J. Math. Math. Sci. (2024) 135–148

A proximal point algorithm for solving a class of implicit price equilibrium models

Nguyen Ngoc Hai (Hanoi, Vietnam)
Le Dung Muu* (Hanoi, Vietnam)

(Received Oct. 07, 2024)

Abstract. We apply the proximal point algorithm to solve a class of implicit price equilibrium models including the Walras supply-demand and competitive equilibrium ones, where both supply and demand are given implicitly as the solution-sets of mathematical programs depending on the price. Such models are formulated as complementarity or variational inequality forms. We employ a monotonicity property of the cost operator to develop proximal point based algorithms to approximate an equilibrium point of the model. Convergence of the algorithm is proved and some computational results with many randomly generated data are reported to show that the proposed algorithms work well for this class of equilibrium models.

1. Introduction

The proximal point algorithm [19] is a fundamental one that has been widely applied to various problems in different fields of Applied mathematics. In this paper we first apply this algorithm for solving the following complementarity problem

$$\text{Find } p^* \in C : \langle F(p^*), p^* \rangle = 0, \qquad CP(C,F)$$

*Corresponding author.
Key words and phrases: Walras, competitive models, implicit supply-demand, proximal point based algorithm, complementarity, variational inequality.
2020 Mathematics Subject Classification: 46N10, 65K10, 47J25, 47J26

where C is a closed convex cone in $\mathbb{R}^n$ and F is an operator from $\mathbb{R}^n$ into itself. Many practical problems can be formulated in the form of this problem, see e.g. [13, 15]. The main difficulty in solving $CP(C, F)$ is the condition $\langle F(p^*), p^*\rangle = 0$, called complementarity condition. A lot of solution methods have been developed for this problem, mainly for the case F is an affine operator satisfying certain properties and is given explicitly, see also [13, 15].

In this paper, we first describe proximal point based algorithms for solving Problem $CP(C, F)$, where F is given implicitly as the solution-sets of certain convex, mathematical programs. Next, we apply the proposed algorithms for finding an equilibrium point to several practical equilibrium models including the Walras and competitive price ones that can be formulated equivalently to Problem $CP(C, F)$, where the cost operator is given implicitly. We also consider the case when the strategy set C is not a cone, but a closed convex set. Such a case arising in some equilibrium models such as competitive ones.

Some computational results by using the proposed algorithms for solving several implicit price equilibrium models are reported at the end of this paper.

2. Preliminaries

First, let us recall some definitions, see e.g. [21], pages 39, 40, and lemmas that will be used in the forthcoming sections.

A set C is a convex cone, if

$$x + y \in C, \lambda x \in C \ \forall x, y \in C, \lambda \geq 0.$$

Let $C \subseteq \mathbb{R}^n$ be a convex set and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) < +\infty$ for all $x \in C$.

(i) The function f is said to be convex on C, if

$$f(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \ \forall x, y \in C, \ \forall \lambda \in [0,1].$$

(ii) f is said to be strictly convex on C if

$$f\big(\lambda x + (1-\lambda)y\big) < \lambda f(x) + (1-\lambda)f(y) \ \forall x, y \in C, \ \forall \lambda \in [0,1].$$

(iii) f is said to be strongly convex on C if there exists a number $\gamma > 0$ such that

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) - \gamma\|x-y\|^2 \ \forall x, y \in C, \ \forall \lambda \in [0,1].$$

It is well known that $(iii) \Rightarrow (ii) \Rightarrow (i)$. Moreover, the minimization problem of a strongly convex function on a closed convex set (not necessarily bounded) always admits a unique solution.

The function f is said to be concave (resp. strictly concave) on C if $-f$ is convex (resp. strictly convex) on C.

The subdifferential of f at a point x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) := \{u \in \mathbb{R}^n : \langle u, y - x\rangle + f(x) \leq f(y)\ \forall y\}.$$

It is well known that if f is differentiable at x, then $\partial f(x) \equiv \{\nabla f(x)\}$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) \neq \emptyset$ and closed convex valued for every $x \in C$. The following concepts for monotonicity of an operator can be found, for example, [4] page 293 and in [2, 3, 6, 10, 14, 13].

(i) T is said to be nonexpansive on C if

$$\|Tx - Ty\| \leq \|x - y\|\ \forall x, y \in C.$$

A typical example for nonexpansive mappings is the metric projection that maps every $x \in \mathbb{R}^n$ onto C by taking

$$P_C(x) := \{y \in C : \|x - y\| \leq \|x - z\|\ \forall z \in C\}.$$

It is well known, see e.g. [4] page 61, that if C is closed, convex, then $P_C(x)$ is singleton and nonexpansive on $\mathbb{R}^n$, that is $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for every $x, y \in \mathbb{R}^n$;

(ii) T is said to be monotone on C if $\langle T(x) - T(y), x - y\rangle \geq 0\ \forall x, y \in C$;

(iii) T is said to be strongly monotone on C with modulus $\eta > 0$ if $\langle T(x) - T(y), x - y\rangle \geq \eta\|x - y\|^2\ \forall x, y \in C$;

(iv) T is said to be co-coercive (strongly inverse monotone) on C with modulus $\xi > 0$ if $\langle T(x) - T(y), x - y\rangle \geq \xi\|T(x) - T(y)\|^2\ \forall x, y \in C$ [25];

(v) T is said to be maximal monotone on C, if it is monotone on C and its graft is not properly contained in that of another monotone operator.

A typical example for maximal monotone operators is the gradient of a lower semi continuous, properly convex function [18].

Below are some lemmas will be used in the proof of convergence theorems for the algorithms to be described.

Lemma 2.1. *[23] Suppose that $\{\alpha_k\}$ is a sequence of nonnegative numbers such that*

$$\alpha_{k+1} \leq (1 - \lambda_k)\alpha_k + \lambda_k\delta_k + \sigma_k\ \forall k,$$

where

(i) $0 < \lambda_k < 1, \ \sum_{k=1}^{+\infty} \lambda_k = +\infty$;
(ii) $\limsup_{k\to+\infty} \delta_k \le 0$;
(iii) $\sum_{k=1}^{+\infty} |\sigma_k| < +\infty$.
Then $\lim_{k\to+\infty} \alpha_k = 0$.

Lemma 2.2. *([4] Theorem 4.17) Let C be a nonempty closed subset in a Hilbert space $\mathcal{H}$ and $T : C \to \mathcal{H}$ be a nonexpansive mapping. Let $\{x^k\}$ be a sequence in C such that $x^k \to x$, $x^k - T(x^k) \to u$. Then $x - T(x) = u$.*

Lemma 2.3. *[16] Let φ be a strongly convex differentiable function with modulus β and L-Lipschitz on C. Then, for any $\alpha > 0$, it holds that*

$$\|\big(x - \frac{1}{\alpha}\nabla\varphi(x)\big) - \big(y - \frac{1}{\alpha}\nabla\varphi(y)\big)\|^2 \le (1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2})\|x-y\|^2 \ \forall x, y \in C.$$

3. The proximal point algorithm

Suppose that F is monotone on C. Since C is a closed convex set, the operator

$$T(x) := F(x) + N_C(x) \text{ if } x \in C \text{ and } T(x) = F(x) \text{ if } x \notin C$$

is maximal monotone, where $N_C(x)$ is the normal cone of C at x. Then the complementarity problem $CP(C,F)$ can be formulated equivalently as the problem of computing a solution of the inclusion $0 \in T(x)$ [19]. It is well known from [11, 19] that the proximal operator $Prox := (I + cT)^{-1}$ is defined everywhere, single values, nonexpansive for any $c > 0$, and its fixed point-set coincides with the solution-set of Problem CP(C,F).

By applying the Rockafellar proximal point regulation algorithm [19] to Problem $CP(C,F)$, we have the following iterative scheme:

At each iteration k, given $x^k \in C$, compute the next iterate by taking

$$x^{k+1} := (I + c_k T)^{-1}(x^k), \qquad (R)$$

where I is the unit operator, and $c_k > 0$ is the coefficient at iteration k. We have the following lemma for computing $(I + c_k T)^{-1}(x^k)$.

Lemma 3.1. *[16] Suppose that T is maximal monotone on $\mathbb{R}^n$ and $x^k \in C$, $c_k > 0$ are given, then computing $x^{k+1} = (I + c_k T)^{-1}(x^k)$ amounts to finding the solution of the monotone complementarity problem*

$$\textit{Find } x^{k+1} \in C : \langle F_k(x^{k+1}), x^{k+1}\rangle = 0, \qquad SCP(C, F_k)$$

where $F_k(x) := F(x) + \frac{1}{2c_k}(x - x^k)$.

Theorem 3.1. *[19] (i) $x^{k+1} = x^k$ if and only if x^k is a solution of Problem CP(C, F);*

(ii) The sequence $\{x^k\}$ converges to a solution of CP(C, F) whenever $c_k > c > 0$.

The main task at each iteration k is to solve the strongly monotone complementarity problem $SCP(C, F_k)$. We have the following lemma for an algorithm to solve strongly monotone complementarity problems.

Lemma 3.2. *[16] Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is monotone on C with modulus τ. Then*

(i) The operator $\phi_r(x) := \phi(x) + \frac{1}{2r}(x - a)$ is strongly monotone with modulus $\tau := \frac{1}{2r}$ for any $r > 0$ and $a \in C$;

(ii) The complementarity problem

$$\text{Find } s(r) \in C : \langle \phi_r(s(r)), s(r)\rangle = 0 \qquad SCP(C, \phi_r)$$

has an unique solution $s(r)$ for any $r > 0$, moreover, if ϕ_r is Lipschitz on C with constant $L > 0$, then one can choose r such that $\tau := \frac{1}{2r} < L/2$, for which the mapping $s(\cdot)$ is a contraction on C with constant $c := 1 - 2r(L/2 - \tau)$. In addition, x^* is a solution of Problem $SCP(C, \phi_r)$ if and only if it is the fixed point of the contraction mapping $s(\cdot)$.

From this lemma, by applying Theorem 2.1 in [16] with cost operator $\phi_r := F_k$ and $r := \frac{1}{2c_k}$, $a := x^k$, since $s(\cdot)$ is a contraction mapping, we can use the Banach iterative scheme $z^{k+1} = s(z^k)$ to approximate the unique fixed point of $s(r)$ with a linear rate.

Avoiding Lipschitz continuity. In the case the cost operator ϕ is not Lipschitz or its Lipschitz constant is difficult to estimate, we propose the following algorithm to solve the complementarity problem $SCP(C, \phi_r)$. The algorithm does not use the Lipschitz continuity of the cost operator ϕ_r. Then the algorithm may not be convergence with a linear rate.

ALGORITHM 1 (for strongly monotone complementarity problem).

Step 1. Choose a sequence $\{r_j\}$ of positive number satisfying

$$0 < r_j < 1, r_j < \frac{1}{2r}\ \forall j, \sum_{j=1}^{\infty} r_j = +\infty,\ \sum_{j=1}^{\infty} r_j^2 < +\infty.$$

Pick a point $x^1 \in C$, and let $j := 1$.

Step 2. If x^j is a solution, then terminate. Otherwise, compute the next iterate by taking $x^{j+1} := Pr_C(x^j - r_j v^j)$ with $v^j := \phi_r(x^j)$. Increase j by one and go back to Step 2.

Theorem 3.2. *Suppose that the algorithm never terminates and the sequence* $\{x^j\}$ *is bounded, then* $x^j \to x^*$ *as* $j \to \infty$.

Proof. Let x^* be the unique solution of Problem $SCP(C, \phi_r)$. Since Pr is the metric projection on C, from $x^{j+1} = Pr_C(x^j - r_j v^j)$, it follows that

$$\|x^{j+1} - x^*\|^2 = \|Pr_C(x^j - r_j v^j) - Pr_C(x^*)\|^2 \leq \|x^j - r_j v^j - x^*\|^2$$
$$= \|x^j - x^*\|^2 - 2r_j \langle v^j, x^j - x^* \rangle + r_j^2 \|v^j\|^2$$

By using the strong monotonicity of ϕ_r with modulus $r > 0$, we have $\langle v^j, x^j - x^* \rangle \geq \langle v^*, x^j - x^* \rangle + r\|x^j - x^*\|^2 \geq 0$ and $-\langle v^j, x^j - x^* \rangle \leq -r\|x^j - x^*\|^2$ with $v^* := \phi_r(x^*)$. Combining these inequalities to obtain

$$\|x^{j+1} - x^*\|^2 \leq (1 - 2r_j r)\|x^j - x^*\|^2 + r_j^2 \|v^j\|.$$

Since $\{x^j\}$ is bounded, we have $\{v^j\}$ is bounded too. Then by applying Lemma 2.1 with $\lambda_j = 2rr_j$, $\alpha_j := \|x^j - x^*\|^2$, $\delta_k \equiv 0$ we can conclude that $\|x^j - x^*\| \to 0$ as $j \to \infty$. ■

Applying this algorithm to strongly monotone complementarity subproblem $SCP(C, F_k)$ needed to solve at each iteration k of the proximal algorithm for Problem $CP(C, F)$ we obtain an approximate solution of the iterate point $x^{k+1} = (I + c_k T)^{-1}(x^k)$. As it has been shown in [19] that if the approximate solution x^j obtained by Algorithm 1 satisfies $\|x^j - (I + c_k T)^{-1}(x^k)\| \leq \epsilon_k$ with

$$\epsilon_k > 0 \ \forall k, \ \sum_{k=1}^{\infty} \epsilon_k < \infty,$$

then $x^k \to x^*$ as $k \to \infty$.

4. An implicit supply-demand Walras equilibrium model

The classical Walras equilibrium price model introduced by E. Walras [22] is a basic model in economics, which has attracted much attention in the literature, see e.g. [20]. In this model there are two sectors, producer and consumer deal with n-commodities $(x_1, ..., x_n) \in \mathbb{R}^n_+$ (the nonnegative orthant of $\mathbb{R}^n$). For a price $p \in \mathbb{R}^n_+$ the supply of producer is given by $S(p) \in X \subseteq \mathbb{R}^n_+$, whereas the demand of consumer is $D(p) \in X \subseteq \mathbb{R}^n_+$. A vector p is said to be an *equilibrium price* of the model if $\langle q^*, p^* \rangle = 0$ with $q^* \in F(p^*) := S(p^*) - D(p^*)$ being the excess demand at price p^*. Mathematically, the problem of finding an equilibrium price can be formulated in the form of a complementarity one given as

$$\text{Find } p^* \in \mathbb{R}^n_+, q^* \in F(p^*) : \langle q^*, p^* \rangle = 0. \qquad CP(\mathbb{R}^n_+, F)$$

An important task for this model is to find its an equilibrium price. In general Problem $CP(\mathbb{R}^n_+, F)$ is difficult to solve even when F is linear, because of the complementarity condition $\langle p, q \rangle = 0$. Some solution-methods have been proposed for solving the complementarity problem $CP(\mathbb{R}^n_+, F)$ when F possesses certain properties see.e.g [10, 15].

In this paper we consider Walras equilibrium models by assuming that they have the following property:

(Q) P can be a closed convex cone or a closed convex set. In addition, the supply $S(p)$ and demand $D(p)$ are given implicitly as optimal solution-sets of certain parametric convex mathematical programs whose objective functions are defined depending on the benefits of producer and consumer, so that both S and D are not given explicitly,

Property (Q) arising in some practical models that have been studied by several authors, see e.g. [12, 15]. In recent paper [12], the supply $S(p)$ and demand $D(p)$ are given implicitly as the benefit of producer and consumer respectively, which are defined as the optimal solution-sets of certain strongly convex mathematical programming problems. The strong convexity condition is not satisfied for several practical models, where the utility functions are certain Cobb-Douglas ones. Here, we suppose that the supply is given as the optimal solution-set of the problem for maximizing the benefit of the producer, which is formulated as the convex (not necessarily strong) mathematical program

$$S(p) := \arg\max_x\{p^T x - c(x)\} \equiv \arg\min_x\{c(x) - p^T x\} \quad \text{subject to } X := \{x : Ax \le b, x \ge 0\} \tag{4.1}$$

where $c(x)$ is the cost for producing x, while $A = (a_{i,j})$ is the coefficient matrix, for example, $a_{i,j}x_j$ being the amount of the material i needed for producing x_j. The demand is defined by the optimal solution-set of another program given as the solution-set of the problem minimizing the total payment of the consumer, that is defined as

$$D(p) := \arg\min_x d(p, x) := \arg\min_x\{p^T x + b(x)\} \text{ subject to } u(x) \ge M, x \in X, \tag{4.2}$$

where the objective function $d(p, x)$ consists of the two parts, the first one $p^T x$ represents the amount that consumer have to pay, and the second one $b(x)$ is the environment tax for using x of the goods, while $u(\cdot)$ is the utility function of the consumer and M denotes the lower bound of her/his desired benefit.

The following lemma shows that the operator $F := S - D$ is continuous, monotone on P.

Lemma 4.1. *Suppose that programs defining S and D are solvable for any $p \in P$, then S and $-D$ are monotone on P, if in addition, X is compact (often in practice), then S and D are continuous.*

Proof. To prove the monotonicity of S, let $p^1, p^2 \in P$, s^1, s^2 is the solutions of the programs defining $S(p^1), S(p^2)$. Then

$$c(s^1) - (p^1)^T s^1 \leq c(s^2) - (p^1)^T s^2,$$
$$c(s^2) - (p^2)^T s^2 \leq c(s^1) - (p^2)^T s^1.$$

Adding these inequalities we obtain

$$\langle s^1 - s^2, p^1 - p^2 \rangle \geq 0,$$

which implies the monotonicity of S. By the same way, we can prove that $-D$ is monotone, so the operator $F := S - D$ is monotone. Suppose that the mathematical programs defining $S(p), D(p)$ solvable. Then continuity of S and D follows from the Berge maximum theorem [5]. ■

Remark 4.1. In this algorithm we do not require that the objective functions of the mathematical programs defining S and D to be strongly convex as in the paper [12].

Now we apply the proximal algorithm for finding an equilibrium point of the Walras equilibrium model, where the strategy set can be a closed convex cone, denoted by P, and $F := S - D$ is a single valued operator. Each iteration k of the algorithm has two loops: the outer loop is the iteration k for the proximal point algorithm to solve the complementarity problem defined as

$$\text{Find } p^* \in P : \langle p^*, F(p^*) \rangle = 0. \qquad CP(P, F)$$

Now we describe a proximal point algorithm for solving Problem $CP(C, F)$. At each iteration k,the main task is to solve the strongly monotone complementarity problem $SCP(P, F_k)$. For solving this subproblem we use Algorithm 1.

ALGORITHM 2.

Initialization. Choose a sequence $\{c_k\}$ of positive numbers satisfying $0 < c < c_k \leq 1$ for every k, where c is given.

Iteration $k = 1, ...,$. At each iteration k, we are given a point $p^k \in P$, a sequence $\{r_{k,j}\}$ of positive numbers satisfying

$$0 < r_{k,j} < c_k, \sum_{j=1}^{\infty} r_{k,j} = \infty, \sum_{j=1}^{\infty} r_{k,j}^2 < \infty,$$

and the strongly monotone operator $F_k(p) := F(p)+c_k(p-p^k)$ with $F := S-D$.

If p^k is a solution of the problem $CP(P,F)$, then terminate, otherwise, solve the strongly monotone problem $SCP(P,F_k)$ by entering into the inner loop below.

Inner loop. $j = 1, .,.$ At each iteration j we are given a solution p^k of Problem $SCP(P,F_k)$. Let $p^{k,1} := p^k$.

Step 1 Compute $p_{k,j+1} := Pr_P\big(p^{k,j} - r_{k,j}F_k(p^{k,j})\big)$, where Pr_P stands for the metric projection onto P.

Step 2. If $p_{k,j+1}$ is a solution of Problem $SCP(P,F)$, then stop. Otherwise, increase j by one and go back to Step 1.

From the proximal point algorithm $p^{k+1} = (I + c_kT)^{-1}$, the sequence $\{p^k\}$ obtained by solving $SCP(P,F_k)$ converges to a solution p^* of Problem $CP(P,F)$.

Remark 4.2. (i) This algorithm seems simple, however, at each iteration we have to solve convex mathematical programming problems defining S and D at each iterate.

(ii) In the case $P \equiv \mathbb{R}^n_+$ the projection on P can be computed explicitly.

The case the prices are not independent

When $P := \mathbb{R}^n_+$, the prices are independent. However in some practical models the prices may not be independent, for example, in competitive models the price-vector is required to be in the unit simplex, or in a closed convex set that is not a cone. In this general case, a vector $p^* \in P$ is said to be an equilibrium point of the model if and only it is a solution of the variational inequality

$$\text{Find } p^* \in P : \langle F(p^*), p - p^*\rangle \geq 0 \ \forall p \in P, \qquad VI(P,F)$$

It is easy to see that when P is a closed convex cone, this problem can be formulated equivalently to a complementarity one.

Since $F := S - D$ is monotone on P, as before, the operator $T := F + N_P$ with N_P being the normal cone of P is maximal monotone, then, for any $c > 0$, its proximal operator $Prox := (I + cT)^{-1}$ is defined everywhere, single values, nonexpansive and the fixed point-set of $Prox$ is the same as the solution-set of Problem $VI(P,F)$ [19]. So starting from any point $p^1 \in P$, the sequence $\{p^k\}$ defined as $p^{k+1} := (I + c_kT)^{-1}(p^k)$ converges to a solution of $VI(P,F)$.

By a simple calculation we can see that computing p^{k+1} can be done by solving the strongly monotone variational inequality

$$\text{find } p^{k+1} \in P : \langle F_(p^{k+1}), p - p^{k+1}\rangle \geq 0 \ \forall p \in P, \qquad (P,F_k)$$

where $F_k(p) := F(p) + \frac{1}{2c_k}(p - p^k)$ with $F := S - D$. To solve this strongly monotone variational inequality we apply the algorithm in[16] to each iteration k of the Rockafellar proximal point algorithm. As in algorithm 2, we choose a sequence $\{c_k\}$ such that $0 < c < c_k \leq 1$ for every $k = 1, 2....$

Step 1. Choose a sequence $\{r_{k,j}\}$ of positive number satisfying

$$0 < r_{k,j} < c_k \ \forall j, \sum_{j=1}^{\infty} r_j = +\infty, \ \sum_{j=1}^{\infty} r_j^2 < +\infty.$$

Choose $p^{k,1} \in P$ and let $j = 1$.

Step 2. Let $v^{k,j} := F_k(p^{k,j})$. If $v^{k,1} = 0$, then $p^{k,j}$ is a solution.

Otherwise, compute the next iterate by taking $p^{k,j+1} := Pr_P(p^{k,j} - r_{k,j} v^{k,j})$.

If $p^{k,j+1} = p^{k,j}$, terminate: $p^{k,j}$ is the solution of Problem $SCP(P, F_k)$. If not, increase j by 1 and go back to Step 2 with $p^{k,j}$ is replaced by $p^{k,j+1}$.

Theorem 4.1. *Suppose that P is convex compact (often in practical models), and the algorithm never terminates, then $p^{k,j}$ tends to a solution p^k of Problem $SCP(P, F_k)$ as $j \to \infty$, and $p^k \to p^*$, an equilibrium point of the model.*

Proof of this convergence result can be done similarly as that of Algorithm 1.

To close this section we would like to mention that another regularization approach for ill-posed problems is the bilevel one that recently has been developed, see e.g.[8, 9, 7, 24]. The bilevel approach for this class of equilibrium models, where either S or D can be multivalued operators is a subject of our forthcoming study.

5. Computational results

In this section, we use the algorithm to solve three examples for the model.

Example 5.1. *In this example $P :\equiv \mathbb{R}^n_+$, $c(x) = x^T M x$, $b(x) = x^T B x, u(x) = l^T x$ and Let $M := M_1^T M_1, B := B_1^T B_1$ whose entries are randomly generated in the interval $[-10, 10]$. Clearly these matrices are symmetric and positive semi-definite. The feasible domain $X := \{x \geq 0 : Ax \leq b\}$ where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$ whose all entries are randomly chosen in $(0, 20)$.*

We choose the parameter $c_k = \frac{2}{3}$ for all k; $r_{k,j} = \frac{1}{(k+j+1)^{0.8}}$ for all k, j.

Example 5.2. *In this example the strategy set P is the unit simplex*

$$P := \{x^T = (x_1, ..., x_n)^T : x_j \geq 0 \ \forall j, \sum_{j=q}^{n} x_j = 1\}.$$

In this example $c(x) = \sum_{i=1}^{n} r_i e^{x_i}$, $b(x) = \sum_{i=1}^{n} s_i e^{x_i}$, *where* $r = (r_1, r_2, ..., r_n)$, $s = (s_1, s_2, ..., s_n); r_i, s_i \in (0, 0.1)$, *other inputs are chosen as in Example 5.1.*

Example 5.3. *In this example the strategy set* P *is a polyhedral convex set defined as*

$$P := \{x_1, x_2, ..., x_n) : \sum_{j=1}^{n} x_j \leq n, x_j \geq 0 \ \forall j\}.$$

For this example other inputs are chosen as in Example 5.2.

We tested these models with Python 3.10 on a computer with the processor: AMD Ryzen 5 1600 Six- Core Processor 3.20 GHz with the installed memory (RAM): 16.0 GB.

We stopped the program when $|\langle p_k, F(p_k)\rangle| < \epsilon$, with $\epsilon = 10^{-4}$;

In the tables below we use the following headings:

- Times: the CPU-computational times (in second);
- Iteration: the number of iterations.

Table 1. Computed Results for Example 5.1

n	m	Times	Iteration
5	3	0.82	10
10	8	2.09	17
30	20	2.56	21
50	30	3.56	25
100	80	4.14	27

Table 2. Computed Results for Example 5.2

n	m	Times	Iteration
5	3	1.25	13
10	8	3.15	16
30	20	3.84	20
50	30	3.34	22
100	80	6.62	24

Table 3. Computed Results for Example 5.3

n	m	Times	Iteration
5	3	1.15	12
10	8	3.34	15
30	20	4.78	24
50	30	5.05	27
100	80	7.54	32

From the computational results reported in the above tables, one can see that the proposed algorithm works well for this class of implicit supply and demand Walras price equilibrium model.

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] **R.Agarwal, D.R. Sahu** (2000) *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer.

[2] **P. N. Anh, H. A. Le Thi** (2013) An Armijo-type method for pseudomonotone 1equilibrium problems and its applications, *J Glob. Optim.* 57:803820 DOI 10.1007/s10898-012-9970-8

[3] **P. N. Anh** (2024), Strong Quasi-nonexpansiveness of solution mappings of equilibrium problems, *Vietnam J. Math.* https://doi.org/10.1007/s10013-024-00697-9.

[4] **H. H. Bauschke, P.L.Combettes** (2011) *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, New York.

[5] **C. Berge** (1968) *Topological Spaces*, MacMillan, New York.

[6] **G. Bigi, , M. Castellani, M. Pappalardo, M. Passacantando** (2019) *Nonlinear Programming Techniques for Equilibria*, Springer.

[7] **F. Deutsch, I. Yamad**a(1998) Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, *Numer. Funct. Anal, Optim.*19,33–56.

[8] **B. V. Dinh, P. G. Hung and L. D. Muu** (2014) Bilevel optimization as a regularization approach to pseudomonotone equilibrium problems, *Numer. Funct. Anal. Optim.* 35, 539-563.

[9] **P. M. Duc, L. D. Muu** (2016) A splitting algorithm for a class of bilevel equilibrium problems involving nonexpansive mappings, *Optimization* 65, 1855-1866.

[10] **F. Facchinei, J.-S. Pang** (2003): *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. 1,2, Springer-Verlag, Berlin.

[11] **E.G. Goldstein, N. V. Tretyakov** (1996), *Modified Lagrangians and Monotone Maps in Optimization*, Wiley, New York, NY.

[12] **N.N. Hai, L.D. Muu, B. V. Dinh** (2023), An algorithm for quasi-convex equilibrium problems and asymptotically nonexpansive mappings: application to a Walras model with implicit supply–demand, *Math. Meth. Oper. Res.*,https://doi.org/10.1007/s00186-023-00837-w

[13] **I. V. Konnov** (2000) *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin.

[14] **J. K. Kim, P.N Anh, H.g. Hyun** (2012) A proximal point-type algorithm for pseudomonotone equilibrium problem, *Bull. Korean Math. Soc.* 49, 749-759.

[15] **I. V. Konnov** (2007): Economics Models and Variational Inequalities, *ELSEVIER*

[16] **L. D. Muu and T. D. Quoc** (2009), Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-Cournot equilibrium model. *Optim. Theory Appl.*,142, 185-204.

[17] **Quoc, T.D., Muu, L.D., Nguyen, V.H.** (2008): Extragradient algorithms extended to equilibrium problems. *Optim.*, 57, 749-776.

[18] R. T. Rockafellar (1970): *Convex Analysis.* Princeton University Press, Princeton, New Jersey.

[19] **R.T. Rockafellar** (1976), Monotone operator and the proximal point algorithm, *SIAM J. control and Optim.* 14, 877-898.

[20] **H.E. Scarf and T. Hansen** (1973) *Computation of Economic Equilibria*, Yale University Press, New Haven.

[21] **H. Tuy** (2016) *Convex Analysis and Global Optimization*, Springer, Berlin.

[22] **L. Walras** (1874) El´ements d'´Economie Politique Pure, L. Corbaz, Lausanne; English translation (1954) *Elements of Pure Economics*, Allen and Unwin, London.

[23] **HK. Xu** (2002) Iterative algorithms for nonlinear operators. *J. London Math. Soc.* 66, 240-256.

[24] **I. Yamada, N. Ogura** (2005) Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numer. Funct. Anal. Optim.* 25, 178-189.

[25] **D. Zhu, P. Marcotte** (1995) A new class of generalized monotonicity, *J. Optim. Theory Appl.* 67, 457-471.

Nguyen Ngoc Hai
Trade Union University
Hanoi
Vietnam
hainn@dhcd.edu.vn

Le Dung Muu
TIMAS, Thang Long University
Hanoi
Vietnam
ldmuu@math.ac.vn, muuld@thanglong.edu.vn