ f_2(x_1, x_2) = x_2,\ g_1(x_1, x_2) = x_1 - x_2^2$.
We have

$$M = \{x \in C \mid g_1(x_1, x_2) \leq 0\} = \{(x_1, x_2) \mid 0 \leq x_1 \leq x_2^2,\ x_2 \geq 0\}$$

Observe that $\overline{x} = (0, 0)$ is a strict local Pareto minimum of order two for (MP). Indeed, for all $x = (x_1, x_2) \in M \setminus \{(0, 0)\}$ and $d = (d_1, d_2) \in \mathbb{R}^2_+$, we get

$$\|f(x) + d - f(\overline{x})\| = \sqrt{(x_1^2 + x_2^2 + d_1)^2 + (x_2 + d_2)^2}$$

$$\geq |x_1^2 + x_2^2 + d_1| \geq x_1^2 + x_2^2 > \frac{1}{2}\|x - \overline{x}\|^2.$$

It is easy to show that, for all $v = (v_1, v_2) \in \mathbb{R}^2$

$$f_1'(\overline{x}) = (0, 0),\ f_2'(\overline{x}) = (0, 1),\ g_1'(\overline{x}) = (1, 0), \tag{4.5}$$

$$f_{1,+}^{00}(\overline{x};v) = 2(v_1^2 + v_2^2),\ f_{2,+}^{00}(\overline{x};v) = 0,\ g_{1,+}^{00}(\overline{x};v) = -2v_2^2.$$

From (4.5), one has

$$f_1'(\overline{x})(u) = 0,\ f_2'(\overline{x})(u) = u_2,\ g_1'(\overline{x})(u) = u_1,$$

therefore $u = (u_1, u_2) \in \mathbb{R}^2_-$ is critical direction at $\overline{x}$ of (MP).
Now, we choose $\lambda_1 = 1,\ \lambda_2 = \mu_1 = 0$, we get

$$f_1'(\overline{x})(v) = 0 \quad \forall v \in \mathbb{R}^2,$$

$$f_{1,+}^{00}(\overline{x};u) = 2(u_1^2 + u_2^2) \geq 0.$$

Hence, (4.1) and (4.2) hold.

5. Second-order sufficient optimality conditions

This section provides Fritz John second-order sufficient optimality conditions.

Definition 5.1. *Let X be a normed space. A function $f : X \to \mathbb{R}$ is called strict regular in the Clarke sense at $\overline{x}$ if for all $u_0 \in X$, $f^0(\overline{x};u_0)$ exist and*

$$f^0(\overline{x};u_0) = \lim_{n\to\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t}, \quad \forall t_n \to 0^+, u_n \to u_0.$$

Remark 5.1. If f is strict regular in the Clarke sense at $\overline{x}$ then f is regular in the Clarke sense at $\overline{x}$. Indeed, choose $u_n = u_0\ \forall n$, we get

$$f^0(\overline{x};u_0) = \lim_{n\to\infty} \frac{f(\overline{x} + t_n u_0) - f(\overline{x})}{t_n} = \lim_{t\to 0^+} \frac{f(\overline{x} + t u_0) - f(\overline{x})}{t}.$$

Theorem 5.1. *Let $X = \mathbb{R}^s$, $\overline{x} \in M$. Assume that the functions $f_k (k \in J)$ and g_i $(i \in I(\overline{x}))$ are locally Lipschitz and strict regular in the Clarke sense at $\overline{x}$, the functions g_i $(i \notin I(\overline{x}))$ are continuous at $\overline{x}$. Suppose also that for every critical direction $u \in T_{\overline{x}}(M)$, there exist $\lambda_k \geqslant 0$ $(k \in J), \mu_i \geqslant 0$ $(i \in I(\overline{x}))$, not all zero, such that*

$$\sum_{k=1}^{m} \lambda_k f_k^0(\overline{x};v) + \sum_{i\in I(\overline{x})} \mu_i g_i^0(\overline{x};v) = 0 \quad \forall v \in T_{\overline{x}}(M), \tag{5.1}$$

$$\sum_{k=1}^{m} \lambda_k f_{k,-}^{00}(\overline{x};u) + \sum_{i\in I(\overline{x})} \mu_i g_{i,-}^{00}(\overline{x};u) > 0. \tag{5.2}$$

Then $\overline{x}$ is a strict local Pareto minimum of order two for problem (MP).

Proof. Assume the contrary, that $\overline{x}$ is not a strict local Pareto minimum for (MP). By virtue of Proposition 3.5(b) [18], there exist $x_n \in M \setminus \{\overline{x}\},\ x_n \to \overline{x}$ and $u \in X$ such that

$$\text{(5.3)} \qquad \lim_{n\to\infty} \frac{f(x_n) - f(\overline{x})}{||x_n - \overline{x}||^2} = e \in [-\infty, 0]^m \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|} = u.$$

Putting $e = (e_1, \dots, e_m),\ t_n = \|x_n - \overline{x}\|$ and $u_n = \dfrac{x_n - \overline{x}}{t_n}$. We have

$$t_n > 0\ \forall n,\ \lim_{n\to\infty} t_n = 0,\ \lim_{n\to\infty} u_n = u,\ x_n = \overline{x} + t_n u_n,$$

therefore $u \in T_{\overline{x}}(M)$.

It follows from (5.3) that

$$\text{(5.4)} \qquad \lim_{n\to\infty} \frac{f_k(x_n) - f_k(\overline{x})}{||x_n - \overline{x}||^2} = \lim_{n\to\infty} \frac{f_k(x_n) - f_k(\overline{x})}{t_n^2} = e_k \in [-\infty, 0].$$

So,

$$\text{(5.5)} \qquad f_k^0(\overline{x}; u) = \lim_{n\to\infty} \frac{f_k(\overline{x} + t_n u_n) - f_k(\overline{x})}{t_n} = \lim_{n\to\infty} \frac{f_k(x_n) - f_k(\overline{x})}{t_n} \le 0.$$

Since $i \in I(\overline{x})$ and $x_n \in M$, we have $g_i(\overline{x}) = 0$ and $g_i(x_n) \le 0$, it follows that

$$\text{(5.6)} \qquad g_i^0(\overline{x}; u) = \lim_{n\to\infty} \frac{g_i(\overline{x} + t_n u_n) - g_i(\overline{x})}{t_n} = \lim_{n\to\infty} \frac{g_i(x_n)}{t_n} \le 0.$$

From (5.1), we get

$$\sum_{k=1}^{m} \lambda_k f_k^0(\overline{x}; u) + \sum_{i\in I(\overline{x})} \mu_i g_i^0(\overline{x}; u) = 0.$$

It follows from (5.5) and (5.6) that u is a critical direction at $\overline{x}$ and

$$\text{(5.7)} \qquad \lambda_k f_k^0(\overline{x}; u) = 0,\ \mu_i g_i^0(\overline{x}; u) = 0,\ \forall k = \overline{1, m},\ i \in I(\overline{x}).$$

One has

$$\begin{aligned}
&\sum_{k=1}^{m}\lambda_k f^{00}_{k,-}(\overline{x};u)+\sum_{i\in I(\overline{x})}\mu_i g^{00}_{k,-}(\overline{x};u)\\
&\le \sum_{k=1}^{m}\lambda_k \liminf_{n\to\infty}\frac{f_k(\overline{x}+t_nu)-f_k(\overline{x})-t_nf^0_k(\overline{x};u)}{t_n^2/2}\\
&+\sum_{i\in I(\overline{x})}\mu_i \liminf_{n\to\infty}\frac{g_i(\overline{x}+t_nu)-g_i(\overline{x})-t_ng^0_i(\overline{x};u)}{t_n^2/2}\\
&=\sum_{k=1}^{m}\liminf_{n\to\infty}\frac{\lambda_k\big(f_k(x_n)-f_k(\overline{x})\big)-t_n\lambda_kf^0_k(\overline{x};u)}{t_n^2/2}\\
&+\sum_{i\in I(\overline{x})}\liminf_{n\to\infty}\frac{\mu_i\big(g_i(x_n)-g_i(\overline{x})\big)-t_n\mu_ig^0_i(\overline{x};u)}{t_n^2/2}\\
&=\sum_{k=1}^{m}\liminf_{n\to\infty}\frac{2\lambda_k\big(f_k(x_n)-f_k(\overline{x})\big)}{t_n^2}+\sum_{i\in I(\overline{x})}\liminf_{n\to\infty}\frac{2\mu_ig_i(x_n)}{t_n^2}\\
&=\sum_{k=1}^{m}2\lambda_ke_k+\sum_{i\in I(\overline{x})}\liminf_{n\to\infty}\frac{2\mu_ig_i(x_n)}{t_n^2}.
\end{aligned}$$

Hence,
(5.8)

$$\sum_{k=1}^{m}\lambda_k f^{00}_{k,-}(\overline{x};u)+\sum_{i\in I(\overline{x})}\mu_i g^{00}_{k,-}(\overline{x};u)\le\sum_{k=1}^{m}2\lambda_ke_k+\sum_{i\in I(\overline{x})}\liminf_{n\to\infty}\frac{2\mu_ig_i(x_n)}{t_n^2}.$$

From (5.4),(5.8) and $g_i(x_n)\le 0$, we have

$$\sum_{k=1}^{m}\lambda_k f^{00}_{k,-}(\overline{x};u)+\sum_{i\in I(\overline{x})}\mu_i g^{00}_{k,-}(\overline{x};u)\le 0.$$

But this contradicts (5.2). Consequently, $\overline{x}$ is a strict local Pareto minimum of (MP). The proof is complete. ■

Example 5.2. Let $X=\mathbb{R}^2,\ C=\mathbb{R}^2_+,\ f_1(x_1,x_2)=x_1^2+x_2^2,\ f_2(x_1,x_2)=2x_1+1,\ g_1(x_1,x_2)=2x_1^2-x_2$.
We have

$$M=\{x\in C\mid g_1(x_1,x_2)\le 0\}=\{(x_1,x_2)\mid x_1\ge 0,\ x_2\ge 2x_1^2\}.$$

Choose $\overline{x} = (0,0)$, it is easy to show that, for all $v = (v_1, v_2) \in \mathbb{R}^2$

$$(5.9) \quad f_1^0(\overline{x}; v) = 0,\ f_2^0(\overline{x}; v) = 2v_1,\ g_1^0(\overline{x}; v) = -v_2,\ T_{\overline{x}}(C) = T_{\overline{x}}^2(C) = \mathbb{R}_+^2,$$

$$f_{1,-}^{00}(\overline{x}; v) = 2(v_1^2 + v_2^2),\ f_{2,-}^{00}(\overline{x}; v) = 0,\ g_{1,-}^{00}(\overline{x}; v) = 4v_1^2.$$

From (5.9), $u = (u_1, u_2) \neq 0$ is a critical direction at $\overline{x}$ of (MP) iff $u_1 = 0,\ u_2 > 0$.

Choose $\lambda_1 = 1,\ \lambda_2 = 0,\ \mu_1 = 0,\ v = (v_1, v_2),\ u = (0, u_2)$, system (5.1)-(5.2) hold because of

$$1 \times 0 + 0 \times 2v_1 + 0 \times (-v_2) = 0,$$

$$1 \times 2(0^2 + u_2^2) + 0 \times 0 + 0 \times 0 > 0.$$

Observe that $\overline{x} = (0,0)$ is a strict local Pareto minimum of order two for (MP). Indeed, for all $x = (x_1, x_2) \in M \setminus \{(0,0)\}$ and $d = (d_1, d_2) \in \mathbb{R}_+^2$, we get

$$\|f(x) + d - f(\overline{x})\| = \sqrt{(x_1^2 + x_2^2 + d_1)^2 + (2x_1 + 1 + d_2 - 1)^2}$$

$$\geq |x_1^2 + x_2^2 + d_1| \geq x_1^2 + x_2^2 > \frac{1}{2}\|x - \overline{x}\|^2.$$

6. Conclusions

Constantin [6] established primal and dual Fritz John second-order necessary conditions for Lipschitz multiobjective optimization problems with Zingwill constraint qualification. In this paper, we establish second-order primal and dual Fritz John optimality conditions to nonsmooth multiobjective optimization problems in terms of the Páles–Zeidan second-order upper generalized directional derivative without any constraint qualifications. Some second-order Karush-Kuhn-Tucker necessary and sufficient conditions for strict local Pareto minima of nonsmooth multiobjective optimization problems with suitable constraint qualification are derived. The results obtained here are new and significant in the theory of optimality conditions.

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