J. Math. Math. Sci. (2025) 31–54

On meromorphic solution of linear difference - differential equation via partially shared values of meromorphic functions and their growth

Ha Tran Phuong (Thai Nguyen, Vietnam)

Nguyen Van Thin* (Thai Nguyen, Vietnam)

(Received Jan. 5, 2025; accepted Mar. 28, 2025)

Abstract. In this paper, we investigate shared value problems related to a meromorphic function of hyper order less than one and its linear difference-differential polynomial. In general, under certain conditions of sharing values of the meromorphic functions and their difference-differential polynomial, a given meromorphic function must satisfy a difference-differential equation. Furthermore, we also study the order of meromorphic solutions of some classes of difference-differential equations.

1. Introduction

We use standard notations from Nevanlinna theory. Denote by $\sigma(f)$ the order of growth of a meromorphic function f on the complex plane $\mathbb{C}$, and also use the notation $\varsigma(f)$ to denote the hyper order of f,

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \ \ \varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

Key words and phrases: Meromorphic function, Nevanlinna theory.
2020 Mathematics Subject Classification: 30D35
Corresponding author*: Nguyen Van Thin
The research results are supported by the Ministry of Education and Training of Vietnam under the project with the name "The Nevanlinna - Cartan's second main theorem and some applications" and grant number B2024-TNA-19.

respectively, where $T(r,f)$ is the characteristic function of f.
A meromorphic function a is said to be small with respect to f if $T(r,a) = o(T(r,f))$, as $r \to +\infty$ possibly outside a set of finite Lebesgue measure. We denote $\mathcal{S}(f)$ by the set of small functions with respect to f and $\widehat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. Let a be a small function with respect to f. The defect $\delta(f,a)$ of f at a is defined by

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,\dfrac{1}{f-a})}{T(r,f)},\ \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\dfrac{1}{f-a})}{T(r,f)}.$$

We can define another defect as follows:

$$\Theta(\infty,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)},\ \delta(\infty,f) = 1 - \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)}.$$

The five-point theorem due to Nevanlinna states that if two non-constant meromorphic functions f and g in $\mathbb{C}$ share five distinct values ignoring multiplicities (IM), then $f \equiv g$. Recently, Halburd, Korhonen, and Tohge [7, 8, 10], Chiang and Feng [3] extended the Nevanlinna theory for difference operator. Difference Nevanlinna theory has emerged as a result of recent interest on value distribution and growth of meromorphic solutions of difference equations [3, 9], also uniqueness of meromorphic functions with difference polynomials.

Definition 1.1. *[15] Let l be a non-negative integer or infinite. Denote by $E_l(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that f and g share (a,l). It is easy to see that if f and g share (a,l), then f and g share (a,p) for $0 \le p \le l$. Also we note that f and g share the value a - IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively.*

Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}(r,\dfrac{1}{f-a})$ to denote the counting function of the zeros of $f-a$, whose multiplicities are not greater than p, $N_{(p+1}(r,\dfrac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not less than $p+1$, and we use $\overline{N}_{p)}(r,\dfrac{1}{f-a})$ and $\overline{N}_{(p+1}(r,\dfrac{1}{f-a})$ to denote their corresponding reduced counting functions (ignoring multiplicities) respectively. We use $\overline{E}_{p)}(a,f)$ ($\overline{E}_{(p+1}(a,f)$) to denote the set of zeros of $f-a$ with multiplicities $\le p$ ($\ge p+1$) (ignoring multiplicity), respectively. We also denote $N_p(r,\dfrac{1}{f-a})$ by

$$N_p(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \cdots + \overline{N}_{(p}(r,\frac{1}{f-a}).$$

Then we define the truncated deficiency as

$$\delta_p(a,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, \dfrac{1}{f-a})}{T(r,f)}.$$

Let f be a nonconstant meromorphic function with hyper-order less than 1, we denote $L(f)$ by

$$L(f) := \sum_{j=1}^{k} a_j f(z+c_j),$$

where $a_j \neq 0, j = 1,\ldots,k$, $c_j \in \mathbb{C}$ $(j = 1,\ldots,k)$ are distinct complex numbers.

In 2015, Li, Korhonen and Yang [13] proved some results uniqueness for entire function f and its linear difference polynomial $L(f)$ which share partially values, and under some conditions about defect values. In 2020, X. Qi and L. Yang [18] investigated the uniqueness problem for derivative of transcendental entire function of finite order f and $f(z+c)$ share 0-CM and a-IM, where a is a nonzero complex. In 2022, S. Chen and A. Xu [2] extended the results of Qi-Yang [18] as follows: Let f be a non-constant meromorphic function of hyper order $\varsigma(f) < 1$, c be a non-zero finite complex number, and k be a positive integer. If $f^{(k)}(z)$ and $f(z+c)$ share $0,\infty$-CM and $1-IM$, then $f^{(k)}(z) \equiv f(z+c)$. Motivate by the results of Li, Korhonen and Yang [13], in this paper, we first prove a result for uniqueness of meromorphic function and its linear difference-differential polynomial $(L(f))^{(n)}$ as follows.

Theorem 1.1. *Let k,n be positive integer numbers. Let $f(z)$ be a non-constant meromorphic function with hyper order less than* 1, *and assume that $(L(f))^{(n)}$ is not a constant function. Suppose that $f-1$ and $(L(f))^{(n)}-1$ share value $(0,l)$, f and $(L(f))^{(n)}$ share $\infty - IM$ and*

$$\overline{E}_{(i}(0,f) \subset \overline{E}_{(i}(0,(L(f))^{(n)})\ (i \geq 2).$$

Then

$$(L(f))^{(n)} \equiv f \tag{1.1}$$

if one of the following assumptions holds:
(1) $l = 0$ *(i.e. $f-1$ and $(L(f))^{(n)}-1$ share the value* 0 *IM) and*

$$2\delta_2(0,f)+3\Theta(0,f)+((2n+4)k+3)\Theta(\infty,f)+2(k-1)\delta(\infty,f) > (2n+6)k+5;$$

(2) $l = 1$ *and*

$$2\delta_2(0,f)+\frac{1}{2}\Theta(0,f)+((n+2)k+\frac{5}{2})\Theta(\infty,f)+(k-1)\delta(\infty,f) > (n+3)k+3;$$

(3) $l \geq 2$ *and*

$$2\delta_2(0,f) + ((n+2)k+2)\Theta(\infty,f) + (k-1)\delta(\infty,f) > (n+3)k+2.$$

Remark 1.1. *In Theorem 1.1, the condition* $\overline{E}_{(i}(0,f) \subset \overline{E}_{(i}(0,(L(f))^{(n)})$ $(i \geq 2)$ *is weaker than condition* f *and* $(L(f))^{(n)}$ *share* $0-CM$. *If* $(L(f))^{(n)}$ *and* f *share* $0-CM$, *then* $\overline{E}_{(i}(0,f) = \overline{E}_{(i}(0,(L(f))^{(n)})$ $(i \geq 1)$. *Then Theorem 1.1 still holds when* $(L(f))^{(n)}$ *and* f *share* 0-*CM.*

From Theorem 1.1, when f is an entire function, we get the following result:

Corollary 1.1. *Let* k, n *be positive integer numbers. Let* $f(z)$ *be an nonconstant entire function with hyper order less than* 1, *and assume that* $(L(f))^{(n)}$ *is not a constant function. Suppose that* $f-1$ *and* $(L(f))^{(n)}-1$ *share value* $(0,l)$ *and*

$$\overline{E}_{(i}(0,f) \subset \overline{E}_{(i}(0,(L(f))^{(n)}) \ (i \geq 2).$$

Then

$$(L(f))^{(n)} \equiv f$$

if one of the following assumptions holds:

(1) $l = 0$ *(i.e.* $f-1$ *and* $(L(f))^{(n)}-1$ *share the value* 0 *IM) and*

$$2\delta_2(0,f) + 3\Theta(0,f) > 4;$$

(2) $l = 1$ *and*

$$2\delta_2(0,f) + \frac{1}{2}\Theta(0,f) > \frac{3}{2};$$

(3) $l \geq 2$ *and* $\delta_2(0,f) > \dfrac{1}{2}$.

The equation $(L(f))^{(n)} \equiv f$ implies also that f is a solution to a linear difference-differential equation with constant coefficients. Therefore, in the principle, we can give some properties of solutions by using the characteristic equation for linear difference-differential equations. Motivate by the works of X. Qi and L. Yang [18] and S. Chen and A. Xu [2], we prove the uniqueness result for derivative of meromorphic function and its difference polynomial as follows:

Theorem 1.2. *Let* k, n *be positive integer numbers. Let* $f(z)$ *be a nonconstant meromorphic function with hyper order less than* 1, *and assume that* $L(f)$ *and* $f^{(n)}$ *are not constant functions. Suppose that* $f^{(n)}-1$ *and* $L(f)-1$ *share value* $(0,l)$, $f^{(n)}$ *and* $L(f)$ *share* ∞-*IM, and*

$$\overline{E}_{(i}(0,f) \subset \overline{E}_{(i}(0,L(f)) \ (i \geq 2).$$

Then

$$L(f) \equiv f^{(n)} \tag{1.2}$$

if one of the following assumptions holds:
(1) $l = 0$ *(i.e.* $f^{(n)} - 1$ *and* $L(f) - 1$ *share the value* 0 *IM) and*

$$(4k+2n+3)\Theta(\infty, f) + 2(k-1)\delta(\infty, f) + 2\Theta(0, f) + \delta_2(0, f) + 2\delta_{n+1}(0, f) \\ + \delta_{n+2}(0, f) > 6k + 2n + 6;$$

(2) $l = 1$ *and*

$$\delta_2(0, f) + \delta_{n+2}(0, f) + \frac{1}{2}\Theta(0, f) + (2k + \frac{5}{2})\Theta(\infty, f) + (k-1)\delta(\infty, f) > 3k + 3;$$

(3) $l \geq 2$ *and*

$$(2k+2)\Theta(\infty, f) + (k-1)\delta(\infty, f) + \delta_2(0, f) + \delta_{n+2}(0, f) > 3k + 2.$$

Since $f^{(n)}(z)$ and $f(z+c)$ share 0-CM implies that $\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, f(z+c))$ $(i \geq 2)$, then Theorem 1.2 still holds when $f^{(n)}(z)$ and $f(z+c)$ share 0-CM and $L(f) = f(z+c), k = 1$. The assumptions in Theorem 1.2 are weaker than those in Theorem D. Namely, we consider that $f^{(n)}$ and $f(z+c)$ share partially value 0 and ∞-IM, $f^{(n)}$ and $f(z+c)$ share $(1, l)$. We note that the method proving Theorem 1.2 is not the same to [2] and [18]. For more results about uniqueness of meromorphic functions and their shift share partially value, we recommend the readers to [4, 11, 12]. Outside that problem, the uniqueness of difference-differential of meromorphic functions sharing values or small functions which was considered by many authors, we refer the readers to [5, 17] for more details. From Theorem 1.2, we get the following result:

Corollary 1.2. *Let* n *be positive integer numbers. Let* $f(z)$ *be a nonconstant meromorphic function with hyper order less than* 1*, and assume that* $f(z+c)$ *and* $f^{(n)}$ *are not constant functions, where* c *is a nonzero complex number. Suppose that* $f^{(n)} - 1$ *and* $f(z+c) - 1$ *share value* $(0, l)$, $f^{(n)}$ *and* $f(z+c)$ *share* ∞*-IM, and*

$$\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, f(z+c)) \ (i \geq 2).$$

Then

$$f(z+c) \equiv f^{(n)}(z)$$

if one of the following assumptions holds:
(1) $l = 0$ *(i.e.* $f^{(n)} - 1$ *and* $L(f) - 1$ *share the value* 0 *IM) and*

$$(2n+7)\Theta(\infty,f)+2\Theta(0,f)+\delta_2(0,f)+2\delta_{n+1}(0,f)$$
$$+\delta_{n+2}(0,f)>2n+12;$$

(2) $l=1$ *and*

$$\delta_2(0,f)+\delta_{n+2}(0,f)+\frac{1}{2}\Theta(0,f)+\frac{9}{2}\Theta(\infty,f)>6;$$

(3) $l\geq 2$ *and*

$$4\Theta(\infty,f)+\delta_2(0,f)+\delta_{n+2}(0,f)>5.$$

From Theorem 1.2, when $k=1$ and $L(f)=f(z+c)$, we get the following result for entire functions:

Corollary 1.3. *Let k,n be positive integer numbers. Let $f(z)$ be a nonconstant entire function with hyper order less than 1, and assume that $f(z+c)$ and $f^{(n)}$ are not constant functions. Suppose that $f^{(n)}-1$ and $f(z+c)-1$ share value $(0,l)$, and*

$$\overline{E}_{(i}(0,f)\subset\overline{E}_{(i}(0,f(z+c))\ (i\geq 2).$$

Then

$$f(z+c)\equiv f^{(n)}(z)$$

if one of the following assumptions holds:

(1) $l=0$ *(i.e. $f^{(n)}-1$ and $f(z+c)-1$ share the value 0 IM) and*

$$2\Theta(0,f)+\delta_2(0,f)+2\delta_{n+1}(0,f)+\delta_{n+2}(0,f)>5;$$

(2) $l=1$ *and*

$$\delta_2(0,f)+\delta_{n+2}(0,f)+\frac{1}{2}\Theta(0,f)>\frac{3}{2};$$

(3) $l\geq 2$ *and*

$$\delta_2(0,f)+\delta_{n+2}(0,f)>1.$$

Finally, we study the growth of solutions to equations (1.1) and (1.2).

Theorem 1.3. *The order of all transcendental meromorphic solutions f of equations (1.1) and (1.2) must satisfy $\sigma(f)\geq 1$.*

Example 1.4. *The function* $f(z) = \sin z$ *has order* $\sigma(f) = 1$ *and* f *is a solution of equation*

$$f'(z) = -2f(z+\pi) + f(z - \frac{\pi}{2}).$$

Here $L(f) = -2f(z+\pi) + f(z - \frac{\pi}{2})$. *We also have that* f *is a solution of*

$$f'(z+\pi) = f(z),$$

where $L(f) = f(z+\pi)$.

2. Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 (Halburd-Korhonen-Tohge [10])**.** *Let* $h : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing continuous function, and let* $s \in (0, +\infty)$. *If the hyper order of* h *is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log h(r)}{\log r} = \varsigma < 1,$$

then

$$h(r+s) = h(r) + o(\frac{h(r)}{r^{1-\varsigma-\varepsilon}}),$$

where $\varepsilon > 0$ *and* $r \to \infty$ *outside of a set of finite logarithmic measure.*

From Lemma 2.1, we get the following corollary.

Corollary 2.1. *[1, 10] Let* f *be a non-constant meromorphic function with* $\varsigma(f) = \varsigma < 1$, *and* $c \in \mathbb{C} \setminus \{0\}$. *Then*

$$N(r, f(z+c)) \le N(r,f) + S(r,f), \qquad \overline{N}(r, f(z+c)) \le \overline{N}(r,f) + S(r,f),$$

$$N(r, \frac{1}{f(z+c)}) \le N(r, \frac{1}{f}) + S(r,f), \qquad \overline{N}(r, \frac{1}{f(z+c)}) \le \overline{N}(r, \frac{1}{f}) + S(r,f),$$

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 2.2. *[19] Let* n *be a postive integer number. Let* f *be a non-constant meromorphic function such that* $f^{(n)} \not\equiv 0$. *Then*

$$N(r, \frac{1}{f^{(n)}}) \leqslant T(r, f^{(n)}) - T(r,f) + N(r, \frac{1}{f}) + S(r,f);$$

$$N(r, \frac{1}{f^{(n)}}) \leqslant n\overline{N}(r,f) + N(r, \frac{1}{f}) + S(r,f).$$

Lemma 2.3. *[21] Let p and k be two positive integers. Let f be a non-constant meromorphic function such that $f^{(k)} \not\equiv 0$. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f);$$
$$N_p(r, \frac{1}{f^{(k)}}) \leqslant k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.4. *[20] Let f and g be two non-constant meromorphic functions, and let $a(z)$ $(a \not\equiv 0, \infty)$ be a small function of both f and g. If f and g share $(a(z), 0)$, then one of the following three cases holds:*

$$(i)\ \ T(r, f) \leqslant N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) + 2(\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f)) + (\overline{N}(r, \frac{1}{g}) + \overline{N}(r, g)) + S(r, f) + S(r, g),$$

and the similar inequality holds for T(r,g);

(ii) $f \equiv g$;

(iii) $fg \equiv a^2$.

Lemma 2.5. *[20] Let f and g be two non-constant meromorphic functions, and let $a(z)$ $(a \not\equiv 0, \infty)$ be a small function of both f and g. If f and g share $(a(z), 1)$, then one of the following three cases holds:*

$$(i)\ \ T(r, f) \leqslant N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) + \frac{1}{2}(\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f)) + S(r, f) + S(r, g),$$

and the similar inequality holds for T(r,g);

(ii) $f \equiv g$;

(iii) $fg \equiv a^2$.

Lemma 2.6. *[16, 20] Let f and g be two non-constant meromorphic functions, and let $a(z)$ $(a \not\equiv 0, \infty)$ be a small function of both f and g. If f and g share $(a(z), l), l \geq 2$, then one of the following three cases holds:*

$$(i)\ \ T(r, f) \leqslant N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) + S(r, f) + S(r, g)$$

and the similar inequality holds for T(r,g);

(ii) $f \equiv g$;

(iii) $fg \equiv a^2$.

Lemma 2.7. *[13] Let f be a non-constant meromorphic function with hyper-order less than* 1*, and $L(f) \not\equiv 0$ be defined as in Theorem A. Then*

$$N(r, \frac{1}{L(f)}) \leq T(r, L(f)) - T(r, f) + N(r, \frac{1}{f}) + S(r, f),$$
$$N(r, \frac{1}{L(f)}) \leq (k-1)N(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

From Lemma 2.7, we get the following result:

Lemma 2.8. *Let n, p be integer numbers. Let f be a non-constant meromorphic function with hyper order less than* 1 *such that $L(f) \not\equiv 0$. Suppose $\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, L(f))$ (all $i \geq p+1$). Then*

$$N_p(r, \frac{1}{L(f)}) \leq T(r, L(f)) - T(r, f) + N_p(r, \frac{1}{f}) + S(r, f),$$
$$N_p(r, \frac{1}{L(f)}) \leq (k-1)N(r, f) + N_p(r, \frac{1}{f}) + S(r, f).$$

Proof. Apply to Lemma 2.7, we have

$$N(r, \frac{1}{L(f)}) \leqslant T(r, L(f)) - T(r, f) + N(r, \frac{1}{f}) + S(r, f). \tag{2.1}$$

We have

$$N(r, \frac{1}{L(f)}) = N_p(r, \frac{1}{L(f)}) + \sum_{j=p+1}^{\infty} \overline{N}_{(j}(r, \frac{1}{L(f)}) \tag{2.2}$$

and

$$N(r, \frac{1}{f}) = N_p(r, \frac{1}{f}) + \sum_{j=p+1}^{\infty} \overline{N}_{(j}(r, \frac{1}{f}). \tag{2.3}$$

Hence, combining (2.1) to (2.3) and by the assumption

$$\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, L(f)) \text{ (all } i \geq p+1),$$

we get $\overline{N}_{(j}(r, \frac{1}{f}) \leq \overline{N}_{(j}(r, \frac{1}{L(f)})$ for all $j \geq p+1$. Using Lemma 2.7 and (2.2), we have

$$\begin{aligned} N_p(r, \frac{1}{L(f)}) \leq T(r, L(f)) - T(r, f) - \sum_{j=p+1}^{\infty} \overline{N}_{(j}(r, \frac{1}{L(f)}) \\ + N(r, \frac{1}{f}) + S(r, f). \end{aligned} \tag{2.4}$$

Combine (2.3) and (2.4) to get

$$\begin{aligned}N_p(r,\frac{1}{L(f)}) &\leq T(r,L(f)) - T(r,f) + N_p(r,\frac{1}{f})\\ &\quad + \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{f}) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{L(f)}) + S(r,f)\\ &\leq T(r,L(f)) - T(r,f) + N_p(r,\frac{1}{f}) + S(r,f).\end{aligned}$$

The remain inequality is similarly proved. For convenience to readers, we write some steps as follows. From (2.1) and Lemma 2.7, we have

(2.5)

$$N_p(r,\frac{1}{L(f)}) \leq (k-1)N(r,f) + N(r,\frac{1}{f}) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{L(f)}) + S(r,f).$$

Then second statement comes from (2.3) and (2.5). ■

Next, we prove some results as following:

Lemma 2.9. *Let n be a integer number. Let f be a non-constant meromorphic function with hyper order less than 1 such that $(L(f))^{(n)} \not\equiv 0$. Then*

$$\begin{aligned}N(r,\frac{1}{(L(f))^{(n)}}) &\leq T(r,(L(f))^{(n)}) - T(r,f) + N(r,\frac{1}{f}) + S(r,f),\\ N(r,\frac{1}{(L(f))^{(n)}}) &\leq nk\overline{N}(r,f) + (k-1)N(r,f) + N(r,\frac{1}{f}) + S(r,f).\end{aligned}$$

Proof. Apply Lemma 2.2, we have

$$N(r,\frac{1}{(L(f))^{(n)}}) \leqslant T(r,(L(f))^{(n)}) - T(r,L(f)) + N(r,\frac{1}{L(f)}) + S(r,f). \tag{2.6}$$

By Lemma 2.7, from (2.6), we get

$$N(r,\frac{1}{L(f)}) \leqslant T(r,L(f)) - T(r,f) + N(r,\frac{1}{f}) + S(r,f). \tag{2.7}$$

Combine (2.6) and (2.7), we get the first inequality. Next, we show the second inequality. By Lemma 2.2, we have

$$N(r,\frac{1}{(L(f))^{(n)}}) \leqslant n\overline{N}(r,L(f)) + N(r,\frac{1}{L(f)}) + S(r,f). \tag{2.8}$$

Combining (2.8), Lemma 2.7 and Corollary 2.1, we obtain

$$N(r, \frac{1}{(L(f))^{(n)}}) \leq nk\overline{N}(r, f) + (k-1)N(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

■

From Lemma 2.9, we get the following result.

Corollary 2.2. *Let n be a integer number. Let f be a non-constant entire function with hyper order less than 1 such that $(L(f))^{(n)} \not\equiv 0$. Then*

$$N(r, \frac{1}{(L(f))^{(n)}}) \leq T(r, (L(f))^{(n)}) - T(r, f) + N(r, \frac{1}{f}) + S(r, f),$$
$$N(r, \frac{1}{(L(f))^{(n)}}) \leq N(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.10. *Let n, p be integer numbers. Let f be a non-constant meromorphic function with hyper order less than 1 such that $(L(f))^{(n)} \not\equiv 0$. Suppose $\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, (L(f))^{(n)})$ (all $i \geq p+1$). Then*

$$N_p(r, \frac{1}{(L(f))^{(n)}}) \leq T(r, (L(f))^{(n)}) - T(r, f) + N_p(r, \frac{1}{f}) + S(r, f),$$
$$N_p(r, \frac{1}{(L(f))^{(n)}}) \leq nk\overline{N}(r, f) + (k-1)N(r, f) + N_p(r, \frac{1}{f}) + S(r, f).$$

Proof. Apply Lemma 2.9, we have

$$N(r, \frac{1}{(L(f))^{(n)}}) \leqslant T(r, (L(f))^{(n)}) - T(r, f) + N(r, \frac{1}{f}) + S(r, f). \tag{2.9}$$

We have

$$N(r, \frac{1}{(L(f))^{(n)}}) = N_p(r, \frac{1}{(L(f))^{(n)}}) + \sum_{j=p+1}^{\infty} \overline{N}_{(j}(r, \frac{1}{(L(f))^{(n)}}) \tag{2.10}$$

and

$$N(r, \frac{1}{f}) = N_p(r, \frac{1}{f}) + \sum_{j=p+1}^{\infty} \overline{N}_{(j}(r, \frac{1}{f}). \tag{2.11}$$

Hence, combining (2.9) to (2.11) and by the assumption

$$\overline{E}_{(i}(0, f) \subset \overline{E}_{(i}(0, (L(f))^{(n)}) \text{ (all } i \geq p+1),$$

we get

$$\begin{aligned}N_p(r,\frac{1}{(L(f))^{(n)}}) &\le T(r,(L(f))^{(n)}) - T(r,f) + N_p(r,\frac{1}{f})\\ &\quad + \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{f}) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{(L(f))^{(n)}}) + S(r,f)\\ &\le T(r,(L(f))^{(n)}) - T(r,f) + N_p(r,\frac{1}{f}) + S(r,f).\end{aligned}$$

By Lemma 2.9, we have

$$N(r,\frac{1}{(L(f))^{(n)}}) \le nk\overline{N}(r,f) + (k-1)N(r,f) + N(r,\frac{1}{f}) + S(r,f). \tag{2.12}$$

Hence, combining (2.9), (2.11) and (2.12), we obtain

$$\begin{aligned}N_p(r,\frac{1}{(L(f))^{(n)}}) &\le nk\overline{N}(r,f) + (k-1)N(r,f) + N_p(r,\frac{1}{f})\\ &\quad + \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{f}) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,\frac{1}{(L(f))^{(n)}}) + S(r,f)\\ &\le nk\overline{N}(r,f) + (k-1)N(r,f) + N_p(r,\frac{1}{f}) + S(r,f).\end{aligned}$$

■

From Lemma 2.10, we get the following result.

Corollary 2.3. *Let n, p be integer numbers. Let f be a non-constant entire function with hyper order less than 1 such that $(L(f))^{(n)} \not\equiv 0$. Suppose $\overline{E}_{(i}(0,f) \subset \overline{E}_{(i}(0,(L(f))^{(n)})$ (all $i \ge p+1$). Then*

$$\begin{aligned}N_p(r,\frac{1}{(L(f))^{(n)}}) &\le T(r,(L(f))^{(n)}) - T(r,f) + N_p(r,\frac{1}{f}) + S(r,f),\\ N_p(r,\frac{1}{(L(f))^{(n)}}) &\le N_p(r,\frac{1}{f}) + S(r,f).\end{aligned}$$

Lemma 2.11. *Let c_1 and c_2 be two arbitrary complex numbers, and let f be a meromorphic function of finite order σ. Assume that $\varepsilon > 0$, then there exists a subset $E \subset \mathbb{R}$ with finite logarithmic measure so that for all $|z| = r \notin E \cup [0,1]$, we have*

$$exp(-r^{\sigma-1+\varepsilon}) \le \left|\frac{f(z+c_1)}{f(z+c_2)}\right| \le exp(r^{\sigma-1+\varepsilon}).$$

Lemma 2.12. *[6, Corollary 1] Assume that f is a transcendental meromorphic function of finite order $\sigma = \sigma(f)$. Let $\varepsilon > 0$, $k > j \geq 0$ be two integers. Then there exists a set $E \subset [0, 2\pi)$ with linear measure zero, so that if $\varphi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\varphi) > 0$ so that for all z verifying $arg z = \varphi$ and $|z| \geq R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.13. *Assume that f is a transcendental meromorphic function of finite order $\sigma = \sigma(f)$. Let c_1 and c_2 be complex numbers and k is a positive integer and $\varepsilon > 0$. Then there is a subset $E_1 \subset \mathbb{R}$ with finite logarithmic measure and set $E \subset [0, 2\pi)$ with linear measure zero so that if $z = re^{i\varphi}$, $\varphi \in [0, 2\pi) \setminus E$, we have that*

$$\left| \frac{f^{(k)}(z+c_1)}{f(z+c_2)} \right| \leq |z|^{k(\sigma-1+\varepsilon)} exp(r^{\sigma-1+\varepsilon})$$

holds for all $|z| = r \geq r_0(\varphi) > 1$ and $|z| \notin E_1$.

Proof. Since f has finite order, then by Corollary 2.1, we have

$$T(r, f(z+c_1)) = T(r, f) + o(T(r, f)).$$

It implies that $f(z+c_1)$ has finite order and $\sigma f(z+c_1) = \sigma(f)$. By Lemma 2.12 for $g(z) = f(z+c_1)$, there is a set $E \subset [0, 2\pi)$ with linear measure zero, so that if $\varphi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\varphi) > 1$ so that

$$\left| \frac{g^{(k)}(z)}{g(z)} \right| \leq |z|^{k(\sigma-1+\varepsilon)} \tag{2.13}$$

holds for all z satisfying $arg z = \varphi$ and $|z| \geq R_0 > 1$. Using Lemma 2.11, there is a subset $E \subset \mathbb{R}$ with finite logarithmic measure so that for all $r \notin E_1 \cup [0, 1]$, we have

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left| \frac{f(z+c_1)}{f(z+c_2)} \right| \leq \exp(r^{\sigma-1+\varepsilon}). \tag{2.14}$$

Combine (2.13) and (2.13), we deduce that

$$\left| \frac{f^{(k)}(z+c_1)}{f(z+c_2)} \right| = \left| \frac{f^{(k)}(z+c_1)}{f(z+c_1)} \frac{f(z+c_1)}{f(z+c_2)} \right| \leq |z|^{k(\sigma-1+\varepsilon)} \exp(r^{\sigma-1+\varepsilon})$$

holds for all $z : arg z = \varphi$ and $|z| \geq R_0 > 1$ and $|z| \notin E_1$. ■

3. Proof of Theorems

3.1. Proof of Theorem 1.1

Proof. From the conditions of Theorem 1.1, we know that f and $(L(f))^{(n)}$ share $(1,l)$. We consider three cases as following of l.

Case 1: $l=0$. Apply Lemma 2.4, we may assume that two following inequalities hold:

$$\begin{aligned} T(r,(L(f))^{(n)}) &\leqslant N_2(r,(L(f))^{(n)})+N_2(r,\frac{1}{(L(f))^{(n)}})+N_2(r,f)+N_2(r,\frac{1}{f}) \\ &\quad +2(\overline{N}(r,\frac{1}{(L(f))^{(n)}})+\overline{N}(r,(L(f))^{(n)}))+(\overline{N}(r,\frac{1}{f})+\overline{N}(r,f))+S(r,f), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} T(r,f) &\leqslant N_2(r,f)+N_2(r,\frac{1}{f})+N_2(r,(L(f))^{(n)})+N_2(r,\frac{1}{(L(f))^{(n)}}) \\ &\quad +2(\overline{N}(r,\frac{1}{f})+\overline{N}(r,f))+(\overline{N}(r,\frac{1}{(L(f))^{(n)}})+\overline{N}(r,(L(f))^{(n)}))+S(r,f). \end{aligned} \tag{3.2}$$

First, from Corollary 2.1, we have

$$N_2(r,(L(f))^{(n)})\leq 2\overline{N}(r,(L(f))^{(n)})=2\overline{N}(r,L(f))\leq 2k\overline{N}(r,f)+S(r,f). \tag{3.3}$$

By Lemma 2.10, we know

$$\begin{aligned} N_2(r,\frac{1}{(L(f))^{(n)}}) &\leq nk\overline{N}(r,f)+(k-1)N(r,f)+N_2(r,\frac{1}{f})+S(r,f), \\ \overline{N}(r,\frac{1}{(L(f))^{(n)}}) &\leq nk\overline{N}(r,f)+(k-1)N(r,f)+\overline{N}(r,\frac{1}{f})+S(r,f). \end{aligned} \tag{3.4}$$

Still using Lemma 2.10 and (3.1), (3.3)-(3.4), we get

$$\begin{aligned} T(r,(L(f))^{(n)}) &\leqslant T(r,(L(f))^{(n)})-T(r,f)+2N_2(r,\frac{1}{f})+3\overline{N}(r,\frac{1}{f}) \\ &\quad +(k(2n+4)+3)\overline{N}(r,f)+2(k-1)N(r,f)+S(r,f). \end{aligned}$$

This implies

$$T(r,f) \leqslant 2N_2(r,\frac{1}{f}) + 3\overline{N}(r,\frac{1}{f}) + (k(2n+4)+3)\overline{N}(r,f) + 2(k-1)N(r,f) + S(r,f). \tag{3.5}$$

Similarly, from Lemma 2.10 and (3.2), we obtain

$$\begin{aligned} T(r,f) &\leqslant 2N_2(r,\frac{1}{f}) + 3\overline{N}(r,\frac{1}{f}) + (k(2n+3)+4)\overline{N}(r,f) \\ &\quad + 2(k-1)N(r,f) + S(r,f) \\ &\leq 2N_2(r,\frac{1}{f}) + 3\overline{N}(r,\frac{1}{f}) + (k(2n+4)+3)\overline{N}(r,f) \\ &\quad + 2(k-1)N(r,f) + S(r,f). \end{aligned} \tag{3.6}$$

Therefore, combining (3.5) and (3.6), we get

$$\begin{aligned} T(r,f) \leqslant\ & 2(1-\delta_2(0,f))T(r,f) + 3(1-\Theta(0,f))T(r,f) \\ & + (k(2n+4)+3)(1-\Theta(\infty,f))T(r,f) \\ & + 2(k-1)(1-\delta(\infty,f))T(r,f) + S(r,f). \end{aligned}$$

This implies $(K_1 - ((2n+6)k+5))T(r,f) \leq S(r,f)$, where

$$\begin{aligned} K_1 =\ & 2\delta_2(0,f) + 3\Theta(0,f) + ((2n+4)k+3)\Theta(\infty,f) \\ & + 2(k-1)\delta(\infty,f) - ((2n+6)k+5) > 0 \end{aligned}$$

since

$$2\delta_2(0,f) + 3\Theta(0,f) + ((2n+4)k+3)\Theta(\infty,f) + 2(k-1)\delta(\infty,f) > (2n+6)k+5.$$

This is a contradiction. Thus, by Lemma 2.4, we must have $f \equiv (L(f))^{(n)}$ or $f.(L(f))^{(n)} \equiv 1$. We consider the case $f.(L(f))^{(n)} \equiv 1$. Since f and $(L(f))^{(n)}$ share $\infty-$ IM, then the case $f.(L(f))^{(n)} \equiv 1$ is impossible. Hence, we obtain

$$f \equiv (L(f))^{(n)}.$$

We have finished the proof of Theorem 1.1 in the case $l = 0$.

Case 2: $l = 1$. Apply to Lemma 2.5, we may assume that two inequality below hold:

$$\begin{aligned} T(r,(L(f))^{(n)}) \leqslant\ & N_2(r,(L(f))^{(n)}) + N_2(r,\frac{1}{(L(f))^{(n)}}) + N_2(r,f) + N_2(r,\frac{1}{f}) \\ & + \frac{1}{2}(\overline{N}(r,\frac{1}{(L(f))^{(n)}}) + \overline{N}(r,(L(f))^{(n)})) + S(r,f), \end{aligned} \tag{3.7}$$

and

$$T(r,f) \leqslant N_2(r,f) + N_2(r,\frac{1}{f}) + N_2(r,(L(f))^{(n)}) + N_2(r,\frac{1}{(L(f))^{(n)}}) + \frac{1}{2}(\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f)) + S(r,f). \tag{3.8}$$

Combine Lemma 2.10 and (3.7), we get

$$T(r,(L(f))^{(n)}) \leqslant T(r,(L(f))^{(n)}) - T(r,f) + 2N_2(r,\frac{1}{f}) + \frac{1}{2}\overline{N}(r,\frac{1}{f}) + ((\frac{n+5}{2})k+2)\overline{N}(r,f) + \frac{k-1}{2}N(r,f) + S(r,f).$$

This implies

$$T(r,f) \leqslant 2N_2(r,\frac{1}{f}) + \frac{1}{2}\overline{N}(r,\frac{1}{f}) + ((\frac{n+5}{2})k+2)\overline{N}(r,f) + \frac{k-1}{2}N(r,f) + S(r,f). \tag{3.9}$$

Similarly, from Lemma 2.10, (3.3)-(3.4) and (3.8), we obtain

$$T(r,f) \leqslant 2N_2(r,\frac{1}{f}) + \frac{1}{2}\overline{N}(r,\frac{1}{f}) + ((n+2)k+\frac{5}{2})\overline{N}(r,f) + (k-1)N(r,f) + S(r,f). \tag{3.10}$$

Since

$$((\frac{n+5}{2}k+2)\overline{N}(r,f) + \frac{k-1}{2}N(r,f) \leq ((n+2)k+\frac{5}{2})\overline{N}(r,f) + (k-1)N(r,f),$$

then, combining (3.9) and (3.10), we get

$$T(r,f) \leqslant 2(1-\delta_2(0,f))T(r,f) + \frac{1}{2}(1-\Theta(0,f))T(r,f) + ((n+2)k+\frac{5}{2})(1-\Theta(\infty,f))T(r,f) + (k-1)(1-\delta(\infty,f))T(r,f) + S(r,f).$$

This implies

$$(K_2 - ((n+3)k+3))T(r,f) \leq S(r,f),$$

where

$$K_2 = 2\delta_2(0,f) + \frac{1}{2}\Theta(0,f) + ((n+2)k+\frac{5}{2})\Theta(\infty,f) + (k-1)\delta(\infty,f).$$

This is a contradiction with

$$2\delta_2(0,f) + \frac{1}{2}\Theta(0,f) + ((n+2)k+\frac{5}{2})\Theta(\infty,f) + (k-1)\delta(\infty,f) > (n+3)k+3.$$

By an argument as Case 1, we have

$$f \equiv (L(f))^{(n)}.$$

Case 3: $l \geq 2$. Apply Lemma 2.6, we may assume that two inequalities below hold.

$$T(r,(L(f))^{(n)}) \leqslant N_2(r,(L(f))^{(n)}) + N_2(r, \frac{1}{(L(f))^{(n)}}) + N_2(r,f) + N_2(r,\frac{1}{f}) + S(r,f), \tag{3.11}$$

and

$$T(r,f) \leqslant N_2(r,f) + N_2(r,\frac{1}{f}) + N_2(r,(L(f))^{(n)}) + N_2(r,\frac{1}{(L(f))^{(n)}}) + S(r,f). \tag{3.12}$$

Using Lemma 2.10, (3.3)-(3.4) and (3.11), (3.12) implies that

$$T(r,f) \leqslant 2N_2(r,\frac{1}{f}) + ((n+2)k+2)\overline{N}(r,f) + (k-1)N(r,f) + S(r,f). \tag{3.13}$$

Indeed, (3.11) implies

$$\begin{aligned} T(r,f) &\leq (2k+2)\overline{N}(r,f) + 2N_2(r,\frac{1}{f}) + S(r,f) \\ &\leq 2N_2(r,\frac{1}{f}) + ((n+2)k+2)\overline{N}(r,f) + (k-1)N(r,f) + S(r,f). \end{aligned}$$

Therefore, from (3.13) we deduce

$$\begin{aligned} T(r,f) \leqslant\ & 2(1-\delta_2(0,f))T(r,f) + ((n+2)k+2)(1-\Theta(\infty,f))T(r,f) \\ & + (k-1)(1-\delta(\infty,f))T(r,f) + S(r,f). \end{aligned}$$

This implies $(K_3 - ((n+3)k+2))T(r,f) \leq S(r,f)$, where

$$K_3 = 2\delta_2(0,f) + ((n+2)k+2)\Theta(\infty,f) + (k-1)\delta(\infty,f).$$

This is a contradiction with

$$2\delta_2(0,f) + ((n+2)k+2)\Theta(\infty,f) + (k-1)\delta(\infty,f) > (n+3)k+2.$$

By an argument as Case 1, we have $f \equiv (L(f))^{(n)}$. ■

3.2. Proof of Theorem 1.2

Proof. From the conditions of Theorem 1.2, we know that $f^{(n)}$ and $L(f)$ share $(1, l)$. We consider three cases as following of l.

Case 1: $l = 0$. Apply Lemma 2.4, we may assume that two following inequalities hold:

$$\begin{aligned}(3.14)\quad & T(r, L(f)) \leqslant N_2(r, L(f)) + N_2(r, \frac{1}{L(f)}) + N_2(r, f^{(n)}) + N_2(r, \frac{1}{f^{(n)}}) \\ & + 2(\overline{N}(r, \frac{1}{L(f)}) + \overline{N}(r, L(f))) + (\overline{N}(r, \frac{1}{f^{(n)}}) + \overline{N}(r, f^{(n)})) + S(r, f),\end{aligned}$$

and

$$\begin{aligned}(3.15)\quad & T(r, f^{(n)}) \leqslant N_2(r, f^{(n)}) + N_2(r, \frac{1}{f^{(n)}}) + N_2(r, L(f)) + N_2(r, \frac{1}{L(f)}) \\ & + 2(\overline{N}(r, \frac{1}{f^{(n)}}) + \overline{N}(r, f^{(n)})) + (\overline{N}(r, \frac{1}{L(f)}) + \overline{N}(r, L(f))) + S(r, f).\end{aligned}$$

From Corrollary 2.1 and (3.14), we have

$$\begin{aligned}(3.16)\quad T(r, L(f)) & \leqslant (2k+2)\overline{N}(r, f) + N_2(r, \frac{1}{L(f)}) + N_2(r, \frac{1}{f^{(n)}}) \\ & + (2k+1)\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{L(f)}) + \overline{N}(r, \frac{1}{f^{(n)}}) + S(r, f),\end{aligned}$$

Using Lemma 2.2 and Lemma 2.8, (3.16) implies that

$$\begin{aligned}T(r, L(f)) & \leqslant (2k+2)\overline{N}(r, f) + T(r, L(f)) - T(r, f) + N_2(r, \frac{1}{f}) \\ & + n\overline{N}(r, f) + N_{n+2}(r, \frac{1}{f}) + (2k+1)\overline{N}(r, f) + 2((k-1)N(r, f) \\ & + \overline{N}(r, \frac{1}{f})) + n\overline{N}(r, f) + N_{n+1}(r, \frac{1}{f}) + S(r, f).\end{aligned}$$

Hence, we deduce

$$\begin{aligned}(3.17)\quad T(r, f) & \leq (4k+2n+3)\overline{N}(r, f) + 2(k-1)N(r, f) + 2\overline{N}(r, \frac{1}{f}) \\ & + N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f}) + 2N_{n+1}(r, \frac{1}{f}) + S(r, f).\end{aligned}$$

From (3.15), using Lemma 2.2 and Lemma 2.8, we have

$$
\begin{aligned}
T(r,f) &\leqslant (2n+3k+4)\overline{N}(r,f)+2(k-1)N(r,f)+\overline{N}(r,\frac{1}{f})\\
&\quad + N_2(r,\frac{1}{f})+2N_{n+1}(r,\frac{1}{f})+N_{n+2}(r,\frac{1}{f})\\
&\le (4k+2n+3)\overline{N}(r,f)+2(k-1)N(r,f)+2\overline{N}(r,\frac{1}{f})\\
&\quad + N_2(r,\frac{1}{f})+2N_{n+1}(r,\frac{1}{f})+N_{n+2}(r,\frac{1}{f})+S(r,f). \qquad (3.18)
\end{aligned}
$$

From (3.17) and (3.18), we have $K_4T(r,f)\le S(r,f)$, where

$$
\begin{aligned}
K_4 &= (4k+2n+3)\Theta(\infty,f)+2(k-1)\delta(\infty,f)+2\Theta(0,f)+\delta_2(0,f)\\
&\quad +2\delta_{n+1}(0,f)+\delta_{n+2}(0,f)-(6k+2n+6).
\end{aligned}
$$

It is a contradiction since

$$
\begin{aligned}
(4k+2n+3)\Theta(\infty,f)+2(k-1)\delta(\infty,f)+2\Theta(0,f)+\delta_2(0,f)+2\delta_{n+1}(0,f)\\
+\delta_{n+2}(0,f)>(6k+2n+6).
\end{aligned}
$$

Thus, by Lemma 2.4, we must have $f^{(n)}\equiv L(f)$ or $f^{(n)}.L(f)\equiv 1$. The equality $f^{(n)}.L(f)\equiv 1$ cannot occur since $f^{(n)}$ and $L(f)$ share ∞-IM. Hence, we obtain

$$
f\equiv (L(f))^{(n)}.
$$

We have finished the proof of Theorem 1.2 in the case $l=0$.

Case 2: $l=1$. Apply Lemma 2.5, we may assume that two inequalities below hold:

$$
\begin{aligned}
T(r,L(f)) &\leqslant N_2(r,L(f))+N_2(r,\frac{1}{L(f)})+N_2(r,f)+N_2(r,\frac{1}{f})\\
&\quad +\frac{1}{2}(\overline{N}(r,\frac{1}{L(f)})+\overline{N}(r,L(f)))+S(r,f), \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
T(r,f^{(n)}) &\leqslant N_2(r,f^{(n)})+N_2(r,\frac{1}{f^{(n)}})+N_2(r,L(f))+N_2(r,\frac{1}{L(f)})\\
&\quad +\frac{1}{2}(\overline{N}(r,\frac{1}{f})+\overline{N}(r,f))+S(r,f). \qquad (3.20)
\end{aligned}
$$

Combine Lemma 2.8 and (3.19), we get

$$T(r, L(f)) \leqslant (2k+2)\overline{N}(r, f) + T(r, L(f)) - T(r, f) + 2N_2(r, \frac{1}{f}) + \frac{1}{2}((k-1)N(r, f) + \overline{N}(r, \frac{1}{f})) + \frac{k}{2}\overline{N}(r, f) + S(r, f).$$

It implies that

$$\begin{aligned} T(r, f) &\leq 2N_2(r, \frac{1}{f}) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) \\ &\quad + (\frac{5k}{2} + 2)\overline{N}(r, f) + \frac{k-1}{2}N(r, f) + S(r, f) \\ &\leq N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f}) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) \\ &\quad + (2k + \frac{5}{2})\overline{N}(r, f) + (k-1)N(r, f) + S(r, f). \end{aligned} \tag{3.21}$$

Similarly, from Lemma 2.3, Lemma 2.8 and (3.20), we obtain

$$T(r, f) \leqslant T(r, f^{(n)}) - T(r, f) + (2k + \frac{5}{2})\overline{N}(r, f) + (k-1)N(r, f) + N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f}) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) + S(r, f).$$

Hence, we deduce

$$\begin{aligned} T(r, f) &\leqslant (2k + \frac{5}{2})\overline{N}(r, f) + (k-1)N(r, f) + N_2(r, \frac{1}{f}) \\ &\quad + N_{n+2}(r, \frac{1}{f}) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) + S(r, f). \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we get $(K_5 - ((3k+3))T(r, f) \leq S(r, f)$, where

$$K_5 = \delta_2(0, f) + \delta_{n+2}(0, f) + \frac{1}{2}\Theta(0, f) + (2k + \frac{5}{2})\Theta(\infty, f) + (k-1)\delta(\infty, f).$$

It is a contradiction with

$$\delta_2(0, f) + \delta_{n+2}(0, f) + \frac{1}{2}\Theta(0, f) + ((2k + \frac{5}{2})\Theta(\infty, f) + (k-1)\delta(\infty, f) > 3k + 3.$$

By an argument as Case 1 of Theorem 1.1, we have

$$f^{(n)} \equiv L(f).$$

Case 3: $l \geq 2$. Apply Lemma 2.6, we may assume that two inequalities below hold.

$$T(r, L(f)) \leqslant N_2(r, L(f)) + N_2(r, \frac{1}{L(f)})$$
$$+ N_2(r, f) + N_2(r, \frac{1}{f}) + S(r, f), \tag{3.23}$$

and

$$T(r, f^{(n)}) \leqslant N_2(r, f^{(n)}) + N_2(r, \frac{1}{f^{(n)}}) + N_2(r, L(f)) + N_2(r, \frac{1}{L(f)}) + S(r, f). \tag{3.24}$$

Combine Lemma 2.8 and (3.23), we get

$$T(r, L(f)) \leqslant T(r, L(f)) - T(r, f) + (2k + 2)\overline{N}(r, f) + 2N_2(r, \frac{1}{f}) + S(r, f).$$

This implies

$$T(r, f) \leqslant 2N_2(r, \frac{1}{f}) + (2k + 2)\overline{N}(r, f) + S(r, f)$$
$$\leq N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f}) + (k - 1)N(r, f) + (2k + 2)\overline{N}(r, f) + S(r, f). \tag{3.25}$$

Using Lemma 2.3, Lemma 2.8 and (3.24), we deduce

$$T(r, f^{(n)}) \leq (2k + 2)\overline{N}(r, f) + (k - 1)N(r, f) + N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f})$$
$$+ T(r, f^{(n)}) - T(r, f) + S(r, f).$$

It implies that

$$T(r, f) \leq (2k + 2)\overline{N}(r, f) + (k - 1)N(r, f) + N_2(r, \frac{1}{f}) + N_{n+2}(r, \frac{1}{f}) + S(r, f). \tag{3.26}$$

From (3.25) and (3.26), we get $(K_6 - (3k + 2))T(r, f) \leqslant S(r, f)$, where

$$K_6 = (2k + 2)\Theta(\infty, f) + (k - 1)\delta(\infty, f) + \delta_2(0, f) + \delta_{n+2}(0, f).$$

This is a contradiction with

$$(2k + 2)\Theta(\infty, f) + (k - 1)\delta(\infty, f) + \delta_2(0, f) + \delta_{n+2}(0, f) > 3k + 2.$$

By an argument as Case 1, we have

$$f^{(n)} \equiv L(f).$$

■

3.3. Proof of Theorem 1.3

Proof. First, we assume that f is a transcendental meromorphic solution of (1.1). It means that

$$(\sum_{j=1}^{k} a_j f(z+c_j))^{(n)} = f. \tag{3.27}$$

Assume that the solution of (3.27) has order $\sigma(f) < 1$, then we can choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - \sigma$. Apply Lemma 2.13, there is a subset $E_1^j \subset \mathbb{R}$ with finite logarithmic measure and set $E_j \subset [0, 2\pi)$ with linear measure zero so that if $z = re^{i\varphi}$, $\varphi \in [0, 2\pi) \setminus E_j$, we have that

$$\left|\frac{f^{(n)}(z+c_j)}{f(z)}\right| \leq |z|^{n(\sigma-1+\varepsilon)} \exp(r^{\sigma-1+\varepsilon}),\ j = 1, \ldots, k, \tag{3.28}$$

hold for all $|z| = r \geq r_j(\varphi) > 1$ and $|z| \notin E_1^j$. We denote $E_1 = \cup_{j=1}^k E_1^j$ and $E = \cup_{j=1}^k E_j$, then E has measure zero in $[0, 2\pi)$ and E_1 has finite logarithmic measure. Denote $r_0 = \max_{j=1,\ldots,k} r_j(\varphi)$, then (3.28) holds for all $j = 1, \ldots, k$ and $z = re^{i\varphi}, \varphi \in [0, 2\pi) \setminus E$ and $|z| > r_0$, $|z| \notin E_1$. Thus, from (3.27) and (3.28), we get

$$1 \leq \sum_{j=1}^{k} |a_j| r^{n(\sigma-1+\varepsilon)} \exp(r^{\sigma-1+\varepsilon}). \tag{3.29}$$

Since $\sigma - 1 + \varepsilon < 0$, let $r \to \infty, r \notin E_1$ in (3.29), the right side tends to zero and we get a contradiction. Hence we get $\sigma(f) \geq 1$. If f is a solution of (1.2), using Lemma 2.13 and by arguments as previous computing, we obtain $\sigma(f) \geq 1$.

■

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Ha Tran Phuong
Thai Nguyen University of Education
Thai Nguyen
Vietnam
hatranphuong@yahoo.com and phuonght@tnue.edu.vn

Nguyen Van Thin
Thai Nguyen University of Education
Thai Nguyen
Vietnam
thinmath@gmail.com and thinnv@tnue.edu.vn