*J. Math. Math. Sci. (2024) 51–66*

# A self-adaptive step size algorithm for solving the split feasibility problem with multiple output sets in Hilbert spaces

**Tran Viet Anh** (Hanoi, Vietnam)

(Received Aug. 25, 2024)

**Abstract.** The purpose of this manuscript is to introduce a new self-adaptive algorithm for solving the split feasibility problem with multiple output sets in real Hilbert spaces. Our algorithm leverages information from previous steps to guide its execution, thereby removing the need to compute or estimate the norm of the given operator. Lastly, we present a simple numerical example to illustrate the performance of our proposed algorithm.

## 1. Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2^j$, for $j = 1, 2, \ldots, N$, denote real Hilbert spaces, and let $A_j : \mathcal{H}_1 \longrightarrow \mathcal{H}_2^j$, for $j = 1, 2, \ldots, N$, be bounded linear operators. Consider nonempty closed convex subset $C \subseteq \mathcal{H}_1$ and $Q_j \subseteq \mathcal{H}_2^j$ for $j = 1, 2, \ldots, N$, respectively. The split feasibility problem with multiple output sets (SFPMOS) is expressed as follows:

$$\text{Find } x^* \in C \text{ such that } A_j(x^*) \in Q_j \ \forall j = 1, 2, \ldots, N. \tag{1.1}$$

A particular case of the SFPMOS arises when $N = 1$, which corresponds to the split feasibility problem (SFP). The SFP is formulated as finding a point $x^* \in C$ such that $A(x^*) \in Q$, where $C$ and $Q$ are nonempty closed

*Key words and phrases*: Split feasibility problem with multiple output sets, split feasibility problem, self-adaptive, strong convergence.
*2020 Mathematics Subject Classification*: 49J40, 90C33, 47H17.

convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2 := \mathcal{H}_2^1$, respectively. Recently, it has been demonstrated that the SFP can serve as a practical model in intensity-modulated radiation therapy [11, 12, 14] and in various other real-world applications. To solve the MSSFP, the SFP, and their generalizations, numerous iterative projection methods have been developed. For more details, see [1–8, 13, 14, 16–18, 22, 25, 27, 28] and the references therein.

The SFPMOS was initially introduced and explored by Reich et al. [24]. In the same work, they also proposed the following algorithm to solve (1.1)

$$\begin{cases} x^0 \in C, \\ u_j^n = A_j(x^n),\ v_j^n = P_{Q_j}(u_j^n)\ \forall j = 1, 2, \ldots, N, \\ y^n = x^n + \delta_n \sum_{j=1}^{N} A_j^*(v_j^n - u_j^n), \\ z^n = P_C(y^n), \\ x^{n+1} = \alpha_n u + (1 - \alpha_n) z^n, \end{cases} \tag{1.2}$$

where $u \in C$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\delta_n\} \subset [a, b] \subset \Big(0, \dfrac{2}{N \max_{1\le j\le N}\{\|A_j\|^2\}}\Big)$. They proved that the sequence $\{x^n\}$ generated by (1.2) converges strongly to $P_{\Omega_{\text{SFPMOS}}}(u)$, provided that the solution set $\Omega_{\text{SFPMOS}} = \{x^* \in C : A_j(x^*) \in Q_j\ \forall j = 1, 2, \ldots, N\}$ of the SFPMOS is nonempty.

One approach to studying the problem (1.1) is through the fixed points of nonexpansive mappings. Specifically, let us define the mappings $T_j : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ for all $j = 1, 2, \ldots, N$ by $T_j = P_C(I^{\mathcal{H}_1} - \mu_j A_j^*(I^{\mathcal{H}_1} - P_{Q_j})A_j)$, where $I^{\mathcal{H}_1}$ is the identity mapping in $\mathcal{H}_1$ and $\mu_j \in \Big(0, \dfrac{2}{\|A_j\|^2}\Big)$. It can be shown that each $T_j$ is a nonexpansive mapping, and the fixed point set $\text{Fix}(T_j)$ of $T_j$ coincides the solution set of the split feasibility problem finding a point $x^* \in C$ such that $A_j(x^*) \in Q_j$. Therefore, the solution set $\Omega_{\text{SFPMOS}}$ of the SFPMOS is the intersection of the fixed point sets of $T_j$ for all $j = 1, 2, \ldots, N$, i.e. $\Omega_{\text{SFPMOS}} = \bigcap_{j=1}^{N} \text{Fix}(T_j)$. This result enables the application of established methods for solving the intersection of fixed point sets of nonexpansive mappings (see [10, 15, 19, 20, 26] and the references therein) to address problem (1.1).

In this manuscript, we present a different approach to solve the problem (1.1). The strong convergence of the proposed algorithm is established without the need for prior knowledge of the norms of the operators $A_j$. A significant limitation of the algorithm presented by Reich et al. in [24, Theorem 3.4] is that it requires prior knowledge of the norms of the operators $A_j$ to select the sequences $\{\delta_n\}$ that satisfy the condition $\{\delta_n\} \subset [a, b] \subset \left(0, \frac{2}{N \max\limits_{1 \le j \le N} \{\|A_j\|^2\}}\right)$. Similarly, the fixed-point approach based on nonexpansive mappings also relies on knowing the norms of the operators $A_j$ to choose $\mu_j \in \left(0, \frac{2}{\|A_j\|^2}\right)$ for each $j = 1, 2, \dots, N$. In contrast, our proposed algorithm does not require any prior knowledge of the norms of the operators $A_j$, making it more practical and easier to implement in real-world scenarios where such information may not be readily available. Furthermore, when the zero element is not in $C$, the algorithm presented by Reich et al. in [24, Theorem 3.4] cannot be used to find the minimum-norm solution of the SFPMOS. In contrast, our proposed algorithm can accomplish this because it allows the choice of $u$ as the zero element in the Hilbert space $\mathcal{H}_1$, whereas Reich et al.'s algorithm requires selecting $u$ from $C$ and cannot use the zero element if it is not in $C$.

The paper is structured as follows. Section 2 provides some essential definitions and preliminary results, which are then utilized in Section 3, where we present the algorithm and establish its strong convergence. Finally, Section 4 includes a numerical example to demonstrate the convergence of the proposed algorithm.

## 2. Preliminaries

Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. It is well-known that for all point $x \in \mathcal{H}$, there exists a unique point $P_C(x) \in C$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}. \tag{2.1}$$

The mapping $P_C : \mathcal{H} \longrightarrow C$ defined by (2.1) is called the metric projection of $\mathcal{H}$ onto $C$. It is well-known that $P_C$ is nonexpansive. Furthermore, the following inequalities hold for all $x \in \mathcal{H}$ and $y \in C$:

$$\langle x - P_C(x), y - P_C(x) \rangle \le 0, \tag{2.2}$$

$$\|P_C(x) - y\|^2 \le \|x - y\|^2 - \|x - P_C(x)\|^2. \tag{2.3}$$

**Definition 2.1.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. An operator $A^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with the property $\langle A(x), y \rangle = \langle x, A^*(y) \rangle$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, is called an adjoint operator.*

The adjoint operator of a bounded linear operator $A$ between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ always exists and is uniquely determined. Furthermore, $A^*$ is a bounded linear operator and $\|A^*\| = \|A\|$.

The following lemma will be needed in the proof of the main result of our paper.

**Lemma 2.1** (see [23]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that for any integer $m$, there exists an integer $p$ such that $p \geq m$ and $a_p \leq a_{p+1}$. Let $n_0$ be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define, for all integer $n \geq n_0$, by*

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_n \leq a_{n+1}\}.$$

*Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty} \tau(n) = \infty$ and the following inequalities are satisfied:*

$$a_{\tau(n)} \leq a_{\tau(n)+1},\ a_n \leq a_{\tau(n)+1}\ \forall n \geq n_0.$$

## 3. The algorithm and convergence analysis

In this section, we propose an algorithm with strong convergence for solving the problem (1.1). The algorithm is presented as follows.

**Algorithm 3.1.**
**Step 0.** Choose $u \in \mathcal{H}_1$ and sequences $\{\rho_n\} \subset [\underline{\rho}, \overline{\rho}] \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.
**Step 1.** Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.
**Step 2.** Compute $u_j^n = A_j(x^n)$ and $v_j^n = P_{Q_j}(u_j^n)$ for all $j = 1, 2, \ldots, N$.
**Step 3.** Choose $j_n$ such that

$$j_n = \text{argmax}\left\{\|v_j^n - u_j^n\| : j = 1, 2, \ldots, N\right\},$$

and set $u^n = u_{j_n}^n$, $v^n = v_{j_n}^n$, and $A = A_{j_n}$ then compute

$$y^n = x^n + \delta_n A^*(v^n - u^n),$$

where the step size $\delta_n$ is defined by

$$\delta_n = \begin{cases} \dfrac{\rho_n \|v^n - u^n\|^2}{\|A^*(v^n - u^n)\|^2} & \text{if } A^*(v^n - u^n) \neq 0, \\ 0 & \text{if } A^*(v^n - u^n) = 0. \end{cases}$$

**Step 4.** Compute $z^n = P_C(y^n)$ and

$$x^{n+1} = \alpha_n u + (1-\alpha_n) z^n.$$

**Step 5.** Set $n := n+1$, and go to **Step 2**.

The strong convergence of the sequence generated through Algorithm 3.1 is established by the following theorem.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of $\mathcal{H}_1$, and let $Q_j$ for $j = 1, 2, \ldots, N$ be $N$ nonempty closed convex subsets of $\mathcal{H}_2^j$, respectively. Then the sequence $\{x^n\}$ generated by Algorithm 3.1 converges strongly to an element $x^* \in \Omega_{SFPMOS}$, where $x^* = P_{\Omega_{SFPMOS}}(u)$, provided the solution set $\Omega_{SFPMOS} = \{x^* \in C : A_j(x^*) \in Q_j \ \forall i = 1, 2, \ldots, N\}$ of the SFPMOS is nonempty.*

**Proof.** The proof of the theorem is divided into several steps.
**Step 1.** We show that, for all $n \geq 0$

$$\frac{\underline{\rho}(1-\overline{\rho})}{1 + \sum_{k=1}^{N} \|A_k\|^2} \|v^n - u^n\|^2 \leq \|x^n - x^*\|^2 - \|y^n - x^*\|^2, \tag{3.1}$$

$$\|y^n - x^n\|^2 \leq \frac{\overline{\rho}^2}{\underline{\rho}(1-\overline{\rho})} \big(\|x^n - x^*\|^2 - \|y^n - x^*\|^2\big). \tag{3.2}$$

Since $x^* \in \Omega_{\text{SFPMOS}}$, we have $x^* \in C$ and $A_j(x^*) \in Q_j$ for all $j = 1, 2, \ldots, N$. Thanks to the nonexpansive property of $P_{Q_j}$ for all $j = 1, 2, \ldots, N$, we have

$$\|v_j^n - A_j(x^*)\| = \|P_{Q_j}(u_j^n) - P_{Q_j}(A_j(x^*))\| \leq \|u_j^n - A_j(x^*)\|. \tag{3.3}$$

From (3.3), we obtain, for all $n \geq 0$

$$\begin{aligned}
\langle x^n - x^*, A_j^*(v_j^n - u_j^n)\rangle &= \langle A_j(x^n - x^*), v_j^n - u_j^n\rangle \\
&= \langle u_j^n - A_j(x^*), v_j^n - u_j^n\rangle \\
&= \langle v_j^n - A_j(x^*), v_j^n - u_j^n\rangle - \|v_j^n - u_j^n\|^2 \\
&= \frac{1}{2}\Big[\big(\|v_j^n - A_j(x^*)\|^2 - \|u_j^n - A_j(x^*)\|^2\big) - \|v_j^n - u_j^n\|^2\Big] \\
&\leq -\frac{1}{2}\|v_j^n - u_j^n\|^2 \ \ \forall j = 1, 2, \ldots, N.
\end{aligned} \tag{3.4}$$

From (3.4), with $j = j_n$, we have

$$\langle x^n - x^*, A^*(v^n - u^n) \leq -\frac{1}{2}\|v^n - u^n\|^2 \ \ \forall n \geq 0. \tag{3.5}$$

Let us consider two cases.
**Case 1.** $A^*(v^n - u^n) = 0$. From (3.5), we imply $\|v^n - u^n\| = 0$. Since $\delta_n = 0$, we get $y^n = x^n$. Thus, (3.1) and (3.2) are proven.
**Case 2.** $A^*(v^n - u^n) \neq 0$.
We have

$$\begin{aligned}\|A^*(v^n - u^n)\|^2 &\le \|A^*\|^2\|v^n - u^n\| = \|A\|^2\|v^n - u^n\|^2 \\ &\le \Big(1 + \sum_{k=1}^{N}\|A_k\|^2\Big)\|v^n - u^n\|^2. \qquad (3.6)\end{aligned}$$

It follows from (3.5) and (3.6), for all $n \ge 0$, that

$$\begin{aligned}\|y^n - x^*\|^2 &= \|(x^n - x^*) + \delta_n A^*(v^n - u^n)\|^2 \\ &= \|x^n - x^*\|^2 + \|\delta_n A^*(v^n - u^n)\|^2 + 2\delta_n\langle x^n - x^*, A^*(v^n - u^n)\rangle \\ &\le \|x^n - x^*\|^2 + \delta_n^2\|A^*(v^n - u^n)\|^2 - \delta_n\|v^n - u^n\|^2 \\ &= \|x^n - x^*\|^2 - \frac{\rho_n(1-\rho_n)\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \\ &\le \|x^n - x^*\|^2 - \frac{\underline{\rho}(1-\overline{\rho})\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \qquad (3.7) \\ &\le \|x^n - x^*\|^2 - \frac{\underline{\rho}(1-\overline{\rho})}{1 + \sum_{k=1}^{N}\|A_k\|^2}\|v^n - u^n\|^2.\end{aligned}$$

Therefore, (3.1) is proven.
From (3.7), we get

$$\begin{aligned}\|y^n - x^n\|^2 = \delta_n^2\|A^*(v^n - u^n)\|^2 &= \frac{\rho_n^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \\ &\le \frac{\overline{\rho}^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \\ &\le \frac{\overline{\rho}^2}{\underline{\rho}(1-\overline{\rho})}\big(\|x^n - x^*\|^2 - \|y^n - x^*\|^2\big).\end{aligned}$$

Thus, (3.2) holds.
**Step 2.** The sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ are bounded.
Using property (2.3) of the metric projection, we get

$$\begin{aligned}\|z^n - x^*\|^2 &= \|P_C(y^n) - x^*\|^2 \\ &\le \|y^n - x^*\|^2 - \|P_C(y^n) - y^n\|^2 \\ &= \|y^n - x^*\|^2 - \|z^n - y^n\|^2.\end{aligned}$$

Therefore

$$0 \le \|z^n - y^n\|^2 \le \|y^n - x^*\|^2 - \|z^n - x^*\|^2. \tag{3.8}$$

Based on (3.2) and (3.8), we obtain

$$\|z^n - x^*\| \le \|y^n - x^*\| \le \|x^n - x^*\| \ \forall n \ge 0. \tag{3.9}$$

From (3.9), we have

$$\begin{aligned} \|x^{n+1} - x^*\| &= \|(1-\alpha_n)(z^n - x^*) + \alpha_n(u - x^*)\| \\ &\le (1-\alpha_n)\|z^n - x^*\| + \alpha_n\|u - x^*\| \\ &\le (1-\alpha_n)\|x^n - x^*\| + \alpha_n\|u - x^*\|. \end{aligned} \tag{3.10}$$

This implies that

$$\|x^{n+1} - x^*\| \le \max\{\|x^n - x^*\|, \|u - x^*\|\} \ \ \forall n \ge 0.$$

So, by induction, we obtain, for every $n \ge 0$ that

$$\|x^n - x^*\| \le \max\{\|x^0 - x^*\|, \|u - x^*\|\}.$$

Hence, the sequence $\{x^n\}$ is bounded and so are the sequences $\{y^n\}$ and $\{z^n\}$ thanks to (3.9).

**Step 3.** We prove that $\{x^n\}$ converges strongly to $x^*$.

We have

$$\begin{aligned} \|x^{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n)z^n - x^*\|^2 \\ &= \|z^n - x^* + \alpha_n(u - z^n)\|^2 \\ &= \|z^n - x^*\|^2 + 2\alpha_n\langle u - z^n, z^n - x^*\rangle + \alpha_n^2\|z^n - u\|^2, \end{aligned} \tag{3.11}$$

which together with (3.9) implies, for all $n \ge 0$

$$\begin{aligned} 0 &\le \|y^n - x^*\|^2 - \|z^n - x^*\|^2 \\ &\le \|x^n - x^*\|^2 - \|z^n - x^*\|^2 \\ &= (\|x^n - x^*\|^2 - \|x^{n+1} - x^*\|^2) + 2\alpha_n\langle u - z^n, z^n - x^*\rangle \\ &\qquad + \alpha_n^2\|z^n - u\|^2. \end{aligned} \tag{3.12}$$

Let us consider two cases.

**Case 1.** There exists $n_0 \ge 0$ such that $\{\|x^n - x^*\|\}$ is decreasing for $n \ge n_0$. In this case, the limit of $\{\|x^n - x^*\|\}$ exists and we denote $\lim\limits_{n\to\infty}\|x^n - x^*\|^2 = \xi \ge 0$. Therefore, it follows from (3.12), $\lim\limits_{n\to\infty}\alpha_n = 0$ and the boundedness of $\{z^n\}$ that

$$\lim_{n\to\infty}(\|y^n - x^*\|^2 - \|z^n - x^*\|^2) = 0, \quad \lim_{n\to\infty}(\|x^n - x^*\|^2 - \|z^n - x^*\|^2) = 0. \tag{3.13}$$

It implies from (3.13) that

$$\lim_{n\to\infty}(\|x^n - x^*\|^2 - \|y^n - x^*\|^2) = 0. \tag{3.14}$$

It follows from (3.1) and (3.14) that $\lim_{n\to\infty}\|v^n - u^n\| = 0$. From the definition of $j_n$, we get $\|v_j^n - u_j^n\| \leq \|v^n - u^n\|$ for all $j = 1, 2, \ldots, N$. This together with $\lim_{n\to\infty}\|v^n - u^n\| = 0$ implies

$$\lim_{n\to\infty}\|v_j^n - u_j^n\| = 0 \ \ \forall j = 1, 2, \ldots, N. \tag{3.15}$$

Using (3.2) and (3.14), we get

$$\lim_{n\to\infty}\|y^n - x^n\| = 0. \tag{3.16}$$

Based on (3.8) and (3.13), we obtain

$$\lim_{n\to\infty}\|z^n - y^n\| = 0. \tag{3.17}$$

From (3.16) and (3.17), we get

$$\|z^n - x^n\| \leq \|z^n - y^n\| + \|y^n - x^n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Now, we prove that

$$\limsup_{n\to\infty}\langle u - x^*, z^n - x^*\rangle \leq 0. \tag{3.19}$$

Choose a subsequence $\{z^{n_k}\}$ of $\{z^n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, z^n - x^*\rangle = \lim_{k\to\infty}\langle u - x^*, z^{n_k} - x^*\rangle.$$

Since $\{z^{n_k}\}$ is bounded, we may assume that $\{z^{n_k}\}$ converges weakly to some $\overline{z} \in \mathcal{H}_1$.
Therefore

$$\limsup_{n\to\infty}\langle u - x^*, z^n - x^*\rangle = \langle u - x^*, \overline{z} - x^*\rangle. \tag{3.20}$$

Since $C$ is closed and convex, it is also weakly closed. So, from $\{z^{n_k} = P_C(y^{n_k})\} \subset C$ and $z^{n_k} \rightharpoonup \overline{z}$, it follows that $\overline{z} \in C$.
From $z^{n_k} \rightharpoonup \overline{z}$ and (3.18), it follows that $x^{n_k} \rightharpoonup \overline{z}$. Morever, since each $A_j$ is a bounded linear operator and $x^{n_k} \rightharpoonup \overline{z}$, we have $u_j^{n_k} = A_j(x^{n_k}) \rightharpoonup A_j(\overline{z})$. This together with (3.15) imply $v_j^{n_k} \rightharpoonup A_j(\overline{z})$ for all $j = 1, 2, \ldots, N$. Since $\{v_j^{n_k}\} \subset Q_j$ and $Q_j$ is weakly closed then $A_j(\overline{z}) \in Q_j$ for all $j = 1, 2, \ldots, N$. Thus, from $\overline{z} \in C$ and $A_j(\overline{z}) \in Q_j$ for all $j = 1, 2, \ldots, N$, we get $\overline{z} \in \Omega_{\text{SFPMOS}}$.

Combining this with $x^* = P_{\Omega_{\text{SFPMOS}}}(u)$ and the property (2.2) of the metric projection, we obtain $\langle u - x^*, \overline{z} - x^*\rangle \le 0$. So, from (3.20), we have $\limsup_{n\to\infty}\langle u - x^*, z^n - x^*\rangle \le 0$. Thus, (3.19) is proven.
It follows from $\lim_{n\to\infty} \|x^n - x^*\|^2 = \xi$ and (3.13) that

$$\lim_{n\to\infty} \|z^n - x^*\|^2 = \xi. \tag{3.21}$$

From $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{z^n\}$, (3.19) and (3.21), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}(2\langle u - z^n, z^n - x^*\rangle + \alpha_n\|z^n - u\|^2)\\
&= 2\limsup_{n\to\infty}\langle u - z^n, z^n - x^*\rangle\\
&= 2\limsup_{n\to\infty}[\langle u - x^*, z^n - x^*\rangle - \|z^n - x^*\|^2] \le -2\xi.
\end{aligned} \tag{3.22}$$

Assume, to get a contradiction, that $\xi > 0$, and choose $\varepsilon = \xi > 0$. It follows from (3.22) that there exists $n_1 \ge 0$ such that

$$2\langle u - z^n, z^n - x^*\rangle + \alpha_n\|z^n - u\|^2 \le -2\xi + \xi = -\xi \ \ \forall n \ge n_1. \tag{3.23}$$

From (3.9) and (3.11), we get

$$\|x^{n+1} - x^*\|^2 \le \|x^n - x^*\|^2 + \alpha_n\left[2\langle u - z^n, z^n - x^*\rangle + \alpha_n\|z^n - u\|^2\right] \ \ \forall n \ge 0,$$

which together with (3.23) implies

$$\|x^{n+1} - x^*\|^2 - \|x^n - x^*\|^2 \le -\alpha_n\xi \ \ \forall n \ge n_1.$$

Thus, after a summation, we obtain

$$\|x^{n+1} - x^*\|^2 - \|x^{n_1} - x^*\|^2 \le -\xi\left(\sum_{j=n_1}^{n}\alpha_j\right) \ \ \forall n \ge n_1.$$

Therefore, we arrive at a contradiction

$$\xi\left(\sum_{j=n_1}^{n}\alpha_j\right) \le \|x^{n_1} - x^*\|^2 \ \ \forall n \ge n_1$$

because of $\sum_{n=0}^{\infty}\alpha_n = \infty$. Hence $\xi = 0$, which implies $x^n \to x^*$.

**Case 2.** Suppose that for any integer $m$, there exists an integer $n$ such that $n \ge m$ and $\|x^n - x^*\| \le \|x^{n+1} - x^*\|$. According to Lemma 2.1, there exists a nondecreasing sequence $\{\tau(n)\}_{n\geq n_2}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}\tau(n)=\infty$ and the following inequalities hold

$$\|x^{\tau(n)}-x^*\|\leq\|x^{\tau(n)+1}-x^*\|,\quad \|x^n-x^*\|\leq\|x^{\tau(n)+1}-x^*\|\ \ \forall n\geq n_2. \tag{3.24}$$

From (3.24) and (3.10), we get

$$\begin{aligned}\|x^{\tau(n)}-x^*\|&\leq\|x^{\tau(n)+1}-x^*\|\\ &\leq(1-\alpha_{\tau(n)})\|z^{\tau(n)}-x^*\|+\alpha_{\tau(n)}\|u-x^*\|\ \ \forall n\geq n_2.\end{aligned} \tag{3.25}$$

From (3.25), we have

$$\|x^{\tau(n)}-x^*\|-\|z^{\tau(n)}-x^*\|\leq\alpha_{\tau(n)}\|u-x^*\|-\alpha_{\tau(n)}\|z^{\tau(n)}-x^*\|\ \ \forall n\geq n_2,$$

which together with (3.9) implies, for all $n\geq n_2$, that

$$\begin{aligned}\alpha_{\tau(n)}\|u-x^*\|-\alpha_{\tau(n)}\|z^{\tau(n)}-x^*\|&\geq\|x^{\tau(n)}-x^*\|-\|z^{\tau(n)}-x^*\|\\ &\geq\|x^{\tau(n)}-x^*\|-\|y^{\tau(n)}-x^*\|\\ &\geq 0.\end{aligned}$$

Then, it follows from the above inequality, the boundedness of $\{z^n\}$ and $\lim_{n\to\infty}\alpha_n=0$ that

$$\begin{aligned}&\lim_{n\to\infty}(\|x^{\tau(n)}-x^*\|-\|z^{\tau(n)}-x^*\|)=0,\\ &\lim_{n\to\infty}(\|x^{\tau(n)}-x^*\|-\|y^{\tau(n)}-x^*\|)=0.\end{aligned} \tag{3.26}$$

From (3.26) and the boundedness of $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$, we obtain

$$\lim_{n\to\infty}(\|x^{\tau(n)}-x^*\|^2-\|z^{\tau(n)}-x^*\|^2)=0,$$

$$\lim_{n\to\infty}(\|x^{\tau(n)}-x^*\|^2-\|y^{\tau(n)}-x^*\|^2)=0.$$

Arguing similarly as in the first case, we can conclude that

$$\limsup_{n\to\infty}\langle u-x^*,z^{\tau(n)}-x^*\rangle\leq 0.$$

Then, the boundedness of $\{z^n\}$ and $\lim_{n\to\infty}\alpha_n=0$ yield

$$\begin{aligned}\limsup_{n\to\infty}\langle u-x^*,x^{\tau(n)+1}-x^*\rangle&=\limsup_{n\to\infty}\langle u-x^*,z^{\tau(n)}-x^*+\alpha_{\tau(n)}(u-z^{\tau(n)})\rangle\\ &=\limsup_{n\to\infty}\langle u-x^*,z^{\tau(n)}-x^*\rangle\leq 0.\end{aligned} \tag{3.27}$$

Using the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \ \ \forall x, y \in \mathcal{H}_1,$$

as well as (3.9) and (3.24), we obtain, for all $n \geq n_2$

$$\begin{aligned}\|x^{\tau(n)+1} - x^*\|^2 &= \|(1 - \alpha_{\tau(n)})(z^{\tau(n)} - x^*) + \alpha_{\tau(n)}(u - x^*)\|^2 \\ &\leq \|(1 - \alpha_{\tau(n)})(z^{\tau(n)} - x^*)\|^2 + 2\langle \alpha_{\tau(n)}(u - x^*), x^{\tau(n)+1} - x^*\rangle \\ &= (1 - \alpha_{\tau(n)})^2\|z^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x^{\tau(n)+1} - x^*\rangle \\ &\leq (1 - \alpha_{\tau(n)})\|z^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x^{\tau(n)+1} - x^*\rangle \\ &\leq (1 - \alpha_{\tau(n)})\|x^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x^{\tau(n)+1} - x^*\rangle \\ &\leq (1 - \alpha_{\tau(n)})\|x^{\tau(n)+1} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x^{\tau(n)+1} - x^*\rangle.\end{aligned}$$

In particular, since $\alpha_{\tau(n)} > 0$

$$\|x^{\tau(n)+1} - x^*\|^2 \leq 2\langle u - x^*, x^{\tau(n)+1} - x^*\rangle \ \ \forall n \geq n_2.$$

Combining the above inequality with (3.24), we get

$$\|x^n - x^*\|^2 \leq 2\langle u - x^*, x^{\tau(n)+1} - x^*\rangle \ \ \forall n \geq n_2. \tag{3.28}$$

Taking the limit in (3.28) as $n \to \infty$, and using (3.27), we obtain

$$\limsup_{n\to\infty} \|x^n - x^*\|^2 \leq 0,$$

which implies $x^n \to x^*$. This complete the proof of Theorem 3.1. ■

**Remark 3.1.** We highlight the advantages of Algorithm 3.1 compared to the algorithm of Reich et al. in [24, Theorem 3.4].

i) In Algorithm 3.1, unlike the result in [24, Theorem 3.4], the step size is selected in such a way that its implementation does not require any prior knowledge of the norms of the given bounded linear operators.

ii) When $u = 0 \in \mathcal{H}_1$, our algorithm becomes the one for finding the minimum-norm solution of the SFPMOS. However, in the algorithm of Reich et al. in [24, Theorem 3.4], we are required to choose $u \in C$, and $u = 0$ cannot be selected if $0 \notin C$.

## 4. Numerical illustrations

**Example 4.1.** We analyze the SFPMOS with

$$\begin{aligned} C &= \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 - x_2 + x_3 \geq 4\}, \\ Q_1 &= \{(u_1, u_2)^T \in \mathbb{R}^2 : u_1 + u_2 \geq 4\}, \\ Q_2 &= \{(u_1, u_2, u_3)^T \in \mathbb{R}^3 : u_1 + u_2 + 2u_3 \leq 5\}, \\ Q_3 &= \{(u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4 : -2u_1 + u_2 + u_3 - u_4 \geq -6\}, \\ Q_4 &= \{(u_1, u_2, u_3, u_4, u_5)^T \in \mathbb{R}^5 : u_1 - 3u_2 - 2u_3 + u_4 - u_5 \geq -24\}. \end{aligned}$$

We also consider the bounded linear operators $A_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $A_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $A_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, and $A_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}^5$, defined by $A_1(x) = M_1 x$, $A_2(x) = M_2 x$, $A_3(x) = M_3 x$, $A_4(x) = M_4 x$ for all $x \in \mathbb{R}^3$, where

$$M_1 = \begin{pmatrix} 3 & 1 & -6 \\ 1 & 1 & -2 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & -6 \end{pmatrix}, \; M_4 = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}.$$

The solutions set of the SFPMOS is given by

$$\begin{aligned} \Omega_{\text{SFPMOS}} &= \{x = (x_1, x_2, x_3)^T \in C : A_j(x) \in Q_j \; \forall j = 1, 2, 3, 4\} \\ &= \left\{ (t + 2, 2t - 2, t)^T : t \leq \frac{3}{10} \right\}. \end{aligned}$$

Suppose $x = (t + 2, 2t - 2, t)^T \in \Omega_{\text{SFPMOS}}$. Then we have

$$\begin{aligned} \|x\| &= \sqrt{(t+2)^2 + (2t-2)^2 + t^2} \\ &= \sqrt{6\left(t - \frac{3}{10}\right)^2 + \frac{2}{5}\left(\frac{3}{10} - t\right) + \frac{734}{100}} \\ &\geq \frac{\sqrt{734}}{10}. \end{aligned}$$

Equality holds if and only if $t = \dfrac{3}{10}$. Therefore, the minimum-norm solution of the SFPMOS is given by $x^* = \left(\dfrac{23}{10}, -\dfrac{14}{10}, \dfrac{3}{10}\right)^T$.

We begin by selecting a starting point $x^0 \in \mathbb{R}^3$, where the elements of $x^0$ are randomly generated within the closed interval $[-10, 10]$. In Algorithm 3.1, we define $\rho_n = 1 - 10^{-3}$ and the sequence $\{\alpha_n\}$ as $\alpha_n = \dfrac{1}{n+2}$, for $n = 0, 1, 2, \ldots$. The stopping criterion is set as $\|x^{n+1} - x^n\| \leq \varepsilon$. Table 1 displays the numerical results of Algorithm 3.1 with varying tolerances. From the numerical results presented, we observe that the approximate solution $x^n$ is a good approximation to the minimum-norm solution $x^* = \left(\dfrac{23}{10}, -\dfrac{14}{10}, \dfrac{3}{10}\right)^T$.

*Table* 1. Algorithm 3.1 for Example 4.1, with different tolerances and $\rho_n = 1 - 10^{-3}$, $\alpha_n = \dfrac{1}{n+2}$

| Tolerance | Iter($n$) | Time(s) | $x^n$ |
|---|---|---|---|
| $\varepsilon = 10^{-4}$ | 721 | 0.0783 | $(2.29962586, -1.3901264, 0.30470758)^T$ |
| $\varepsilon = 10^{-5}$ | 8090 | 0.899 | $(2.29991637, -1.39921942, 0.30036984)^T$ |
| $\varepsilon = 10^{-6}$ | 81574 | 8.72283 | $(2.29999252, -1.39992094, 0.3000375)^T$ |

We conducted the iterative schemes in Python, utilizing version 3.11, on a 2017 MacBook Pro featuring a 2.3 GHz Intel Core i5 processor, an Intel Iris Plus Graphics 640 graphics card with 1536 MB of memory, and 8 GB of 2133 MHz LPDDR3 RAM.

## 5. Conclusion

This paper presents a self-adaptive step size algorithm for solving the split feasibility problem with multiple output sets in real Hilbert spaces. Under certain conditions on the parameters, we have established a strong convergence theorem for the algorithm, which does not require the computation or estimation of the norms of the given bounded linear operators. A simple numerical example is provided to illustrate the effectiveness of the proposed algorithm.

## References


[1] **Anh P.K., Anh T.V. and Muu, L.D.,** On bilevel split pseudomonotone variational inequality problems with applications. *Acta Math, Vietnam*, **42**, 413-429 (2017)

[2] **Anh T.V.,** An extragradient method for finding minimum-norm solution of the split equilibrium problem. *Acta Math. Vietnam*, **42**, 587-604 (2017).

[3] **Anh T.V.,** A parallel method for variational inequalities with the multiple-sets split feasibility problem constraints. *J. Fixed Point Theory Appl.* **19**, 2681-2696 (2017).

[4] **Anh T.V.,** Linesearch methods for bilevel split pseudomonotone variational inequality problems. *Numer. Algorithms* **81**, 1067-1087 (2019).

[5] **Anh T.V. and Muu L.D.,** A projection-fixed point method for a class of bilevel variational inequalities with split fixed point constraints. *Optimization* **65**, 1229-1243 (2016).

[6] **Buong N.,** Iterative algorithms for the multiple-sets split feasibility problem in Hilbert spaces. *Numer. Algorithms* **76**, 783-798 (2017).

[7] **Byrne C.,** Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **18**, 441-453 (2002).

[8] **Byrne C., Censor Y., Gibali A. and Reich S.,** The split common null point problem. *J. Nonlinear Convex Anal.* **13**, 759-775 (2012).

[9] **Ceng L.C., Ansari Q.H. and Yao J.C.,** Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem. *Nonlinear Anal.* **75**, 2116-2125 (2012).

[10] **Ceng L.C., Petruşel A., Qin X. and Yao J.C.,** Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints. *Optimization* **70**, 1337-1358 (2021).

[11] **Censor Y., Bortfeld T., Martin B. and Trofimov A.,** A unified approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **51**, 2353-2365 (2006).

[12] **Censor Y. and Elfving T.,** A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **8**, 221-239 (1994).

[13] **Censor Y., Elfving T., Kopf N. and Bortfeld T.,** The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Prob.* **21**, 2071-2084 (2005).

[14] **Censor Y. and Segal A.,** Iterative projection methods in biomedical inverse problems, in: Y. Censor, M. Jiang, A.K. Louis (Eds.), *Mathematical Methods in Biomedical Imaging and Intensity-Modulated Therapy*, IMRT, Edizioni della Norale, Pisa, Italy, 2008, pp. 65-96.

[15] **Eslamian M.,** A hierarchical variational inequality problem for generalized demimetric mappings with applications. *J. Nonlinear Var. Anal.* **5**, 965-979 (2021).

[16] **Hai N.M., Van L.H.M. and Anh T.V.,** An Algorithm for a Class of Bilevel Variational Inequalities with Split Variational Inequality and Fixed Point Problem Constraints. *Acta Math. Vietnam*, **46**, 515–530 (2021).

[17] **Huy P.V., Hien N.D. and Anh T.V.,** A strongly convergent modified Halpern subgradient extragradient method for solving the split variational inequality problem. *Vietnam J. Math.* **48**, 187-204 (2020).

[18] **Huy P.V., Van L.H.M., Hien N.D. and Anh T.V.,** Modified Tseng's extragradient methods with self-adaptive step size for solving bilevel split variational inequality problems. *Optimization* **71**, 1721-1748 (2022).

[19] **Iiduka H.,** Fixed point optimization algorithm and its application to power control in CDMA data networks. *Math. Program.* **133**, 227-242 (2012).

[20] **Iiduka H.,** Fixed point optimization algorithm and its application to network bandwidth allocation. *J. Comput. Appl. Math.* **236**, 1733-1742 (2012).

[21] **Konnov I.V.,** *Combined Relaxation Methods for Variational Inequalities.* Springer, Berlin (2000).

[22] **Liu B., Qu B. and Zheng N.,** A Successive Projection Algorithm for Solving the Multiple-Sets Split Feasibility Problem. *Numer. Funct. Anal. Optim.* **35**, 1459-1466 (2014).

[23] **Maingé P.E.,** A hybrid extragradient-viscosity method for monotone operators and fixed point problems. *SIAM J. Control Optim.* **47**, 1499-1515 (2008).

[24] **Reich S., Truong M.T. and Mai T.N.H.,** The split feasibility problem with multiple output sets in Hilbert spaces. *Optim. Lett.* **14**, 2335–2353 (2020).

[25] **Wen M., Peng J.G. and Tang Y.C.,** A cyclic and simultaneous iterative method for solving the multiple-sets split feasibility problem. *J. Optim. Theory Appl.* **166**(3), 844-860 (2015).

[26] **Yamada I. and Ogura N.,** Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings. *Numer. Funct. Anal. Optim.* **25**, 619-655 (2004).

[27] **Zhao J.L. and Yang Q.Z.,** A simple projection method for solving the multiple-sets split feasibility problem. *Inverse Probl. Sci. Eng.* **21**, 537-546 (2013).

[28] **Zhao J.L., Zhang Y.J. and Yang Q.Z.,** Modified projection methods for the split feasibility problem and the multiple-sets split feasibility problem. *Appl. Math. Comput.* **219**, 1644-1653 (2012).

**Tran Viet Anh**
Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology
Hanoi
Vietnam
tranvietanh@outlook.com; tvanh@ptit.edu.vn