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On invariants of plane curve singularities in positive characteristic

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Abstract. In this survey paper we give an overview on some aspects of singularities of algebraic plane curves over an algebraically closed field of arbitrary characteristic. We review, in particular, classical results and recent developments on invariants of plane curve singularities.

1. Introduction

The study of plane curve singularities started with fundamental work of Heisuke Hironaka on the resolution of singularities ([20]1964), Oskar Zariski's studies in equisingularity ([30] 1965-1968), Michael Artin's paper on isolated rational singularities of surfaces ([4] 1966), and the work by René Thom, Bernard Malgrange, John Mather,... on singularities of differentiable mappings. It culminated in the 1970ties and 1980ties with the work of John Milnor and Pierre Deligne, who intorduced what is now called the Milnor fibration, Milnor number and the Milnor's formula ([23] 1968, [11] 1973), Egbert Brieskorn's discovery of exotic spheres as neighborhood boundaries of isolated hypersurface singularities (1966) and the connection to Lie groups (1971), Vladimir Arnold's classification of singularities ([1, 2, 3] 1972-1976), and many others, e.g. Andrei Gabrielov, Sabir Gusein-Zade, Ignaciao Luengo, Seidenberg, Walker, Antonio Campillo, C.T.C. Wall, A. Melle-Hernández, Johnatan Wahl, Le Dung Trang, Bernard Teissier, Dierk Siersma, Joseph Steenbrink, Gert-Martin Greuel, Yousra Boubakri, Thomas Markwig, Félix Delgado de la

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Mata, P. Cassou-Noguès, E. Garcıa Barroso, Arkadiusz Ploski, Hefez Abramo, Olmedo Rodrigues, Rodrigo Salomão ... (see [8, 9, 28, 27, 12, 29, 24, 10, 7, 13, 14, 18]). In this survey paper we give an overview on some aspects of singularities of algebraic plane curves over an algebraically closed field of arbitrary characteristic. We review, in particular, classical results and recent developments on invariants of plane curve singularities that should serve as a quick guide to references.

In this note, by a plane curve singularity we mean a non-unit formal power series in $\mathbb{k}[\![x,y]\!]$. Invariants of a plane curve singularity f will be quantities (e.g. integers) associated to f which is stabe in right or contact equivalent classes. Recall that two plane curve singularities f and g are right equivalent if $f = \Phi(g)$ for some automorphism of local $\mathbb{k}$-algebra $\mathbb{k}[\![x,y]\!]$. They are called contact equivalent if $f = u \cdot \Phi(g)$ for some automorphism of local $\mathbb{k}$-algebra $\mathbb{k}[\![x,y]\!]$ and for some unit $u \in \mathbb{k}[\![x,y]\!]$. We denote by $f \sim_r g$ and by $f \sim_c g$ respectively.

We study classical invariants of plane curve singularities such as multiplicity, Milnor number ($\mu(f)$), delta and kappa invariants ($\delta(f), \kappa(f)$), semi-group ($S(f)$) and their relations. Especially we are interested in studying the Milnor formula in positive characteristic, which states, in characteristic zero, that for any reduced plane curve singularity f

$$\mu(f) = 2\delta(f) - r(f) + 1, \tag{1.1}$$

where $r(f)$ denotes the number of branches of f. More precisely, we give some partial answers to the following problem.

Problem 1. *Is there at least a "reasonable" characterization of those plane curve singularities such that Equation 1.1 holds?*

2. Preliminaries

2.1. Resolution of singularities

Let $0 \neq f \in \mathfrak{m} \subset \mathbb{k}[\![x,y]\!]$. Then $R := R_f = \mathbb{k}[\![x,y]\!]/(f)$ (or f) is called a *plane curve singularity*. There is a unique (up to multiplication with units) decomposition

$$f = f_1^{\rho_1} \cdot \ldots \cdot f_r^{\rho_r},$$

with $f_i \in \mathfrak{m}$ irreducible in $\mathbb{k}[\![x,y]\!]$ and $\rho_i \geq 1$ for all $i = 1, \ldots, r$. The series f_i resp. the rings R_{f_i} are called the *branches* of f resp. of R_f. The plane curve singularity f is said *reduced* if $\rho_i = 1$ for $i = 1, \ldots, r$. It is *irreducible* if it reduced and $r = 1$. Recall that the multiplicity of f, denoted by $\mathrm{mt}(f)$, is the minimal degree of the homogeneous part of f. So

$$f = \sum_{k \geq m := \mathrm{mt}(f)} f_k(x, y),$$

where f_k is is either zero or a homogeneous polynomial of degree k and $f_m \neq 0$. Then f_m is decomposed into linear factors,

$$f_m = \prod_{i=1}^{s} (\alpha_i x - \beta_i y)^{r_i},$$

with $(\beta_i : \alpha_i) \in \mathbb{P}^1$ pairwise distinct. We call f_m the *tangent cone* of f. The points $P_i := (\beta_i : \alpha_i), i = 1, \ldots, s$, are the *tangent directions* or the *infinitely near points in the* 1*-st neighbourhood of* 0 of f. For each i, the number r_i is called the multiplicity of P_i, and denoted by m_{P_i}. Note that $m = r_1 + \ldots + r_s$.

For each tangent direction $P := (\beta : \alpha)$ of f, we define a morphism $\iota_P \colon \mathbb{k}[\![x, y]\!] \to \mathbb{k}[\![x_P, y_P]\!]$ and a series $f_P \in \mathbb{k}[\![x_P, y_P]\!]$ as follows

- if $\alpha \neq 0$ then

$$\iota_P(x) = \frac{x_P y_P + \beta y_P}{\alpha}, \iota_P(y) = y_P, \text{ and } \iota_P(f) = y_P^m f_P$$

- if $\alpha = 0$ then

$$\iota_P(x) = x_P, \iota_P(y) = \frac{\alpha x_P + x_P y_P}{\beta}, \text{ and } \iota_P(f) = x_P^m f_P.$$

The series f_P is called the *local equation of the strict transform of* f *at* P. For each $n \geq 1$, if P is an infinitely near points in the n-th neighbourhood of 0, and if Q is a tangent direction of $f_P(x_P, y_P)$, then Q is called *an infinitely near point in the* $(n+1)$*-th neighbourhood of* 0, denoted by $Q \overset{n+1}{\to} 0$ or simply $Q \to 0$. We also denote by $f_Q(x_Q, y_Q)$ the local equation of the strict transform of f_P at Q. Note that, by definition, if $Q \to P$ then $\mathrm{mt}(f_Q) \leq \mathrm{mt}(f_P)$. The following lemma can be proved easily by using induction.

Lemma 2.1. *Let $f, g \in \mathbb{k}[\![x, y]\!]$ be plane curve singularities and let P be a tangent direction of f of multiplicity r. Then*

(i) If $m = \mathrm{mt}(f) = 1$, then $\mathrm{mt}(f_P) = 1$ for all $P \to 0$.

(ii) We have $\mathrm{mt}(f_P) \leq r \leq m$. In particular, if f has at least two tangent directions, then $\mathrm{mt}(f_P) < m$.

(iii) Assume g is an irreducible component of f. Then if $Q \stackrel{n}{\to} 0$ for g then $Q \stackrel{n}{\to} 0$ for f, and moreover, g_Q is an irreducible component of f_Q.

We denote by $R^{(n)}$ the ring

$$R^{(n)} := R_f^{(n)} := \bigoplus_{Q \stackrel{n}{\to} 0} \mathbb{k}[\![x_Q, y_Q]\!]/f_Q(x_Q, y_Q),$$

and call it the *n-th strict transform of f*. Then we have the following inclusions

$$R = R^{(0)} \hookrightarrow R^{(1)} \hookrightarrow \ldots \hookrightarrow R^{(n)} \to \ldots \tag{2.1}$$

defined inductively as

$$R = R^{(0)} \hookrightarrow R^{(1)} = \bigoplus_{P \stackrel{1}{\to} 0} \mathbb{k}[\![x_P, y_P]\!]/f_P(x_P, y_P), g \mapsto \oplus \iota_P(g).$$

Theorem 2.1. *Let $f \in \mathbb{k}[\![x, y]\!]$ be a reduced plane curve singularities. Then*

(i) the sequence of injective morphisms (2.1) stabilizes. More precisely, there exists $k \geq 1$ such that

$$R^{(n)} \cong \bigoplus_{i=1}^{r} \mathbb{k}[\![t]\!],$$

for all $n \geq k$;

(ii) the morphisms $R^{(i)} \hookrightarrow R^{(i+1)}$ are integral extensions in the quotient ring $Q(R)$ of R;

(iii) the ring

$$R^{(n)} \cong \bigoplus_{i=1}^{r} \mathbb{k}[\![t]\!],$$

for all $n \gg 1$ is the integral closure of R, is also called the normalization *of R and denoted by $\bar{R}$.*

Proposition 2.2. *Any plane curve singularity $f \in \mathbb{k}[\![x, y]\!]$ can be factorized as*

$$f = \prod_{P \stackrel{1}{\to} 0} \bar{f}_P$$

in $\mathbb{k}[\![x, y]\!]$ such that $\bar{f}_P$ has a unique tangent direction, and the $\bar{f}_P$ are pairwise coprime.

In particular, if $f \in \mathfrak{m} \subset \mathbb{k}[\![x, y]\!]$ is irreducible, then f has a unique tangent direction.

2.2. Parametrization equivalence

Definition 2.1. Let $0 \neq f \in \mathfrak{m} \subset \Bbbk[\![x, y]\!]$ be reduced and $R \hookrightarrow \bar{R}$ be its normalization. A composition of the natural projection $\Bbbk[\![x, y]\!] \twoheadrightarrow R$, the normalization $R \hookrightarrow \bar{R}$ and an isomorphism $\bar{R} \cong \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$,

$$\psi\colon \Bbbk[\![x, y]\!] \twoheadrightarrow R \hookrightarrow \bar{R} \cong \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$$

is called a *(primitive) parametrization* of f (or of R). More precisely,

(a) if f is irreducible, then a parametrization of f is given by a map

$$\psi\colon \Bbbk[\![x, y]\!] \longrightarrow \Bbbk[\![t]\!], (x, y) \mapsto (x(t), y(t)),$$

(b) if f decomposes into several branches, then a parametrization of R is given by a set of parametrizations of the branches. More precisely, if $f = f_1 \cdot \ldots \cdot f_r$ is a decomposition of f into irreducible factors, then $\bar{R} \cong \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$ is the normalization of R and a parametrization ψ of R can be represented as a matrix of the form:

$$\psi(t) = (\psi_1(t), \ldots, \psi_r(t)),$$

where for $i = 1, \ldots, r, (\psi_i(t) = (x_i(t), y_i(t))$ represents a parametrization of the i-th branch.

A parametrization of a reduced plane curve singularity has the following properties:

Proposition 2.3. *Let $0 \neq f \in \mathfrak{m} \subset \Bbbk[\![x, y]\!]$ be reduced and $\psi\colon \Bbbk[\![x, y]\!] \twoheadrightarrow R \hookrightarrow \bar{R} \cong \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$ be its parametrization. Then*

(i) $ker(\psi) = (f)$,

(ii) ψ satisfies the following universal factorization property: Each $\psi' : \Bbbk[\![x, y]\!] \to \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$ such that $\psi'(f) = 0$, factorizes in a unique way through ψ, that is there exists the unique morphism $\phi : \bigoplus_{i=1}^{r} \Bbbk[\![t]\!] \to \bigoplus_{i=1}^{r} \Bbbk[\![t]\!]$ such that $\psi' = \phi \circ \psi$. Moreover, if ψ' is also a parametrization of f, then ϕ is an isomorphism.

Proposition 2.4. *Let $f \in \Bbbk[\![x, y]\!]$ be irreducible such that $m := \mathrm{mt}(f) = \mathrm{ord} f(0, y)$. Assume that m is not divisible by $\mathrm{char}(\Bbbk)$, then f has a* **Puiseux parametrization**, *i.e. a parametrization of the form*

$$(x(t)|y(t)) := (t^m | \sum_{k \geq m} c_k t^k).$$

Moreover, there exists a unit $u \in \mathbb{k}[\![x,y]\!]$ such that

$$f = u \cdot \prod_{j=1}^{m} (y - y(\xi^j x^{1/m})),$$

where ξ is a primitive m-th root of unity.

Now we define the notion of parametrization equivalence.

Definition 2.2. Let $\psi, \psi' \colon \mathbb{k}[\![x,y]\!] \to \bar{R} = \bigoplus_{i=1}^{r} \mathbb{k}[\![t]\!]$. Then ψ is said to be *equivalent* to ψ', $\psi \sim \psi'$, if there exist a reparametrization $\phi \in Aut_{\mathbb{k}}(\bar{R})$ and a coordinate change $\Phi \in Aut_{\mathbb{k}}(\mathbb{k}[\![x,y]\!])$ such that $\psi' \circ \Phi = \phi \circ \psi$.

Let $f, g \in \mathbb{k}[\![x,y]\!]$ be reduced. Then f is said to be *parametrization equivalent* to g, $f \sim_p g$, if there exist a parametrization ψ of f and a parametrization ψ' of g such that $\psi \sim \psi'$.

Note that, if $f \sim_p g$, then for any parametrization ψ (resp. ψ') of f (resp. g) we have $\psi \sim \psi'$ by Proposition 2.3(ii).

Proposition 2.5. *Let f, g be two given plane curve singularities. Then*

$$f \sim_p g \Leftrightarrow f \sim_c g.$$

Proof.

cf. [25, Proposition 1.2.10]. ■

2.3. Intersection multiplicity and classical invariants

Definition 2.3. Let $f \in \mathbb{k}[\![x,y]\!]$ be reduced and let $\psi \colon \mathbb{k}[\![x,y]\!] \twoheadrightarrow R \hookrightarrow \bar{R} \cong \bigoplus_{i=1}^{r} \mathbb{k}[\![t]\!]$ be a parametrization of f.

(a) We call $\delta(f) := \dim_{\mathbb{k}} \bar{R}/R$ the *δ-invariant* of f.

(b) We introduce the *valuation map*

$$v := (v_1, ..., v_r) \colon R \to (\mathbb{Z}_{\geq 0} \cup \infty)^r, g \mapsto \mathrm{ord}(g(x_i(t), y_i(t)))_{i=1,\ldots,r}.$$

Its image $\Gamma(R) := \Gamma(f) := v(R)$ is a semigroup, called the *semigroup of values* of f.

(c) Let $\mathscr{C} := (R : \bar{R}) := \{u \in R \mid u\bar{R} \subset R\}$ be the *conductor ideal* of $\bar{R}$ in R (cf. [31]). Then $\mathscr{C}$ is an ideal of both R and $\bar{R}$. So one has $\mathscr{C} = (t^{c_1}) \times \cdots \times (t^{c_r})$ for some $\mathbf{c} := (c_1, \ldots, c_r) \in \mathbb{Z}^r_{\geq 0}$. We call $\mathbf{c}$ the *conductor (exponent)* of f. One obviously has $\mathbf{c} + \mathbb{Z}^r_{\geq 0} \subset S(f)$ and $\mathbf{c}$ is the minimum element in $S(f)$ with this property w.r.t. the product ordering on $\mathbb{Z}^r_{\geq 0}$, i.e. the partial ordering given by: if $\alpha = (\alpha_1, \ldots, \alpha_r), \beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r_{\geq 0}$ the $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for every $i = 1, \ldots, r$.

Definition 2.4. Let $g \in \Bbbk[\![x,y]\!]$ be irreducible and $(x(t), y(t))$ its parametrization. Then the intersection multiplicity of any $f \in \Bbbk[\![x,y]\!]$ with g is given by

$$i_0(f,g) := \mathrm{ord} f(x(t), y(t)).$$

If u is a unit then we define $i_0(f,u) := 0$.

The intersection multiplicity of f with a plane curve singularity $g = g_1 \cdot \ldots \cdot g_s$, g_i irreducible, is defined to be the sum

$$i(f,g) := i_0(f,g_1) + \ldots + i(f,g_s).$$

The Milnor number $\mu(f)$ and kappa invariant $\kappa(f)$ of f are defined respectively as

$$\mu(f) := i_0(f_x, f_y);\ \ \kappa(f) := i_0(f, \alpha f_x + \beta f_y),$$

where $(\alpha : \beta) \in \mathbb{P}^1$ is generic.

Proposition 2.6. *Let $f, g \in \Bbbk[\![x,y]\!]$. Then*

$$i_0(f,g) = i_0(g,f) = \dim \Bbbk[\![x,y]\!]/(f,g).$$

Proof.

cf. [15, Proposition I.3.12] ■

Corollary 2.1. *Let $f \in \Bbbk[\![x,y]\!]$ be an irreducible plane curve singularity. A couple $(x(t), y(t))$ of two power series is a parametrization of f if and only if*

$$f\left(x(t), y(t)\right) = 0 \ \textit{and}\ \min\{\mathrm{ord} x(t), \mathrm{ord} y(t)\} = \mathrm{mt}(f).$$

Proposition 2.7. *Let $f, g \in \Bbbk[\![x,y]\!]$ be two reduced power series which have no factor in common. Then*

$$\delta(fg) = \delta(f) + \delta(g) + i_0(f,g)$$

and

$$\kappa(fg) = \kappa(f) + \kappa(g) + i_0(f,g).$$

Proof.

cf. [15, Proposition I.3.32, Corollary 3.39] ■

Proposition 2.8. *Let $f \in \Bbbk[\![x,y]\!]$ be a reduced plane curve singularity. Then*

$$\delta(f) = \sum_{Q \to 0} \frac{\mathrm{mt}(f_Q)\left(\mathrm{mt}(f_Q) - 1\right)}{2}.$$

Proof.

cf. [15, Proposition I.3.34] ■

Proposition 2.9. *Let $f = f_1 \cdot \ldots \cdot f_r$ with f_i irreducible and let $\mathbf{c} = (c_1, \ldots, c_r)$ its conductor. Then, for any $i = 1, \ldots, r$ one has*

$$\begin{aligned} c_i &= 2\delta(f_i) + \sum_{j \neq i} i_0(f_i, f_j)) \\ &= c(f_i) + \sum_{j \neq i} i_0(f_i, f_j) \end{aligned}$$

and therefore $2\delta(f) = c(f) := c_1 + \cdots + c_r$.

Proof.

cf. [19]. ■

Lemma 2.2 (Dedekind's formula)**.** *Suppose that $i_0(f, x) = \operatorname{ord}(f) \neq 0 \mod p$. Then*

$$i_0\left(f, \frac{\partial f}{\partial y}\right) = c(f) + \operatorname{ord}(f) - 1.$$

For more facts on the conductor see [15], [12], [19]. The following proposition says that the δ-invariant, the conductor and the maximal contact multiplicity are invariant under contact equivalence, and by Proposition 2.5, they are also invariant under parametrization equivalence.

Proposition 2.10. *Let $f, g \in \Bbbk[[x, y]]$, let $u, v \in \Bbbk[[x, y]]^*$ be unit and let $\Phi \in \operatorname{Aut}_{\Bbbk}(\Bbbk[[x, y]])$. Then $i(f, g) = i(u \cdot \Phi(f), v \cdot \Phi(g))$. Moreover, if $f \sim_c g$, then*

(i) $\delta(f) = \delta(g)$.

(ii) $\kappa(f) = \kappa(g)$.

(iii) $\mathbf{c}(f) = \mathbf{c}(g)$ *(up to a permutation of the indices $\{1, \ldots, r\}$).*

Proof.

cf. [25, Proposition 1.2.19]. ■

For reduced plane curve $f = f_1 \cdot \ldots \cdot f_r$ with f_i irreducible we define

1. $\underline{\mathrm{mt}}(f) := (\mathrm{mt}(f_1), \ldots, \mathrm{mt}(f_r)) \in \mathbb{Z}^r$ the *multi-multiplicity* of f,

2. $\underline{c}(f) := (c(f_1), \ldots, c(f_r)) = (2\delta(f_1), \ldots, 2\delta(f_r)) \in \mathbb{Z}^r$ the *multi-conductor* of f.

These tuples are invariant under parametrization and contact equivalence as the following corollary shows.

Corollary 2.2. *If $f \sim_c g$ then $\underline{\mathrm{mt}}(f) = \underline{\mathrm{mt}}(g)$ and $\underline{\mathrm{c}}(f) = \underline{\mathrm{c}}(g)$ (up to a permutation of the indices $\{1, \ldots, r\}$).*

Proof.

Follows from Proposition 2.10. ■

We recall that if f is a plane curve singularity then its Milnor number $\mu(f)$ is $\dim \Bbbk[\![x,y]\!]/(f_x, f_y)$, where f_x, f_y be the partials of f. Proposition 2.6 yields that the Milnor number can be computed as an intersection multiplicity: $\mu(f) = i(f_x, f_y)$.

2.4. Newton diagrams and Newton factorizations

Let us recall the definition of the Newton diagram of a plane curve singularity. To each power series $f = \sum_{(\alpha,\beta)} c_{\alpha,\beta} x^\alpha y^\beta \in \Bbbk[\![x,y]\!]$ we can associate its *Newton polyhedron* $\Gamma_+(f)$ as the convex hull of the set

$$\bigcup_{\alpha \in \mathrm{supp}(f)} ((\alpha, \beta) + \mathbb{R}^2_{\geq 0}).$$

where $\mathrm{supp}(f) = \{\alpha | c_{\alpha,\beta} \neq 0\}$ denotes the support of f. This is an unbounded polytope in $\mathbb{R}^n$. We call the union $\Gamma(f)$ of its compact faces the *Newton diagram* of f. By $\Gamma_-(f)$ we denote the union of all line segments joining the origin to a point on $\Gamma(f)$. For each subset Δ in $\mathbb{R}^2_{\geq 0}$ we denote

$$\mathrm{in}_\Delta(f) := \sum_{(\alpha,\beta)\in\Delta} c_{\alpha,\beta} x^\alpha y^\beta \in \Bbbk[\![x,y]\!].$$

The initial part of f is defined to be

$$f_{in} := \mathrm{in}_{\Gamma(f)}(f).$$

Proposition 2.11. *Let $f \in \mathfrak{m} \subset \Bbbk[\![x,y]\!]$ be an irreducible plane curve singularity such tha $i_0(f,x) = n$ and $i_0(f,x) = m$. Let $(x(t), y(t))$ be parametrization of f. Then*

(i) $\mathrm{ord}(x(t)) = n$ *and* $\mathrm{ord}(y(t)) = m$.

(ii) The Newton diagram of f is a straight line segment.

(iii) There exist $\xi, \lambda \in \Bbbk^*$ such that

$$f_{in}(x,y) = \xi \cdot (x^{m/q} - \lambda y^{n/q})^q,$$

where $q = (m,n)$.

Proof.

cf. [9, Lemma 3.4.3, 3.4.4, 3.4.5]. ■

Proposition 2.12. *[8, Lemma 3] Let $f \in \Bbbk[\![x,y]\!]$ and let $E_i, i = 1, \ldots, k$ be the edges of its Newton diagram. Then there is a factorization of f:*

$$f = \text{monomial} \cdot \bar{f}_1 \cdot \ldots \cdot \bar{f}_k$$

such that $\bar{f}_i$ is convenient, and $\mathrm{in}_{E_i}(f) = \text{monomial} \times (\bar{f}_i)_{in}$. In particular, if f is convenient then $f = \bar{f}_1 \cdot \ldots \cdot \bar{f}_k$.

3. Milnor numbers and delta invariants

3.1. Milnor numbers

We first introduce the different notions of non-degeneracy originated by Kouchnirenko and Wall. For this let

$$f = \sum_\alpha c_\alpha x^\alpha \in \mathfrak{m} \subset \Bbbk[\![\mathbf{x}]\!] := \Bbbk[\![x_1, \ldots, x_n]\!]$$

be a power series, let P be a C-polytope, i.e. a compact rational polytope P of dimension $n-1$ in the positive orthant $\mathbb{R}^n_{\geq 0}$ and the region above P is convex and every ray in the positive orthant emanating from the origin meets P in exactly one point. For each subset $\Delta \subset \mathbb{R}^n_{\geq 0}$ we denote by $f_\Delta := \mathrm{in}_\Delta(f) := \sum_{\alpha \in \Delta} c_\alpha x^\alpha$ the initial form or principal part of f along Δ. Following Kouchnirenko we call f *non-degenerate* (ND) *along* Δ if the Jacobian ideal* $\mathrm{j}(\mathrm{in}_\Delta(f))$ has no zero in the torus $(\Bbbk^*)^n$. f is then said to be *Newton non-degenerate* (NND) if f is non-degenerate along each face (of any dimension) of the Newton diagram $\Gamma(f)$. We do not require f to be convenient.

To define inner non-degeneracy we need to fix two more notions. The face Δ is an *inner face* of P if it is not contained in any coordinate hyperplane. Each point $q \in \Bbbk^n$ determines a coordinate hyperspace $H_q = \bigcap_{q_i = 0} \{x_i = 0\} \subset \mathbb{R}^n$ in $\mathbb{R}^n$. We call f *inner non-degenerate* (IND) *along* Δ if for each zero q of the Jacobian ideal $\mathrm{j}(\mathrm{in}_\Delta(f))$ the polytope Δ contains no point on H_q. f is called *inner Newton non-degenerate* (INND) *w.r.t. a C-polytope P* if no point of $\mathrm{supp}(f)$ lies below P and f is IND along each inner face of P. We call f simply *inner Newton non-degenerate* (INND) if it is INND w.r.t some C-polytope.

*The Jacobian ideal $j(f)$ denotes the ideal generated by all partials of $f \in \Bbbk[\![x]\!]$.

Finally, we call f *weakly non-degenerate* (WND) *along* Δ if the Tjurina ideal† $tj(\mathrm{in}_\Delta(f))$ has no zero in the torus $(\mathbb{k}^*)^n$, and f is called *weakly Newton non-degenerate* (WNND) if f is weakly non-degenerate along each facet of $\Gamma(f)$. Note that NND implies WNND while NND does not imply INND. See [7, Remark 3.1] for facts on and relations between the different types of non-degeneracy.

For any compact polytope Q in $\mathbb{R}^n_{\geq 0}$ we denote by $V_k(Q)$ the sum of the k-dimensional Euclidean volumes of the intersections of Q with the k-dimensional coordinate subspaces of $\mathbb{R}^n$ and, following Kouchnirenko, we then call

$$\mu_N(Q) = \sum_{k=0}^{n}(-1)^{n-k}k!V_k(Q)$$

the Newton number of Q. For a power series $f \in \mathbb{k}[\![\mathbf{x}]\!]$ we define the *Newton number* of f to be

$$\mu_N(f) = \sup\{\mu_N(\Gamma_-(f_m))|f_m := f + x_1^m + \ldots + x_n^m, m \geq 1\}.$$

If f is convenient then

$$\mu_N(f) = \mu_N(\Gamma_-(f)).$$

The following theorem was proved by Kouchnirenko in arbitrary characteristic. We recall that $\mu(f) := \dim \mathbb{k}[\![\mathbf{x}]\!]/j(f)$ is the *Milnor number* of f.

Theorem 3.1. *[21] For $f \in \mathbb{k}[\![\mathbf{x}]\!]$ we have $\mu_N(f) \leq \mu(f)$, and if f is NND and convenient then $\mu_N(f) = \mu(f) < \infty$.*

Since Theorem 3.1 does not cover all semi-quasihomogeneous singularities, Wall introduced the condition INND (denoted by NPND* in [29]). Using Theorem 3.1, Wall proved the following theorem for $\mathbb{k} = \mathbb{C}$ which was extended to arbitrary $\mathbb{k}$ in [7].

Theorem 3.2. *[29], [7] If $f \in \mathbb{k}[\![\mathbf{x}]\!]$ is INND, then*

$$\mu(f) = \mu_N(f) = \mu_N(\Gamma_-(f)) < \infty.$$

Kouchnirenko proved that the condition "convenient" is not necessary in Theorem 3.1. In the planar case, the authors in [7] show that Kouchnirenko's result holds in arbitrary characteristic without the assumption that f is convenient (allowing $\mu(f) = \infty$):

Proposition 3.3. *[7, Proposition 4.5] Suppose that $f \in \mathbb{k}[\![x, y]\!]$ is NND, then $\mu_N(f) = \mu(f)$.*

†For $f \in \mathbb{k}[\![x]\!]$ we call $tj(f) = (f) + j(f)$ the Tjurina ideal of f.

Theorem 3.4. *Let $f \in \mathfrak{m} \subset \Bbbk[\![x, y]\!]$. Then the following are equivalent*

(i) $\mu(f) = \mu_N(f) < \infty$.

(ii) f *is INND.*

Proof.

The theorem follows from the following lemmas (for proofs, see [16]):

Lemma 3.1. *Let $f, g \in \Bbbk[\![x, y]\!]$ be convenient such that $\Gamma_-(f) \subseteq \Gamma_-(g)$. Then*

(i) $\mu_N(f) \leq \mu_N(g)$.

(ii) The equality holds if and only if $\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} = \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}$, *where*

$$\mathbb{R}^2_{\geq 1} = \{(x, y) \in \mathbb{R}^2 | x \geq 1, y \geq 1\}.$$

Note that Part (i) of the lemma holds true in many variables by [6, Cor. 5.6]. Let us denote by $\Gamma_1(f)$ the cone joining the origin with $\Gamma(f) \cap \mathbb{R}^2_{\geq 1}$. (cf. Fig. 1).

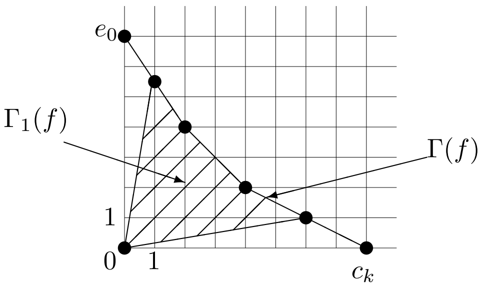

Fig. 1.

Lemma 3.2. *Let $f = \sum c_{ij} x^i y^j \in K[\![x, y]\!]$ be convenient and let $(0, n)$ (resp. $(m, 0)$) be the vertex on the y-axis (resp. on the x-axis) of $\Gamma(f)$. Assume that $m = n = 0 \mod p$ then $\mu(f) > \mu_N(f)$.*

Lemma 3.3. *Let $f \in K[\![x, y]\!]$ be convenient. If f is degenerate along some edge or some inner vertex of $\Gamma(f)$ then $\mu(f) > \mu_N(f)$.*

■

Corollary 3.1. *Let $f \in \Bbbk[\![x, y]\!]$ and let $M \in \mathbb{N}$ such that $\Gamma(f) \subset \Gamma(f_M)$ with $f_M := f + x^M + y^M$. Then f is INND if and only if it is INND w.r.t. $\Gamma(f_m)$ for some (equivalently for all) $m > M$.*

3.2. Delta-invariants

We consider now another important invariant of plane curve singularities, the invariant δ and its combinatorial counterpart, the Newton invariant δ_N. We show that both coincide iff f is weighted homogeneous Newton non-degenerate (WHNND), a new non-degenerate condition introduced below.

Let $E_1, \dots, E_k$ be the edges of the Newton diagram of f. We denote by $l(E_i)$ the lattice length of E_i, i.e. the number of integral points on E_i minus one and by $s(\mathrm{in}_{E_i}(f))$ the number of non-monomial irreducible (reduced) factors of $\mathrm{in}_{E_i}(f)$. We set

(a) If f is convenient, we define

$$\delta_N(f) := V_2(\Gamma_-(f)) - \frac{V_1(\Gamma_-(f))}{2} + \frac{\sum_{i=1}^k l(E_i)}{2},$$

and otherwise we set $\delta_N(f) := \sup\{\delta_N(f_m) | f_m := f + x^m + y^m, m \in \mathbb{N}\}$ and call it the *Newton δ-invariant* of f.

(b) $r_N(f) := \sum_{i=1}^k l(E_i) + \max\{j | x^j \text{ divides } f\} + \max\{l | y^l \text{ divides } f\}$.

(c) $s_N(f) := \sum_{i=1}^k s(\mathrm{in}_{E_i}(f)) + \max\{j | x^j \text{ divides } f\} + \max\{l | y^l \text{ divides } f\}$.

Note that all of these numbers depend on the Newton diagram of f and hence are coordinate-dependent.

Proposition 3.5. *For $0 \neq f \in (x, y)$ we have $r(f) \leq r_N(f)$, and if f is WNND then $r(f) = r_N(f)$.*

Proof.

cf. [7, Lemma 4.10] ∎

Let E be an edge of the Newton diagram of f. Then we can write f_E as follows,

$$\mathrm{in}_E(f) = \prod_{i=1}^{s} (a_i x^{m_0} - b_i y^{n_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct; $m_0, n_0, r_i \in \mathbb{N}_{>0}$, $\gcd(m_0, n_0) = 1$. It easy to see that

$$s = s(\mathrm{in}_E(f)) \text{ and } l(E) = \sum_{i=1}^{s} r_i.$$

This implies $s(\mathrm{in}_E(f)) \leq l(E)$ and hence $s_N(f) \leq r_N(f)$.

Let $f = f_d^{\mathrm{w}} + f_{d+1}^{\mathrm{w}} + \dots$ with $f_d^{\mathrm{w}} \neq 0$ be the (n_0, m_0)-weighted homogeneous decomposition of f.

Definition 3.1. We say that f is *weighted homogeneous non-degenerate* (WHND) *along* E if either $r_i = 1$ for all $i = 1, \dots, s$ or $(a_i x^{m_0} - b_i y^{n_0})$ does not divide f^{w}_{d+1} for each $r_i > 1$.

f is called *weighted homogeneous Newton non-degenerate* (WHNND) if its Newton diagram has no edge or if it is WHND along each edge of its Newton diagram.

Remark 3.1. (a) In [22] the author introduced superisolated singularities to study the μ-constant stratum. We recall that $f \in \Bbbk[\![x, y]\!]$ is *superisolated* if it becomes regular after only one blowing up. By ([22, Lemma 1]), this is equivalent to: $f_{m+1}(\beta_i, \alpha_i) \neq 0$ for all tangent directions $(\beta_i : \alpha_i)$ of f with $r_i > 1$, where $f = f_m + f_{m+1} + \dots$ is the homogeneous decomposition of f and

$$f_m = \prod_{i=1}^{s} (\alpha_i x - \beta_i y)^{r_i}.$$

Note that this condition concerns all factors of f_m including monomials. For WHNND singularities we require a similar condition, but for "all weights" and without any condition on the monomial factors of the first term of the weighted homogeneous decomposition of f.

(b) Since a plane curve singularity is superisolated iff it becomes regular after only one blowing up, we have $\delta(f) = m(m-1)/2$ and hence $\delta(f) = \delta_N(f) = m(m-1)/2$, by Proposition 4.1. It follows from Theorem 4.3 that

(c) A superisolated plane curve singularity is WHNND.

(d) The plane curve singularity $x^2 + y^5$ is WHNND but not superisolated.

Proposition 3.6. *With notations as above, f is WND along E if and only if $s(f_E) = l(E)$ or, equivalently, iff $r_i = 1$ for all $i = 1, \dots, s$. In particular, WNND implies WHNND.*

Proof.

cf. [16, Proposition 3.5] ∎

Proposition 3.7. *For $0 \neq f \in (x, y)$ we have $s_N(f) \leq r(f)$ and if f is WHNND then $s_N(f) = r(f)$.*

Proof.

cf. [16, Proposition 3.7] ∎

Proposition 3.8. *For $0 \neq f \in (x, y)$ we have $s_N(f) \leq r(f) \leq r_N(f)$, and both equalities hold if and only if f is WNND.*

Proof.

The inequalities follow from Proposition 3.5 and Proposition 3.7. For each edge E of $\Gamma(f)$, by Proposition 3.6, f is WND along E iff $s(f_E) = l(E)$. This implies that f is WNND if and only if $s_N(f) = r_N(f)$ since $s(f_E) \leq l(E)$ and both sides are additive with respect to edges of $\Gamma(f)$. ■

We investigate now the relations between $\nu(f)$, $\delta_N(f)$ and $\delta(f)$, which were studied in [5] and [7].

Proposition 3.9. *[7, Prop. 4.9] For $0 \neq f \in (x, y)$ we have $\delta_N(f) \leq \delta(f)$, and if f is WNND then $\delta_N(f) = \delta(f)$.*

Hence WNND is sufficient but, by the following example, not necessary for $\delta_N(f) = \delta(f)$.

Example 3.10. Let $f(x, y) = (x + y)^2 + y^3 \in \Bbbk[[x, y]]$. Then f is not WNND but $\delta_N(f) = \delta(f) = 1$. This easy example shows also that WNND depends on the coordinates since $x^2 + y^3$ is WNND. Note that f is WHNND.

Theorem 3.11. *[16, Theorem 3.12] Let $f \in \Bbbk[[x, y]]$ be reduced. Then $\delta(f) = \delta_N(f)$ if and only if f is WHNND.*

Proof.

Recall that, if E is an edge of the Newton diagram of f. Then we can write f_E as follows,

$$\text{in}_E(f) = \prod_{i=1}^{s} (a_i x^{m_0} - b_i y^{n_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct; $m_0, n_0, r_i \in \mathbb{N}_{>0}$, $\gcd(m_0, n_0) = 1$. It easy to see that

$$s = s(\text{in}_E(f)) \text{ and } l(E) = \sum_{i=1}^{s} r_i.$$

The theorem is then based on the following lemmas. We refer to [16] for detail proofs.

Lemma 3.4. *There exist an integer n and an infinitely near point $P \xrightarrow{n} 0$ in the n-th neighbourhood of 0, such that*

$$\text{in}_{E_P}(f_P)(u, v) = \textit{monomial} \times \prod_{i=1}^{s} (a_i u - b_i v)^{r_i},$$

where f_P is the local equation of the strict transform of f at P and E_P is some edge of its Newton diagram $\Gamma(f_P)$. Moreover, f is WHND along E if and only if f_P is WHND along E_P.

Let us denote by Q_i the points $(a_i : b_i)$ and by m_{Q_i} the multiplicity of f_{Q_i}. Then

Lemma 3.5. *The following are equivalent*

(i) f *is WHND along* E.

(ii) $m_{Q_i} = 1$ *for all* i.

■

3.3. Milnor formula

We recall that if f is a plane curve singularity then its Milnor number $\mu(f)$ is $\dim \mathbb{k}[\![x,y]\!]/(f_x, f_y)$, where f_x, f_y be the partials of f. Proposition 2.6 yields that the Milnor number can be computed as the intersection multiplicity of f_x and f_y: $\mu(f) = i(f_x, f_y)$. Moreover if $\mathbb{k} = \mathbb{C}$, the Milnor's famous formula (see, [23, Thm 10.5], or also [15, Prop. 3.35]) gives a relation between the Milnor number, the δ-invariant:

$$\mu(f) = 2\delta(f) - r(f) + 1.$$

This also holds true in characteristic zero. But in positive characteristic, it is in general not true as the following example shows: $f = x^3 + x^4 + y^6 + y^7 \in \mathbb{k}[\![x,y]\!]$ with $\mathrm{char}(\mathbb{k}) = 3$. Then

$$r(f) = 1;\ \mu(f) = 18;\ \delta(f) = 6.$$

In positive characteristic the equality holds under certain conditions of the Newton diagram, e.g. NND ([7, Thm. 9]) or INND ([16, Cor. 3.16]). However without the assumption of Newton non-degeneracy one has at least an inequality as proven by Pierre Deligne [11], see also [24]:

$$\mu(f) \geq 2\delta(f) - r(f) + 1.$$

The difference of the two sides is measured by the so called Swan character, denoted by $\mathrm{Sw}(f)$, which counts wild vanishing cycles that can only occur in positive characteristic.

However it still holds true if f is NND by [7, Thm. 4.13]. Using the general inequality

$$\mu_N(f) = 2\delta_N(f) - r_N(f) + 1 \leq 2\delta(f) - r(f) + 1 \leq \mu(f)$$

from [7], then Theorem 3.4, Proposition 3.5 and Proposition 4.2 imply.

Although we can compute the number of wild vanishing cycles, it seems hard to understand them. In [16] we have posed the following problems.

Problem 3.12. *Is there any "geometric" way to understand the wild vanishing cycles, distinguishing them from the ordinary vanishing cycles counted by $2\delta - r + 1$? Is there at least a "reasonable" characterization of those singularities without wild vanishing cycles?*

Problem 3.13. *Find an "elementary proof" for the inequality*

$$\mu(f) \geq 2\delta(f) - r(f) + 1.$$

We will discuss more carefully about this topic in the last two sections.

4. Gamma and kappa invariants

The results in this section are borrowed from [26].

4.1. Gamma invariants

Following [26, Section 2] we introduce and study new (gamma) invariants $\gamma, \tilde{\gamma}$ of plane curve singularities which have not been considered before. In characteristic zero, these invariants coincide and are equal to the Milnor number (see Remark 4.1). So they may be considered as generalizations of the Milnor number in positive characteristic and are believed to be useful in studying classical invariants. In this section we use them to connect the delta and kappa invariant. We will show, in Proposition 4.1, that

$$\kappa(f) \geq \gamma(f) + \mathrm{mt}(f) - 1$$

and in Theorem 4.4, that

$$\gamma(f) \geq 2\delta(f) - r(f) + 1$$

and obtain the inequality in the main result (Theorem 4.5) of the section:

$$\kappa(f) \geq 2\delta(f) + \mathrm{mt}(f) - 1$$

with equalitity if and only if p is m-good for f (see, Definition 4.3 for the notion of m-goodness).

Definition 4.1. Let $f \in \Bbbk[[x, y]]$ be reduced. The number $\tilde{\gamma}_{x,y}(f)$ (or $\tilde{\gamma}(f)$, if the coordinate $\{x, y\}$ is fixed) of f, is defined as follows:

(a) $\tilde{\gamma}(x) := 0$, $\tilde{\gamma}(y) := 0$.

(b) If f is irreducible and *convenient* (i.e. $i_0(f,x), i_0(f,y) < \infty$), then

$$\tilde{\gamma}(f) := \min\{i_0(f,f_x) - i_0(f,y) + 1, i_0(f,f_y) - i_0(f,x) + 1\}.$$

(c) If $f = f_1 \cdot \ldots \cdot f_r$, then

$$\tilde{\gamma}(f) := \sum_{i=1}^{r} \Big(\tilde{\gamma}(f_i) + \sum_{j\neq i} i_0(f_i,f_j)\Big) - r + 1.$$

Definition 4.2. The *gamma invariant* of a reduced plane curve singularity f, denoted by $\gamma(f)$, is the minimum of $\tilde{\gamma}_{X,Y}(f)$ for all coordinates X, Y.

Remark 4.1. (a) In characteristic zero, $\gamma(f) = \tilde{\gamma}(f) = \mu(f)$ due to Theorems 4.3, 4.4 and the Milnor formula.

(b) In general we have, by definition, that $\gamma(f) \le \tilde{\gamma}(f)$ (with equality if p is im-good for f, see Definition 4.3 and Corollary 4.1) and that $\gamma(f) = \tilde{\gamma}(g)$ for some g right equivalent to f (f is called *right equivalent* to g, denoted by $f \sim_r g$, if there is an automorphism $\Phi \in \mathrm{Aut}_{\Bbbk}(\Bbbk[\![x,y]\!])$ such that $f = \Phi(g)$).

(c) The number $\tilde{\gamma}$ depends on the choice of coordinates, i.e. it is not invariant under right equivalence. E.g. $f = x^3+x^4+y^5$ and $g = (x+y)^3+(x+y)^4+y^5$ in $\Bbbk[\![x,y]\!]$ with $\mathrm{char}(\Bbbk) = 3$ and then $f \sim_r g$, but $\tilde{\gamma}(f) = 8$, $\tilde{\gamma}(g) = 10$. However, as we will see in Proposition 4.2, if the characteristic p is multiplicity good for f then $\tilde{\gamma}(f) = \tilde{\gamma}(g)$ for all g contact equivalent to f. Recall that f, g are *contact equivalent* if there is an automorphism $\Phi \in \mathrm{Aut}_{\Bbbk}(\Bbbk[\![x,y]\!])$ and a unit $u \in \Bbbk[\![x,y]\!]$ such that $f = u \cdot \Phi(g)$, and we denote this by $f \sim_c g$.

(d) It follows from the definition that $\tilde{\gamma}(u) = 1$ and $\tilde{\gamma}(u \cdot f) = \tilde{\gamma}(f)$ for every unit u and therefore γ is invariant under contact equivalence.

(e) The Milnor number μ is invariant under right equivalence. The numbers $\delta, \kappa, \mathrm{mt}, r, i$ are invariant under contact equivalence (see, for instance [25], Prop. 1.2.19 for the invariance of δ). This means that, if $f \sim_c g$ then

$$\delta(f) = \delta(g),\ \kappa(f) = \kappa(g),\ \mathrm{mt}(f) = \mathrm{mt}(g) \text{ and } r(f) = r(g).$$

Moreover, for any $\Phi \in \mathrm{Aut}_{\Bbbk}(\Bbbk[\![x,y]\!])$ and units u, v, one has

$$i_0(f,h) = i_0\left(u \cdot \Phi(f), v \cdot \Phi(h)\right).$$

Before studying in detail gamma invariants, we collect several facts on invariants of plane curve singularities which we use later. For proofs, we refer to [15] and [25].

Remark 4.2. (a) If f is irreducible, then

$$\kappa(f) = \min\{i_0(f, f_x), i_0(f, f_y)\}.$$

Indeed, taking a parametrization $(x(t), y(t))$ of f we obtain that

$$\kappa(f) = \text{ord}\ \left(\alpha f_x(x(t), y(t)) + \beta f_y(x(t), y(t))\right),$$

which equals to the minimum of $i(f, f_x)$ and $i(f, f_y)$ since $(\alpha : \beta)$ is generic.

(b) If f is convenient, then

$$\tilde{\gamma}(f) \quad = \quad i_0(f, \alpha x f_x + \beta y f_y) - i_0(f, x) - i_0(f, y) + 1,$$

where $(\alpha : \beta) \in \mathbb{P}^1$ is generic.

Definition 4.3. Let $\text{char}(\Bbbk) = p \geq 0$ and let $f = f_1 \cdot \ldots \cdot f_r \in \Bbbk[\![x, y]\!]$ be reduced with f_i irreducible. The characteristic p is said to be

(a) *multiplicity good* (*m-good*) for f if the multiplicities $\text{mt}(f_i) \neq 0 \pmod p$ for all $i = 1, \ldots, r$;

(b) *intersection multiplicity good* (*im-good*) for f if for all $i = 1, \ldots, r$, either $i(f_i, x) \neq 0 \pmod p$ or $i(f_i, y) \neq 0 \pmod p$;

(c) *right intersection multiplicity good* (*right im-good*) for f if it is im-good for f after some change of coordinate. That is, it is im-good for some g right equivalent to f.

Note that these notions are trivial in characteristic zero, i.e. if $p = 0$ then it is always m-good, im-good and right im-good for f. In general we have

$$\text{“m-good”} \Longrightarrow \text{“im-good”} \Longrightarrow \text{“right im-good”}.$$

The following proposition gives us the first relations between the gamma invariants and classical invariants.

Proposition 4.1. *Let $f \in \Bbbk[\![x, y]\!]$ be reduced. Then*

$$\gamma(f) \leq \tilde{\gamma}(f) \leq \kappa(f) - \text{mt}(f) + 1$$

with equality if p is m-good for f.

Proof.

cf. [26, Proposition 2.6]. ∎

The following proposition says that the number $\tilde{\gamma}$ is invariant under contact equivalence in the class of singularities for which p is m-good. It will be shown in Corollary 4.1 that $\tilde{\gamma}$ is invariant under contact equivalence in the class of singularities for which p is im-good.

Proposition 4.2. *Let $f \in \mathbb{k}[\![x,y]\!]$ be reduced such that p is m-good for f and let $g \sim_c f$. Then $\tilde{\gamma}(g) = \tilde{\gamma}(f)$. In particular, $\gamma(f) = \tilde{\gamma}(f)$.*

Proof.

This follows from Proposition 4.1 and Remark 4.1(e). See [25, Lemma 2.3.4] for a direct proof. ■

Theorem 4.3. *Let $f \in \mathbb{k}[\![x,y]\!]$ be reduced. Then*

$$\tilde{\gamma}(f) \geq 2\delta(f) - r(f) + 1.$$

Equality holds if and only if the characteristic p is im-good for f.

Proof.

cf. [26, Theorem 2.11]. ■

Corollary 4.1. *Assume that p is im-good for f. Then*

$$\gamma(f) = \tilde{\gamma}(f).$$

The following simple corollary should be useful in computation, since the number in the left side is easily computed.

Corollary 4.2. *Assume that $p > \mathrm{mt}(f)$. Then*

$$\mu(f) - \tilde{\gamma}(f) = \mathrm{Sw}(f).$$

Theorem 4.4. *Let $f \in \mathbb{k}[\![x,y]\!]$ be reduced. Then*

$$\gamma(f) \geq 2\delta(f) - r(f) + 1.$$

Equality holds if and only if the characteristic p is right im-good for f.

Proof.

Taking g right equivalent to f such that $\gamma(f) = \tilde{\gamma}(g)$ and combining Theorem 4.3 and Remark 4.1 we get

$$\gamma(f) = \tilde{\gamma}(g) \geq 2\delta(g) - r(g) + 1 = 2\delta(f) - r(f) + 1$$

with equality if and only if p is im-good for g. It remains to show that if p is right im-good for f, then

$$\gamma(f) = 2\delta(f) - r(f) + 1.$$

Indeed, by definition, p is im-good for some h right equivalent to f. Again combining Theorem 4.3 and Remark 4.1 we get

$$\gamma(f) = \gamma(h) \leq \tilde{\gamma}(h) = 2\delta(h) - r(h) + 1 = 2\delta(f) - r(f) + 1 \leq \gamma(f).$$

This implies that

$$\gamma(f) = 2\delta(f) - r(f) + 1,$$

which completes the theorem. ■

4.2. Kappa invariants and Plücker formulas

We prove in this section the main result (Theorem 4.5) and apply it to Plücker formulas (Corollaries 4.4, 4.5). Furthermore we show, in Corollary 4.3 (resp. Corollary 4.5), that if p is "big" for f (resp. for a plane curve C), then f (resp. C) has no wild vanishing cycle.

Theorem 4.5. *Let $f \in \mathbb{k}[\![x, y]\!]$ be reduced. One has*

$$\kappa(f) \geq 2\delta(f) + \mathrm{mt}(f) - r(f)$$

with equality if and only if p is m-good for f.

The following interesting corollary says that if the characteristic p is "big" for f, then f has no wild vanishing cycle.

Corollary 4.3. *Assume that $p > \kappa(f)$. Then f has no wild vanishing cycle, i.e.* $\mathrm{Sw}(f) = 0$. *Moreover one has*

$$\begin{aligned} \kappa(f) &= 2\delta(f) + \mathrm{mt}(f) - r(f) \\ &= \mu(f) + \mathrm{mt}(f) - 1. \end{aligned}$$

Let C be a irreducible curve of degree d in $\mathbb{P}^2$ defined by a homogeneous polynomial $F \in \mathbb{k}[x, y, z]$. Let $\mathrm{Sing}(C)$ resp. $C^* := C \setminus \mathrm{Sing}(C)$ the singular resp. smooth locus of C, and let $s(C) := \sharp\mathrm{Sing}(C)$ the number of singular points. Let $\rho\colon C^* \to \check{\mathbb{P}}^2, P = (x\colon y\colon z) \mapsto (F_x(P)\colon F_y(P)\colon F_z(P))$ the dual (Gauss) map and $\deg(\rho)$ its degree. We call the closure of the image of ρ in $\check{\mathbb{P}}^2$ the *dual curve* of C denoted by $\check{C}$. We denote by $\check{d}$ the degree of $\check{C}$. For each singular point $P \in \mathrm{Sing}(C)$ take a local equation $f_P = 0$ of C at P, and define

$$\begin{array}{lllllll} \delta(C) & := & \sum \delta(f_P), & & \mathrm{mt}(C) & := & \sum \mathrm{mt}(f_P), \\ \mu(C) & := & \sum \mu(f_P), & & r(C) & := & \sum r(f_P), \\ \mathrm{Sw}(C) & := & \sum \mathrm{Sw}(f_P). & & & & \end{array}$$

where all the sums are taken over $P \in \mathrm{Sing}(C)$.

Corollary 4.4. *Using the above notions, we have*

$$\begin{aligned}\deg(\rho)\cdot \check{d} &\leq d(d-1)-2\delta(C)+r(C)-\mathrm{mt}(C)\\ &= d(d-1)-\mu(C)-\mathrm{mt}(C)+s(C)+\mathrm{Sw}(C),\end{aligned}$$

with equality if and only if p is multiplicity good (m-good) *for C, i.e. p is m-good for all the f_P.*

Combining Corollaries 4.3 and 4.3 we obtain

Corollary 4.5. *With the above notions, assume that*

$$\max_{P\in \mathrm{Sing}(C)}\{\kappa(f_P)\}<p,$$

(for example, $d(d-1)<p$). Then C has no wild vanishing cycle, i.e. $\mathrm{Sw}(C)=0$*. Moreover one has*

$$\begin{aligned}\deg(\rho)\cdot \check{d} &= d(d-1)-2\delta(C)+r(C)-\mathrm{mt}(C)\\ &= d(d-1)-\mu(C)-\mathrm{mt}(C)+s(C).\end{aligned}$$

5. Semigroup of a plane algebroid branch

In this section we study the semigroup of a given irreducible plane curve singularity and apply it to study Problem 3.12 proposed in Section 3. The proofs can be found in [13, 14].

5.1. Semigroups

We say that a subset G of $\mathbb{N}$ is a *semigroup* if it contains 0 and if it is closed under addition. Let $n>0$ be an integer. A sequence of positive integers $(v_0,\dots,v_h)$ is said to be a *Seidenberg n-characteristic sequence* or *n-characteristic sequence* if $v_0=n$ and it satisfies the following two axioms

(a) Set $d_i=\gcd(v_0,\dots,v_i)$ for $0\leq i\leq h$ and $n_i=\frac{d_{i-1}}{d_i}$ for $1\leq i\leq h$. Then $d_h=1$ and $n_i>1$ for $1\leq i\leq h$.

(b) $n_{i-1}v_{i-1}<v_i$ for $2\leq i\leq h$.

Note that condition (b) implies that the sequence $(v_1,\dots,v_h)$ is strictly increasing. If $n>1$ then $h\geq 1$. If $h=1$ then the sequence (v_0,v_1) is a Seidenberg

n-characteristic sequence if and only if $v_0 = n$ and $\gcd(v_0, v_1) = 1$. There is exactly one 1-sequence which is (1).

Let G be a nonzero semigroup and let $n \in G$, $n > 0$. Then there exists (cf. [17], Chapter 6, Proposition 6.1) a unique sequence $v_0, \ldots, v_h$ such that $v_0 = n$, $v_k = \min(G \setminus v_0\mathbb{N} + \cdots + v_{k-1}\mathbb{N})$ for $k \in \{1, \ldots, h\}$ and $G = v_0\mathbb{N} + \cdots + v_h\mathbb{N}$. We call the sequence $(v_0, \ldots, v_h)$ the *n-minimal system of generators of G*. If $n = \min(G\setminus\{0\})$ then we say that $(v_0, \ldots, v_h)$ is the *minimal set* of generators of G. We will study semigroups generated by n-characteristic sequences.

Proposition 5.1. *Let $G = v_0\mathbb{N} + \cdots + v_h\mathbb{N}$ where $(v_0, \ldots, v_h)$ is an n-characteristic sequence. Then*

(i) *The sequence $(v_0, \ldots, v_h)$ is the n-minimal system of generators of G.*

(ii) $\min(G\setminus\{0\}) = \min(v_0, v_1)$.

(iii) *The minimal system of generators of G is $(v_0, v_1, \ldots, v_h)$ if $v_0 < v_1$, $(v_1, v_0, \ldots, v_h)$ if $v_1 < v_0$ and $v_0 \not\equiv 0$ (mod v_1) and $(v_1, v_2, \ldots, v_h)$ if $v_0 \equiv 0$ (mod v_1). Moreover, the minimal system of generators of G is a $\min(G\setminus\{0\})$-characteristic sequence.*

(iv) *Let $c = \sum_{k=1}^{h}(n_k - 1)v_k - v_0 + 1$. Then for every $a, b \in \mathbb{Z}$: if $a + b = c - 1$ then exactly one element of the pair (a, b) belongs to G. Consequently c is the smallest element of G such that all integers bigger than or equal to it are in G.*

(v) *c is an even number and $\sharp(\mathbb{N}\setminus G) = \dfrac{c}{2}$.*

The number c is called the conductor *of the semigroup G.*

5.2. Polar factorization theorems

The aim of this section is to study the structure of the semigroup associated with a plane branch and its relation to the factorization theorems.

Let $f = f(x, y) \in \Bbbk[\![x, y]\!]$ be an irreducible power series and let $S(f)$ be the semigroup associated with the branch $\{f = 0\}$. Suppose that $\{f = 0\} \neq \{x = 0\}$ and put $n = i_0(f, x)$. That is,

$$S(f) = \{i(f, g) \mid g \in \Bbbk[\![x, y]\!] \setminus (f)\Bbbk[\![x, y]\!]\}.$$

Let $(\bar{b}_0, \ldots, \bar{b}_h)$, $\bar{b}_0 = n$ be the n-minimal system of generators of $S(f)$. We define

$$e_0 = n, e_k = \gcd(e_{k-1}, \bar{b}_k) \text{ and } n_k = \frac{e_{k-1}}{e_k} \text{ for } k \in \{1, \ldots, h\}.$$

Lemma 5.1. *We have $e_h = 1$.*

Proof.

It follows from Theorem 2.1 that $\pi\colon \Bbbk[\![x,y]\!]/(f) \to \Bbbk[\![t]\!]$ is the normalization, and hence

$$Q\left(\Bbbk[\![x,y]\!]/(f)\right) \cong \Bbbk(\!(t)\!).$$

This implies that, there exist $p, q \in \Bbbk[\![x,y]\!]/(f)$ such that

$$\frac{p\left(x(t), y(t)\right)}{q\left(x(t), y(t)\right)} = t,$$

where $x(t) = \pi(x)$ and $y(t) = \pi(y)$. Taking order of both sides we get

$$\text{ord}\ p\left(x(t), y(t)\right) - \text{ord}\ q\left(x(t), y(t)\right) = 1.$$

Since ord $p\left(x(t), y(t)\right)$ and ord $q\left(x(t), y(t)\right)$ are elements in $S(f)$, it follows that $\gcd(S(f)) = 1$ and hence $e_h = 1$. ■

Corollary 5.1 (Conductor formula)**.** *One has*

$$c(f) = \sum_{k=1}^{h} (n_k - 1)\bar{b}_k - \bar{b}_0 + 1.$$

Theorem 5.2 (Semigroup Theorem)**.** *Let $\{f = 0\}$ be a branch such that $\{f = 0\} \neq \{x = 0\}$. Set $n = i_0(f, x)$ and let $\bar{b}_0, \ldots, \bar{b}_h$ be the n-minimal system of generators of the semigroup $S(f)$. There exists a sequence of monic polynomials $f_0, f_1, \ldots, f_{h-1} \in \Bbbk[\![x]\!][y]$ such that for $k \in \{1, \ldots, h\}$:*

(a_k) $\deg_y(f_{k-1}) = \frac{n}{e_{k-1}}$,

(b_k) $i_0(f, f_{k-1}) = \bar{b}_k$ *for* $k \in \{1, \ldots, h\}$,

(c_k) *if* $k > 1$ *then* $n_{k-1}\bar{b}_{k-1} < \bar{b}_k$.

Moreover $n_k > 1$ for all $k \in \{1, \ldots, h\}$.

Theorem 5.3 (Merle-Granja's Factorization Theorem)**.** *Let $\{f = 0\}$ be a branch such that $\{f = 0\} \neq \{x = 0\}$. Set $n = i_0(f, x)$ and let $\bar{b}_0, \ldots, \bar{b}_h$ be the n-minimal system of generators of the semigroup $S(f)$. Fix k, $1 \leq k \leq h$. Let $g = g(x, y) \in \Bbbk[\![x,y]\!]$ be a power series such that*

(i) $i_0(g, x) = \frac{n}{e_k} - 1$,

(ii) $i_0(f, g) = \sum_{i=1}^{k} (n_i - 1)\bar{b}_i$.

Then there is a factorization $g = g_1 \cdots g_k \in \Bbbk[\![x,y]\!]$ such that

1. $i_0(g_i, x) = \frac{n}{e_i} - \frac{n}{e_{i-1}}$ *for* $i \in \{1, \ldots, k\}$,
2. *if* $\phi \in \Bbbk[\![x,y]\!]$ *is an irreducible factor of* g_i, $i \in \{1, \ldots, k\}$ *then*
   (a) $\frac{i_0(f,\phi)}{i_0(\phi,x)} = \frac{e_{i-1}\bar{b}_i}{n}$,
   (b) $i_0(\phi, x) \equiv 0 \mod \frac{n}{e_{i-1}}$.

Theorem 5.4 (Merle's factorization theorem)**.** *Let $\{f = 0\}$ be a branch such that $\{f = 0\} \neq \{x = 0\}$. Set $n = i_0(f, x)$ and let $\bar{b}_0, \ldots, \bar{b}_h$ be the n-minimal system of generators of the semigroup $S(f)$. Suppose that $n > 1$ and $n \not\equiv 0$* mod (char $\Bbbk$)*. Then $\frac{\partial f}{\partial y} = g_1 \cdots g_h$ in $\Bbbk[\![x,y]\!]$, where*

(i) $i_0(g_i, x) = \frac{n}{e_i} - \frac{n}{e_{i-1}}$ *for* $i \in \{1, \ldots, h\}$.

(ii) *If* $\phi \in \Bbbk[\![x,y]\!]$ *is an irreducible factor of* g_i, $i \in \{1, \ldots, h\}$, *then*

$$\frac{i_0(f,\phi)}{i_0(\phi,x)} = \frac{e_{i-1}\bar{b}_i}{n} \text{ and } i_0(\phi, x) \equiv 0 \mod \frac{n}{e_{i-1}}.$$

Proof. Since $n \not\equiv 0$ (mod char $\Bbbk$) we have $i_0\left(\frac{\partial f}{\partial y}, x\right) = n-1$. By the Dedekind formula and the Conductor formula we have $i_0\left(f, \frac{\partial f}{\partial y}\right) = c(f) + n - 1 = \sum_{k=1}^{h}(n_k - 1)\bar{b}_k$. The theorem is then proved by applying Theorem 5.3 to the series $g = \frac{\partial f}{\partial y}$. ∎

5.3. García Barroso-Ploski's theorem

In this section, we give a proof of the following theorem.

Theorem 5.5. *[14, Theorem 1.1] Let $f \in \Bbbk[\![x,y]\!]$ be an irreducible singularity and let $(\bar{b}_0, \ldots, \bar{b}_h)$ be the minimal system of generators of $S(f)$. Suppose that $p = \operatorname{char} \Bbbk > \operatorname{ord} f$. Then the following two conditions are equivalent:*

(i) $\bar{b}_k \not\equiv 0 \mod p$, *for* $k \in \{1, \ldots, h\}$;

(ii) $\mu(f) = c(f)$.

Proof. Since $p > \operatorname{ord} f$, it follows from Dedekind's formula (Lemma 2.2) that

$$i_0(f, \frac{\partial f}{\partial y}) = c(f) + \operatorname{ord} f - 1.$$

It remains to prove that

$$i_0(f, \frac{\partial f}{\partial y}) = \mu(f) + \mathrm{ord} f - 1$$

if and only if $\bar{b}_k \not\equiv 0 \pmod{p}$ for $k \in \{1, \ldots, h\}$.

In fact, let ϕ be an irreducible factor of $\frac{\partial f}{\partial y}$ and let $(x(t), y(t))$ be a parametrization of $\phi = 0$. Then

$$\mathrm{ord} x(t) = i_0(x, \phi) = \mathrm{ord}\phi < \mathrm{ord} f < p$$

and, consequently, ord $x(t) \not\equiv 0 (\mathrm{mod} p)$, which implies ord $x'(t) = \mathrm{ord}\, x(t) - 1$. We have

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t).$$

This yields

$$\mathrm{ord}\ f(x(t), y(t)) - 1 \leq \mathrm{ord}\ \frac{\partial f}{\partial x}(x(t), y(t)) + \mathrm{ord}\, x(t) - 1$$

with equality if and only if $i_0(f, \phi) = \mathrm{ord} f(x(t), y(t)) \not\equiv 0 \mod p$. Taking the sum over all irreducible factors $\frac{\partial f}{\partial y}$ gives us

$$i_0(f, \frac{\partial f}{\partial y}) \leq i_0(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) + \mathrm{ord} f - 1 = \mu(f) + \mathrm{ord} f - 1 \tag{5.1}$$

with equality if and only if $i_0(f, \phi) \not\equiv 0 \mod p$ for all ϕ.

Let $\frac{\partial f}{\partial y} = g_1 \cdots g_h$ be the Merle factorization of the polar $\frac{\partial f}{\partial y}$ and assume that ϕ is an irreducible factor of g_k. Then by Theorem 5.4, we can write ord $\phi = m_k \frac{n}{e_{k-1}}$, where $m_k \geqslant 1$ is an integer. Since ord $\phi < p$, it yields that $m_k < p$ and therefore $m_k \not\equiv 0 \mod p$. Again, by Theorem 5.4

$$i_0(f, \phi) = \frac{e_{k-1}\bar{b}_k}{n} \mathrm{ord}\ \phi = m_k \bar{b}_k.$$

Therefore, $i(f, \phi) \not\equiv 0 \pmod{p}$ if and only if $\bar{b}_k \not\equiv 0 \pmod{p}$. The theorem hence follows from (5.1). ■

Remark 5.1. *If $p <$ ord f, then the proof of Theorem 5.5 fails, even if* ord $f \not\equiv 0$ (mod p). *Take $f = x^{p+2} + y^{p+1} + x^{p+1}y$.*

Conjecture 5.6. *Let $f \in \Bbbk[\![x, y]\!]$ be an irreducible singularity and let $\bar{b}_0, \ldots, \bar{b}_g$ be the minimal system of generators of $S(f)$. Then the following two conditions are equivalent:*

(i) $\bar{b}_k \not\equiv 0 \mod p$, *for $k \in \{1, \ldots, g\}$;*

(ii) $\mu(f) = c(f)$.

The conjecture is true if $S(f)$ is generated by $\bar{b}_0$ and $\bar{b}_1$. Recently, Hefez, Rodrigues and Salomao in [18] have proved that (i) implies (ii).

Theorem 5.7 (Hefez, Rodrigues and Salomao)**.** *Let $f \in \Bbbk[\![x, y]\!]$ be an irreducible singularity and let $\bar{b}_0, \ldots, \bar{b}_g$ be the minimal system of generators of $S(f)$. If $\bar{b}_k \not\equiv 0 \mod p$, for $k \in \{1, \ldots, g\}$ then $\mu(f) = c(f)$.*

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