An algorithm for solving the variational inequality problem over the solution set of the split variational inequality and fixed point problem

Tran Viet Anh (Hanoi, Vietnam)

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Abstract. In this paper, we introduce a new algorithm for solving strongly monotone variational inequality problem, where the constraint set is the solution set of the split variational inequality and fixed point problem. Our method uses dynamic step sizes selected based on information of the previous step, which gives strong convergence result without the prior knowledge of the given bounded linear operator's norm. In addition, using our method, we do not require any information of the Lipschitz and strongly monotone constants of the mappings. Several corollaries of our main result are also presented. Finally, a numerical example has been given to illustrate the effectiveness of our proposed algorithm.

1. Introduction

Consider two real Hilbert spaces, denoted as $\mathcal{H}_1$ and $\mathcal{H}_2$, with a bounded linear operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$. Let C be a nonempty closed convex subset of $\mathcal{H}_1$. Additionally, let $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $T : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ be given mappings. The Split Variational Inequality and Fixed Point Problem (SVIFPP) aim to find a solution x^* in the space $\mathcal{H}_1$ for which the image $A(x^*)$, under the operator A, serves as a fixed point for another mapping in $\mathcal{H}_2$.

To be more specific, the SVIFPP can be formulated as follows:

$$\text{Find } x^* \in C : \langle F(x^*), x - x^* \rangle \geq 0 \ \ \forall x \in C \tag{1.1}$$

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such that

$$T(A(x^*)) = A(x^*). \tag{1.2}$$

A particular instance of the SVIFPP, denoted by equations (1.1)-(1.2) with $F = 0$ and $T = P_Q$, corresponds to the Split Feasibility Problem (SFP). In short, the SFP can be stated as follows:

$$\text{Find } x^* \in C \text{ such that } A(x^*) \in Q, \tag{1.3}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Recently, it has been demonstrated that the SFP can serve as a practical model in intensity-modulated radiation therapy [10, 11, 13] and in various other real-world applications. To solve the SFP and their generalizations, numerous iterative projection methods have been developed. For more details, see [1–9, 12–16, 18, 21, 23, 24] and the references therein.

To find a specific solution to the SVIFPP, Hai et al. [14] investigated the following variational inequality problem

$$\text{Find } x^* \in \Omega_{\text{SVIFPP}} \text{ such that } \langle S(x^*), x - x^* \rangle \geq 0 \; \forall x \in \Omega_{\text{SVIFPP}}, \tag{1.4}$$

where $S : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitz continuous on $\mathcal{H}_1$, $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is pseudomonotone on C and L-Lipschitz continuous on $\mathcal{H}_1$, $T : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is γ-demicontractive and demi-closed at zero, $\Omega_{\text{SVIFPP}} = \{x^* \in \text{Sol}(C, F) : A(x^*) \in \text{Fix}(T)\}$ defines the solution set of the SVIFPP. As detailed in [14], the authors recommended the subgradient extragradient method to solve problem (1.4) (refer to Algorithm 1 in [14])

$$\begin{cases} x^0 \in \mathcal{H}_1, \\ u^n = A(x^n), \\ v^n = T(u^n), \\ y^n = x^n + \delta_n A^*(v^n - u^n), \\ z^n = P_C(y^n - \mu_n F(y^n)), \\ t^n = P_{C_n}(y^n - \mu_n F(z^n)), \\ x^{n+1} = t^n - \varepsilon_n S(t^n) \end{cases} \tag{1.5}$$

where $C_n = \{\omega \in \mathcal{H}_1 : \langle y^n - \mu_n F(y^n) - z^n, \omega - z^n \rangle \leq 0\}$, $\{\delta_n\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \dfrac{1-\gamma}{\|A\|^2 + 1}\right)$, $\{\mu_n\} \subset [a, b] \subset \left(0, \dfrac{1}{L}\right)$, $\{\varepsilon_n\} \subset (0, 1)$, $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=0}^{\infty} \varepsilon_n = \infty$. In [14], the authors proved that the sequence $\{x^n\}$, generated by (1.5), converges strongly to the unique solution x^* of the variational inequality problem (1.4), assuming the solution set Ω_{SVIFPP} of the SVIFPP is nonempty.

In extragradient methods, performing two projections onto the constrained set C per iteration can hinder the algorithm's efficiency. To overcome this challenge, Tseng's extragradient method [20] reduces the computational burden by performing only one projection onto C in each iteration. The formulation of Tseng's extragradient method is outlined as follows:

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \mu F(x^n)), \\ x^{n+1} = y^n - \mu(F(y^n) - F(x^n)), \end{cases} \tag{1.6}$$

where F is L-Lipschitz continuous, and $\mu \in \left(0, \dfrac{1}{L}\right)$. It is important to highlight that the main drawback of Algorithms (1.5) and (1.6) is the need to know the Lipschitz constants of the operator F, or at the very least, to have estimates of this parameter.

In this paper, motivated by the previously discussed works, we propose a novel algorithm designed to solve the variational inequality problem over the solution set of the split variational inequality and fixed point problem (1.4). The main contribution of the algorithm is the replacement of the subgradient extragradient method in Algorithm (1.5) with a modified version of Tseng's extragradient methods, which use self-adaptive step sizes. By implementing this modification, the need for the Lipschitz constant of the cost operator F is removed, resulting in a faster convergence rate. Additionally, our method does not require any prior information regarding the norm of the operator A.

The paper is structured as follows. Section 2 presents key definitions and preliminary results, which are utilized in Section 3, where the algorithm is introduced, its strong convergence is established, and several corollaries are discussed. In the final section, a numerical example is provided to compare the performance of the proposed algorithm with that of Hai et al. [14].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. It is well-known that for all point $x \in \mathcal{H}$, there exists a unique point $P_C(x) \in C$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}. \tag{2.1}$$

The mapping $P_C : \mathcal{H} \longrightarrow C$ defined by (2.1) is called the metric projection of $\mathcal{H}$ onto C. Notably, P_C is nonexpansive. Additionally, the following inequality holds for all for all $x \in \mathcal{H}$ and $y \in C$:

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0.$$

Definition 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. An operator $A^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with the property $\langle A(x), y \rangle = \langle x, A^*(y) \rangle$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, is called an adjoint operator.*

The adjoint operator of a bounded linear operator A between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ always exists and is uniquely determined. Additionally, A^* is a bounded linear operator and the equality $\|A^*\| = \|A\|$ holds true.

Definition 2.2 (see [17])**.** *A mapping $S : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be*

(i) η-strongly monotone on $\mathcal{H}$ if there exists $\eta > 0$ such that

$$\langle S(x) - S(y), x - y \rangle \geq \eta \|x - y\|^2 \ \ \forall x, y \in \mathcal{H};$$

(ii) κ-Lipschitz continuous on $\mathcal{H}$ if

$$\|S(x) - S(y)\| \leq \kappa \|x - y\| \ \ \forall x, y \in \mathcal{H}.$$

Definition 2.3. *A mapping $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be*

(i) γ-demicontractive if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a constant $\gamma \in [0, 1)$ such that

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 + \gamma \|T(x) - x\|^2 \ \ \forall x \in \mathcal{H}, \forall x^* \in \mathrm{Fix}(T);$$

(ii) demi-closed at zero if, for every sequence $\{x^n\}$ in $\mathcal{H}$, the following implication holds

$$\begin{cases} x^n \rightharpoonup x \\ \lim_{n \to \infty} \|T(x^n) - x^n\| = 0 \end{cases} \Rightarrow x \in \mathrm{Fix}(T).$$

The subsequent lemmas are essential for establishing the main result in our paper.

Lemma 2.1 (see [16])**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping such that $\limsup_{n \to \infty} \langle F(x^n), y - y^n \rangle \leq \langle F(\overline{x}), y - \overline{y} \rangle$ for every sequences $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively. Assume that $\mu_n \geq a > 0$ for all n, $\{x^n\}$ is a sequence in $\mathcal{H}$ satisfying $x^n \rightharpoonup \overline{x}$ and $\lim_{n \to \infty} \|x^n - y^n\| = 0$, where $y^n = P_C(x^n - \mu_n F(x^n))$ for all n. Then $\overline{x} \in \mathrm{Sol}(C, F)$.*

Lemma 2.2 (see [22]). *Let $\{u_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{v_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} v_n \le 0$. Suppose that*

$$u_{n+1} \le (1-\alpha_n)u_n + \alpha_n v_n \ \forall n \ge 0.$$

Then $\lim_{n\to\infty} u_n = 0$.

Lemma 2.3 (see [19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that for any integer m, there exists an integer p such that $p \ge m$ and $a_p \le a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \le a_{n_0+1}$ and define, for all integer $n \ge n_0$, by*

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \le k \le n, a_k \le a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty} \tau(n) = \infty$ and the following inequalities are satisfied:

$$a_{\tau(n)} \le a_{\tau(n)+1}, \ a_n \le a_{\tau(n)+1} \ \forall n \ge n_0.$$

3. The algorithm and convergence analysis

In this section, we propose an algorithm with strong convergence for solving the problem (1.4). We specify the following assumptions related to the mappings S, F and T involved in the formulation of the problem (1.4).

(A_1): $S : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitz continuous on $\mathcal{H}_1$.

(A_2): $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is pseudomonotone on C and L-Lipschitz continuous on $\mathcal{H}_1$.

(A_3): $\limsup_{n\to\infty} \langle F(x^n), y - y^n\rangle \le \langle F(\overline{x}), y - \overline{y}\rangle$ for every sequence $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}_1$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively.

(A_4): $T : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is γ-demicontractive and demi-closed at zero.

The algorithm is presented as follows.

Algorithm 3.1.

Step 0. Choose $\mu_0 > 0$, $\mu \in (0,1)$, $\{\rho_n\} \subset [a,b] \subset \big(0, 1-\gamma\big)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.
Step 2. Compute $u^n = A(x^n)$, $v^n = T(u^n)$ and

$$y^n = x^n + \delta_n A^*(v^n - u^n),$$

where the step size δ_n is chosen in such a way that

$$\delta_n = \begin{cases} \dfrac{\rho_n \|v^n - u^n\|^2}{\|A^*(v^n - u^n)\|^2} & \text{if } A^*(v^n - u^n) \neq 0, \\ 0 & \text{if } A^*(v^n - u^n) = 0. \end{cases}$$

Step 3. Compute

$$\begin{aligned} z^n &= P_C(y^n - \mu_n F(y^n)), \\ t^n &= z^n - \mu_n (F(z^n) - F(y^n)), \end{aligned}$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu \|y^n - z^n\|}{\|F(y^n) - F(z^n)\|}, \mu_n \right\} & \text{if } F(y^n) \neq F(z^n), \\ \mu_n & \text{if } F(y^n) = F(z^n). \end{cases}$$

Step 4. Compute

$$x^{n+1} = t^n - \varepsilon_n S(t^n).$$

Step 5. Set $n := n + 1$, and go to **Step 2**.

The strong convergence of the sequence generated through Algorithm 3.1 is established by the following theorem.

Theorem 3.1. *Assuming that conditions* (A_1), (A_2), (A_3) *and* (A_4) *hold, the sequence* $\{x^n\}$ *generated by Algorithm 3.1 converges strongly to the unique solution of problem* (1.4)*, provided that the solution set* Ω_{SVIFPP} *of the SVIFPP is nonempty.*

Proof. Since $\Omega_{\text{SVIFPP}} \neq \emptyset$, the problem (1.4) has a unique solution, denoted by x^*. In particular, $x^* \in \Omega_{\text{SVIFPP}}$, which implies that $x^* \in \text{Sol}(C, F)$ and $A(x^*) \in \text{Fix}(T)$. The proof of the theorem is divided into several steps.
Step 1. For all $n \geq 0$, we show that

$$\left(1 - \mu^2 \frac{\mu_n^2}{\mu_{n+1}^2}\right) \|y^n - z^n\|^2 \leq \|y^n - x^*\|^2 - \|t^n - x^*\|^2. \tag{3.1}$$

Given that $z^n = P_C(y^n - \mu_n F(y^n))$ and $x^* \in C$, by utilizing the properties of the projection mapping, we have

$$\langle y^n - \mu_n F(y^n) - z^n, x^* - z^n \rangle \leq 0$$

or, equivalently

$$-\langle y^n - z^n, z^n - x^*\rangle \le -\mu_n \langle F(y^n), z^n - x^*\rangle. \tag{3.2}$$

By applying the equality

$$\|y\|^2 = \|x+y\|^2 - \|x\|^2 - 2\langle x, y\rangle \ \ \forall x, y \in \mathcal{H}_1$$

and taking (3.2) into consideration, we derive

$$\begin{aligned}\|z^n - x^*\|^2 &= \|(y^n - z^n) + (z^n - x^*)\|^2 - \|y^n - z^n\|^2 - 2\langle y^n - z^n, z^n - x^*\rangle \\ &\le \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - 2\mu_n \langle F(y^n), z^n - x^*\rangle. \end{aligned} \tag{3.3}$$

Since $x^* \in \mathrm{Sol}(C, F)$, it follows that $\langle F(x^*), z - x^*\rangle \ge 0$ for all $z \in C$. By applying the pseudomonotonicity of F on C, we deduce that $\langle F(z), z - x^*\rangle \ge 0$ for all $z \in C$. Taking $z = z^n \in C$, we obtain

$$\langle F(z^n), z^n - x^*\rangle \ge 0. \tag{3.4}$$

From the definition of μ_{n+1}, it follows that

$$\|F(y^n) - F(z^n)\| \le \frac{\mu}{\mu_{n+1}} \|y^n - z^n\|. \tag{3.5}$$

Indeed, if $F(y^n) = F(z^n)$, then the inequality (3.5) is satisfied. Otherwise, we derive the following

$$\mu_{n+1} = \min\left\{ \frac{\mu\|y^n - z^n\|}{\|F(y^n) - F(z^n)\|}, \mu_n \right\} \le \frac{\mu\|y^n - z^n\|}{\|F(y^n) - F(z^n)\|},$$

which implies (3.5).
From (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}\|t^n - x^*\|^2 &= \|z^n - x^* - \mu_n(F(z^n) - F(y^n))\|^2 \\ &= \|z^n - x^*\|^2 - 2\mu_n \langle F(z^n) - F(y^n), z^n - x^*\rangle \\ &\qquad\qquad + \mu_n^2 \|F(z^n) - F(y^n)\|^2 \\ &\le \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - 2\mu_n \langle F(z^n), z^n - x^*\rangle \\ &\qquad\qquad + \mu_n^2 \|F(z^n) - F(y^n)\|^2 \\ &\le \|y^n - x^*\|^2 - \|y^n - z^n\|^2 + \mu_n^2 \|F(z^n) - F(y^n)\|^2 \\ &\le \|y^n - x^*\|^2 - \left(1 - \mu^2 \frac{\mu_n^2}{\mu_{n+1}^2}\right) \|y^n - z^n\|^2. \end{aligned}$$

As a result, we get

$$\left(1 - \mu^2 \frac{\mu_n^2}{\mu_{n+1}^2}\right) \|y^n - z^n\|^2 \le \|y^n - x^*\|^2 - \|t^n - x^*\|^2 \ \ \forall n \ge 0.$$

Step 2. For all $n \geq 0$, we have

$$\langle x^n - x^*, A^*(v^n - u^n)\rangle \leq -\frac{1-\gamma}{2}\|v^n - u^n\|^2. \tag{3.6}$$

Thanks to the γ-demicontractivity of T, we get

$$\begin{aligned}
&\langle x^n - x^*, A^*(v^n - u^n)\rangle \\
&\quad = \langle A(x^n - x^*), v^n - u^n\rangle \\
&\quad = \langle v^n - A(x^*), v^n - u^n\rangle - \|v^n - u^n\|^2 \\
&\quad = \frac{1}{2}\left[\left(\|v^n - A(x^*)\|^2 - \|u^n - A(x^*)\|^2\right) - \|v^n - u^n\|^2\right] \\
&\quad = \frac{1}{2}\left[\left(\|T(u^n) - A(x^*)\|^2 - \|u^n - A(x^*)\|^2\right) - \|v^n - u^n\|^2\right] \\
&\quad \leq \frac{1}{2}\left[\gamma\|T(u^n) - u^n\|^2 - \|v^n - u^n\|^2\right] \\
&\quad = -\frac{1-\gamma}{2}\|v^n - u^n\|^2.
\end{aligned}$$

Step 3. We show that

$$\mu_{n+1} \leq \mu_n,\ \mu_n \geq \min\left(\frac{\mu}{L}, \mu_0\right) \quad \forall n \geq 0, \lim_{n\to\infty} \mu_n = \mu^* \geq \min\left(\frac{\mu}{L}, \mu_0\right). \tag{3.7}$$

Since F is L-Lipschitz continuous on $\mathcal{H}_1$, we have

$$\|F(y^n) - F(z^n)\| \leq L\|y^n - z^n\|.$$

Thus, when $F(y^n) \neq F(z^n)$, it follows that

$$\frac{\mu\|y^n - z^n\|}{\|F(y^n) - F(z^n)\|} \geq \frac{\mu}{L}.$$

By induction, we obtain

$$\mu_n \geq \min\left(\frac{\mu}{L}, \mu_0\right) \ \ \forall n \geq 0.$$

From the definition of μ_{n+1}, it is clear that $\mu_{n+1} \leq \mu_n$ for all $n \geq 0$. Therefore, together with the fact that $\mu_n \geq \min\left(\frac{\mu}{L}, \mu_0\right)$ for all $n \geq 0$, it follows that the sequence $\{\mu_n\}$ has a limit, denoted by μ^*, and we conclude that $\lim_{n\to\infty} \mu_n = \mu^* \geq \min\left(\frac{\mu}{L}, \mu_0\right)$.

Step 4. We show that, for all $n \geq 0$

$$\begin{aligned}
&\frac{a^2}{(\|A\|+1)^2}\|v^n - u^n\|^2 \leq \|y^n - x^n\|^2, \\
&\|y^n - x^n\|^2 \leq \frac{b}{1-\gamma-b}\left(\|x^n - x^*\|^2 - \|y^n - x^*\|^2\right).
\end{aligned} \tag{3.8}$$

We now consider two distinct cases.

Case 1. $A^*(v^n - u^n) = 0$. From (3.6), we deduce that $\|v^n - u^n\| = 0$. Since $\delta_n = 0$, it follows that $y^n = x^n$. Therefore, (3.8) holds.

Case 2. $A^*(v^n - u^n) \neq 0$. It follows from (3.6) that

$$\begin{aligned}
\|y^n - x^*\|^2 &= \|(x^n - x^*) + \delta_n A^*(v^n - u^n)\|^2 \\
&= \|x^n - x^*\|^2 + \|\delta_n A^*(v^n - u^n)\|^2 + 2\delta_n \langle x^n - x^*, A^*(v^n - u^n)\rangle \\
&\leq \|x^n - x^*\|^2 + \delta_n^2 \|A^*(v^n - u^n)\|^2 - \delta_n(1-\gamma)\|v^n - u^n\|^2 \\
&= \|x^n - x^*\|^2 - \frac{\rho_n^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \cdot \frac{1-\gamma-\rho_n}{\rho_n} \\
&\leq \|x^n - x^*\|^2 - \frac{\rho_n^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \cdot \frac{1-\gamma-b}{b} \quad \forall n \geq 0. \qquad (3.9)
\end{aligned}$$

By applying (3.9), we get

$$\begin{aligned}
\|y^n - x^n\|^2 &= \delta_n^2 \|A^*(v^n - u^n)\|^2 \\
&= \frac{\rho_n^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^4}\|A^*(v^n - u^n)\|^2 \\
&= \frac{\rho_n^2\|v^n - u^n\|^4}{\|A^*(v^n - u^n)\|^2} \qquad (3.10)\\
&\leq \frac{b}{1-\gamma-b}\left(\|x^n - x^*\|^2 - \|y^n - x^*\|^2\right) \quad \forall n \geq 0.
\end{aligned}$$

On the other hand

$$\|A^*(v^n - u^n)\| \leq \|A^*\|\|v^n - u^n\| = \|A\|\|v^n - u^n\| \leq (\|A\|+1)\|v^n - u^n\|. \qquad (3.11)$$

By using (3.10) and (3.11) together, we obtain

$$\|y^n - x^n\|^2 \geq \frac{\rho_n^2\|v^n - u^n\|^4}{(\|A\|+1)^2\|v^n - u^n\|^2} \geq \frac{a^2}{(\|A\|+1)^2}\|v^n - u^n\|^2 \quad \forall n \geq 0.$$

Therefore, the inequalities in (3.8) are proven.

Now, choose $\varepsilon \in \left(0, \frac{2\eta}{\kappa^2}\right)$. From $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty}\left(1-\mu^2\frac{\mu_n^2}{\mu_{n+1}^2}\right) = 1-\mu^2 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon_n < \varepsilon \ \ \forall n \geq n_0, \ \ 1-\mu^2\frac{\mu_n^2}{\mu_{n+1}^2} > \frac{1-\mu^2}{2} > 0 \ \ \forall n \geq n_0. \qquad (3.12)$$

Step 5. For all $n \geq n_0$, we show that

$$\|t^n - \varepsilon_n S(t^n) - x^* + \varepsilon_n S(x^*)\| \leq \left(1 - \frac{\varepsilon_n \tau}{\varepsilon}\right)\|t^n - x^*\|, \qquad (3.13)$$

where $\tau = 1 - \sqrt{1-\varepsilon(2\eta - \varepsilon\kappa^2)} \in (0,1]$.
Given the κ-Lipschitz continuity and η-strong monotonicity of S on $\mathcal{H}_1$, we deduce

$$\begin{aligned}\|t^n - x^* - &\varepsilon(S(t^n) - S(x^*))\|^2 \\ &= \|t^n - x^*\|^2 - 2\varepsilon\langle t^n - x^*, S(t^n) - S(x^*)\rangle + \varepsilon^2\|S(t^n) - S(x^*)\|^2 \\ &\le \|t^n - x^*\|^2 - 2\varepsilon\eta\|t^n - x^*\|^2 + \varepsilon^2\kappa^2\|t^n - x^*\|^2 \\ &= \left[1 - \varepsilon(2\eta - \varepsilon\kappa^2)\right]\|t^n - x^*\|^2.\end{aligned}$$

From (3.12) and the inequality above, it follows that

$$\begin{aligned}\|t^n - &\varepsilon_n S(t^n) - x^* + \varepsilon_n S(x^*)\| \\ &= \left\|\left(1 - \frac{\varepsilon_n}{\varepsilon}\right)(t^n - x^*) + \frac{\varepsilon_n}{\varepsilon}\left[t^n - x^* - \varepsilon(S(t^n) - S(x^*))\right]\right\| \\ &\le \left(1 - \frac{\varepsilon_n}{\varepsilon}\right)\|t^n - x^*\| + \frac{\varepsilon_n}{\varepsilon}\|t^n - x^* - \varepsilon(S(t^n) - S(x^*))\| \\ &\le \left(1 - \frac{\varepsilon_n}{\varepsilon}\right)\|t^n - x^*\| + \frac{\varepsilon_n}{\varepsilon}\sqrt{1 - \varepsilon(2\eta - \varepsilon\kappa^2)}\|t^n - x^*\| \\ &= \left[1 - \frac{\varepsilon_n}{\varepsilon}\left(1 - \sqrt{1 - \varepsilon(2\eta - \varepsilon\kappa^2)}\right)\right]\|t^n - x^*\| \\ &= \left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|t^n - x^*\| \quad \forall n \ge n_0.\end{aligned}$$

Step 6. The sequences $\{x^n\}$, $\{y^n\}$, $\{t^n\}$ and $\{S(t^n)\}$ are bounded.
From inequality (3.13), we obtain

$$\begin{aligned}\|x^{n+1} - x^*\| &= \|t^n - \varepsilon_n S(t^n) - x^* + \varepsilon_n S(x^*) - \varepsilon_n S(x^*)\| \\ &\le \|t^n - \varepsilon_n S(t^n) - x^* + \varepsilon_n S(x^*)\| + \varepsilon_n\|S(x^*)\| \\ &\le \left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|t^n - x^*\| + \varepsilon_n\|S(x^*)\| \quad \forall n \ge n_0.\end{aligned} \tag{3.14}$$

Using (3.1), (3.8) and (3.12), we get

$$\|t^n - x^*\| \le \|y^n - x^*\| \le \|x^n - x^*\| \quad \forall n \ge n_0. \tag{3.15}$$

By applying (3.14) and (3.15), we derive

$$\begin{aligned}\|x^{n+1} - x^*\| &\le \left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|x^n - x^*\| + \varepsilon_n\|S(x^*)\| \\ &= \left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|x^n - x^*\| + \frac{\varepsilon_n\tau}{\varepsilon}\cdot\frac{\varepsilon\|S(x^*)\|}{\tau} \quad \forall n \ge n_0.\end{aligned}$$

In particular,

$$\|x^{n+1} - x^*\| \le \max\left\{\|x^n - x^*\|, \frac{\varepsilon\|S(x^*)\|}{\tau}\right\} \quad \forall n \ge n_0,$$

and thus, by induction, we have

$$\|x^n - x^*\| \le \max\left\{\|x^{n_0} - x^*\|, \frac{\varepsilon\|S(x^*)\|}{\tau}\right\} \;\; \forall n \ge n_0.$$

Therefore, the sequence $\{x^n\}$ is bounded and this is true for the sequences $\{y^n\}$, $\{t^n\}$ and $\{S(t^n)\}$ as well, thanks to (3.15) and the Lipschitz continuity of S.

Step 7. We prove that $\{x^n\}$ converges strongly to x^*.

Based on (3.13), we deduce, for every $n \ge n_0$, that

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 &\le \|x^{n+1} - x^*\|^2 + \varepsilon_n^2\|S(x^*)\|^2 \\
&= \|x^{n+1} - x^* + \varepsilon_n S(x^*)\|^2 - 2\langle \varepsilon_n S(x^*), x^{n+1} - x^*\rangle \\
&= \|t^n - \varepsilon_n S(t^n) - x^* + \varepsilon_n S(x^*)\|^2 - 2\varepsilon_n\langle S(x^*), x^{n+1} - x^*\rangle \\
&\le \left[\left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|t^n - x^*\|\right]^2 - 2\varepsilon_n\langle S(x^*), x^{n+1} - x^*\rangle \\
&\le \left(1 - \frac{\varepsilon_n\tau}{\varepsilon}\right)\|t^n - x^*\|^2 - 2\varepsilon_n\langle S(x^*), x^{n+1} - x^*\rangle. \qquad (3.16)
\end{aligned}$$

We will consider two cases.

Case 1. Let us consider the case where there exists n_* such that $\{\|x^n - x^*\|\}$ is decreasing for $n \ge n_*$. As a result, the limit of $\{\|x^n - x^*\|\}$ exists. Consequently, from (3.15) and (3.16), we deduce, for all $n \ge n_0$, that

$$\begin{aligned}
0 &\le \|y^n - x^*\|^2 - \|t^n - x^*\|^2 \\
&\le \|x^n - x^*\|^2 - \|t^n - x^*\|^2 \\
&\le \left(\|x^n - x^*\|^2 - \|x^{n+1} - x^*\|^2\right) - 2\varepsilon_n\langle S(x^*), x^{n+1} - x^*\rangle.
\end{aligned}$$

Given that $\|x^n - x^*\|$ has a limit, with $\lim_{n\to\infty} \varepsilon_n = 0$, and the sequence $\{x^n\}$ is bounded, the above inequalities yield that

$$\lim_{n\to\infty}(\|y^n - x^*\|^2 - \|t^n - x^*\|^2) = 0, \qquad (3.17)$$

$$\lim_{n\to\infty}(\|x^n - x^*\|^2 - \|t^n - x^*\|^2) = 0. \qquad (3.18)$$

It follows from (3.1), (3.12) and (3.17) that

$$\lim_{n\to\infty}\|y^n - z^n\| = 0. \qquad (3.19)$$

From (3.17) and (3.18), we have

$$\lim_{n\to\infty}(\|x^n - x^*\|^2 - \|y^n - x^*\|^2) = 0. \qquad (3.20)$$

Consequently, from (3.8) and (3.20), we get

$$\lim_{n\to\infty} \|y^n - x^n\| = 0, \tag{3.21}$$

$$\lim_{n\to\infty} \|v^n - u^n\| \Rightarrow \lim_{n\to\infty} \|T(u^n) - u^n\| = 0. \tag{3.22}$$

Applying the triangle inequality along with the L-Lipschitz continuity of F on $\mathcal{H}_1$, we have

$$\begin{aligned}\|x^n - t^n\| &\leq \|x^n - y^n\| + \|y^n - z^n\| + \|z^n - t^n\| \\ &= \|x^n - y^n\| + \|y^n - z^n\| + \|\mu_n(F(z^n) - F(y^n))\| \\ &\leq \|x^n - y^n\| + \|y^n - z^n\| + \mu_n L\|z^n - y^n\| \\ &\leq \|x^n - y^n\| + (1+\mu_0 L)\|y^n - z^n\|.\end{aligned}$$

Therefore, using (3.19) and (3.21), it follows that

$$\lim_{n\to\infty} \|x^n - t^n\| = 0. \tag{3.23}$$

Now, we prove that

$$\limsup_{n\to\infty} \langle S(x^*), x^* - x^{n+1}\rangle \leq 0. \tag{3.24}$$

Choose a subsequence $\{x^{n_\nu}\}$ from $\{x^n\}$ such that

$$\limsup_{n\to\infty} \langle S(x^*), x^* - x^{n+1}\rangle = \lim_{\nu\to\infty} \langle S(x^*), x^* - x^{n_\nu}\rangle.$$

As $\{x^{n_\nu}\}$ is bounded, we can assume without loss of generality that $x^{n_\nu} \rightharpoonup \overline{x}$. Hence

$$\limsup_{n\to\infty} \langle S(x^*), x^* - x^{n+1}\rangle = \langle S(x^*), x^* - \overline{x}\rangle. \tag{3.25}$$

Using the weak convergence $x^{n_\nu} \rightharpoonup \overline{x}$ and (3.21), we infer $y^{n_\nu} \rightharpoonup \overline{x}$. From (3.19), we have $\lim_{\nu\to\infty} \|y^{n_\nu} - z^{n_\nu}\| = 0$. Since $z^{n_\nu} = P_C(y^{n_\nu} - \mu_{n_\nu} F(y^{n_\nu}))$, $y^{n_\nu} \rightharpoonup \overline{x}$, $\mu_{n_\nu} \geq \min\left(\frac{\mu}{L}, \mu_0\right) > 0$. By Lemma 2.1, we obtain $\overline{x} \in \mathrm{Sol}(C, F)$.
From $x^{n_\nu} \rightharpoonup \overline{x}$, we imply $u^{n_\nu} = A(x^{n_\nu}) \rightharpoonup A(\overline{x})$. Together with (3.22) and the demiclosedness of T, it follows that $A(\overline{x}) \in \mathrm{Fix}(T)$. Taking into account that $\overline{x} \in \mathrm{Sol}(C, F)$, we conclude that $\overline{x} \in \Omega_{\mathrm{SVIFPP}}$. Consequently, $\langle S(x^*), \overline{x} - x^*\rangle \geq 0$, and combined with (3.25), this gives (3.24).
By applying (3.15) and (3.16), we get

$$\|x^{n+1} - x^*\|^2 \leq \left(1 - \frac{\varepsilon_n \tau}{\varepsilon}\right)\|x^n - x^*\|^2 + \frac{\varepsilon_n \tau}{\varepsilon} b_n \ \ \forall n \geq n_0, \tag{3.26}$$

where

$$b_n = \frac{2\varepsilon\langle S(x^*), x^* - x^{n+1}\rangle}{\tau}.$$

Using (3.24), we conclude that $\limsup_{n\to\infty} b_n \le 0$. Since $\varepsilon_n < \varepsilon$ $\forall n \ge n_0$ and $0 < \tau \le 1$, it follows that $\left\{\frac{\varepsilon_n \tau}{\varepsilon}\right\}_{n\ge n_0} \subset (0,1)$. As a result, from (3.26), $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, $\limsup_{n\to\infty} b_n \le 0$ and Lemma 2.2, we deduce that $\lim_{n\to\infty} \|x^n - x^*\|^2 = 0$, which implies $x^n \to x^*$ as $n \to \infty$.

Case 2. Assume that for every integer m, there exists an integer n such that $n \ge m$ and $\|x^n - x^*\| \le \|x^{n+1} - x^*\|$. By applying Lemma 2.3, we can define a nondecreasing sequence $\{\tau(n)\}_{n\ge N}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \tau(n) = \infty$ and the following inequalities hold

$$\|x^{\tau(n)} - x^*\| \le \|x^{\tau(n)+1} - x^*\|, \quad \|x^n - x^*\| \le \|x^{\tau(n)+1} - x^*\| \;\; \forall n \ge N. \tag{3.27}$$

Select $n_* \ge N$ such that $\tau(n) \ge n_0$ for all $n \ge n_*$. Using (3.27) and (3.14), we get

$$\begin{aligned}
\|x^{\tau(n)} - x^*\| &\le \|x^{\tau(n)+1} - x^*\| \\
&\le \left(1 - \frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\right)\|t^{\tau(n)} - x^*\| + \varepsilon_{\tau(n)}\|S(x^*)\| \\
&\le \|t^{\tau(n)} - x^*\| + \varepsilon_{\tau(n)}\|S(x^*)\| \;\; \forall n \ge n_*,
\end{aligned}$$

which together with (3.15) implies, for all $n \ge n_*$, that

$$\begin{aligned}
0 &\le \|y^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\| \\
&\le \|x^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\| \le \varepsilon_{\tau(n)}\|S(x^*)\|.
\end{aligned} \tag{3.28}$$

Then, it follows from (3.28) and $\lim_{n\to\infty} \varepsilon_n = 0$ that

$$\begin{aligned}
&\lim_{n\to\infty} (\|y^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\|) = 0, \\
&\lim_{n\to\infty} (\|x^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\|) = 0.
\end{aligned} \tag{3.29}$$

Using (3.29) and the fact that the sequences $\{x^n\}$, $\{y^n\}$ and $\{t^n\}$ are bounded, we derive

$$\begin{aligned}
&\lim_{n\to\infty} (\|y^{\tau(n)} - x^*\|^2 - \|t^{\tau(n)} - x^*\|^2) = 0, \\
&\lim_{n\to\infty} (\|x^{\tau(n)} - x^*\|^2 - \|t^{\tau(n)} - x^*\|^2) = 0.
\end{aligned}$$

Applying the same reasoning as in the first case, it follows that

$$\lim_{n\to\infty} \|x^{\tau(n)} - t^{\tau(n)}\| = 0, \quad \limsup_{n\to\infty} \langle S(x^*), x^* - x^{\tau(n)}\rangle \le 0. \tag{3.30}$$

We now observe that

$$\begin{aligned}\|x^{\tau(n)+1} - x^{\tau(n)}\| &= \|t^{\tau(n)} - x^{\tau(n)} - \varepsilon_{\tau(n)} S(t^{\tau(n)})\| \\ &\leq \|t^{\tau(n)} - x^{\tau(n)}\| + \varepsilon_{\tau(n)} \|S(t^{\tau(n)})\|,\end{aligned}$$

which, in combination with (3.30), $\lim_{n\to\infty} \varepsilon_n = 0$ and the boundedness of $\{S(t^{\tau(n)})\}$, implies

$$\lim_{n\to\infty} \|x^{\tau(n)+1} - x^{\tau(n)}\| = 0. \tag{3.31}$$

Using (3.31) along with the Cauchy-Schwarz inequality, we get

$$\lim_{n\to\infty} \langle S(x^*), x^{\tau(n)} - x^{\tau(n)+1}\rangle = 0. \tag{3.32}$$

By combining (3.32) and (3.30), we conclude that

$$\begin{aligned}&\limsup_{n\to\infty} \langle S(x^*), x^* - x^{\tau(n)+1}\rangle \\ &\quad= \limsup_{n\to\infty} \left[\langle S(x^*), x^* - x^{\tau(n)}\rangle + \langle S(x^*), x^{\tau(n)} - x^{\tau(n)+1}\rangle\right] \\ &\quad= \limsup_{n\to\infty} \langle S(x^*), x^* - x^{\tau(n)}\rangle \leq 0.\end{aligned} \tag{3.33}$$

Also, from (3.16) and (3.15), we get

$$\|x^{n+1} - x^*\|^2 \leq \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big)\|x^n - x^*\|^2 + 2\varepsilon_n \langle S(x^*), x^* - x^{n+1}\rangle \quad \forall n \geq n_0, \tag{3.34}$$

Since $\tau(n) \geq n_0$ holds for all $n \geq n_*$, we can conclude from (3.34) and (3.27) that for all $n \geq n_*$

$$\begin{aligned}\|x^{\tau(n)+1} - x^*\|^2 &\leq \Big(1 - \frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\Big)\|x^{\tau(n)} - x^*\|^2 + 2\varepsilon_{\tau(n)} \langle S(x^*), x^* - x^{\tau(n)+1}\rangle \\ &\leq \Big(1 - \frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\Big)\|x^{\tau(n)+1} - x^*\|^2 + 2\varepsilon_{\tau(n)} \langle S(x^*), x^* - x^{\tau(n)+1}\rangle.\end{aligned}$$

As a result, since $\varepsilon_{\tau(n)} > 0$

$$\|x^{\tau(n)+1} - x^*\|^2 \leq \frac{2\varepsilon}{\tau} \langle S(x^*), x^* - x^{\tau(n)+1}\rangle \quad \forall n \geq n_*.$$

By combining this inequality with (3.27), given that $n_* \geq N$, we have

$$\|x^n - x^*\|^2 \leq \frac{2\varepsilon}{\tau} \langle S(x^*), x^* - x^{\tau(n)+1}\rangle \quad \forall n \geq n_*. \tag{3.35}$$

Taking the limit in (3.35) as $n \to \infty$ and applying (3.33), we arrive at

$$\limsup_{n\to\infty} \|x^n - x^*\|^2 \leq 0.$$

Therefore, it follows that $x^n \to x^*$ as $n \to \infty$. This completes the proof of Theorem 3.1. ■

Remark 3.1. We highlight the advantages of Algorithm 3.1 compared to the algorithm of Hai et al. in [14, Algorithm 1].

i) In Algorithm 3.1, unlike the result in [14, Algorithm 1], the step size is selected in such a way that its implementation does not require any prior knowledge of the norms of the given bounded linear operators.

ii) Algorithm 1 in [14] requires computing or estimating the Lipschitz constant of the mapping F, which is generally a challenging task in practice. In contrast, our Algorithm 3.1 removes this restriction.

When F is set to zero and T is defined as P_Q, the SVIFPP described by equations (1.1)-(1.2) reduces to the SFP given in (1.3). Consequently, utilizing the results from Algorithm 1 and Theorem 3.1, we derive the following result for solving the variational inequality problem over the solution set of the SFP. It is important to note that the proposed algorithm requires only two projections per iteration, and notably, its implementation does not rely on any information about the norm of the operator A.

Algorithm 3.2.
Step 0. Choose $\{\rho_n\} \subset [a,b] \subset (0,1)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.
Step 1. Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.
Step 2. Compute $u^n = A(x^n)$, $v^n = P_Q(u^n)$ and

$$y^n = x^n + \delta_n A^*(v^n - u^n),$$

where the stepsize δ_n is chosen in such a way that

$$\delta_n = \begin{cases} \dfrac{\rho_n \|v^n - u^n\|^2}{\|A^*(v^n - u^n)\|^2} & \text{if } A^*(v^n - u^n) \neq 0, \\ 0 & \text{if } A^*(v^n - u^n) = 0. \end{cases}$$

Step 3. Compute $z^n = P_C(y^n)$.
Step 4. Compute

$$x^{n+1} = z^n - \varepsilon_n S(z^n).$$

Step 5. Set $n := n + 1$, and go to **Step 2**.

Corollary 3.1. *Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $S : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ be a strongly monotone and Lipschitz continuous mapping. Suppose that the solution set $\Omega_{SFP} = \{x^* \in C : A(x^*) \in Q\}$ of the SFP is nonempty. Then the sequence*

$\{x^n\}$ generated by Algorithm 3.2 converges strongly to $x^* \in \Omega_{SFP}$, which is the unique solution of the variational inequality problem

$$\langle S(x^*), x - x^* \rangle \geq 0 \ \forall x \in \Omega_{SFP}, \tag{3.36}$$

provided that the solution set Ω_{SFP} of the SFP is nonempty.

Assume the following conditions to be satisfied:
(B_1): $S : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone and Lipschitz continuous on $\mathcal{H}$.
(B_2): $F : \mathcal{H} \longrightarrow \mathcal{H}$ is pseudomonotone on C and Lipschitz continuous on $\mathcal{H}$.
(B_3): $\limsup_{n\to\infty} \langle F(x^n), y - y^n \rangle \leq \langle F(\overline{x}), y - \overline{y} \rangle$ for every sequence $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively.

When $\mathcal{H}_1 = \mathcal{H}_2 := \mathcal{H}$, and both T and A are the identity mappings in $\mathcal{H}$, the SVIFPP reduces to the variational inequality problem (1.1). Consequently, by applying Algorithm 3.1 and utilizing Theorem 3.1, we obtain the following result for solving the variational inequality problem over the solution set of another VIP. It is important to emphasize that the proposed algorithm requires only one projection onto the feasible set at each iteration, and its implementation does not require any information about the Lipschitz constants of the mappings S and F, nor the modulus of strong monotonicity of S.

Algorithm 3.3.
Step 0. Choose $\mu_0 > 0$, $\mu \in (0, 1)$ and $\{\varepsilon_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.
Step 1. Let $x^0 \in \mathcal{H}$. Set $n := 0$.
Step 2. Compute

$$y^n = P_C(x^n - \mu_n F(x^n)),$$
$$z^n = y^n - \mu_n(F(y^n) - F(x^n)),$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu\|x^n - y^n\|}{\|F(x^n) - F(y^n)\|}, \mu_n \right\} & \text{if } F(x^n) \neq F(y^n), \\ \mu_n & \text{if } F(x^n) = F(y^n). \end{cases}$$

Step 3. Compute

$$x^{n+1} = z^n - \varepsilon_n S(z^n).$$

Step 4. Set $n := n + 1$, and go to **Step 2**.

Corollary 3.2. *Under the assumption that conditions (B_1), (B_2) and (B_3) hold, the sequence $\{x^n\}$ generated by Algorithm 3.3 converges strongly to a* *point* $x^* \in \mathrm{Sol}(C, F)$*, which is the unique solution of the variational inequality*

$$\langle S(x^*), x - x^* \rangle \geq 0 \; \forall x \in \mathrm{Sol}(C, F), \tag{3.37}$$

provided that $\mathrm{Sol}(C, F) \neq \emptyset$.

4. Numerical illustrations

In this section, we present numerical experiments to demonstrate the effectiveness of the proposed algorithm. The Python scripts were run on a 2017 MacBook Pro, featuring a 2.3 GHz Intel Core i5 processor, an Intel Iris Plus Graphics 640 with 1536 MB of memory, and 8 GB of 2133 MHz LPDDR3 RAM. The experiments were conducted using Python version 3.11.

Example 4.1. ([14]) Let $\mathbb{R}^K$ be endowed with the standard Euclidean norm $\|x\| = \left(x_1^2 + x_2^2 + \cdots + x_K^2\right)^{\frac{1}{2}}$ for all $x = (x_1, x_2, \ldots, x_K)^T \in \mathbb{R}^K$. We consider the SVIFPP with the mapping $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $F(x) = (\sin \|x\| + 2)a^0$ for all $x \in \mathbb{R}^4$, where $a^0 = (12, -4, 4, -4)^T \in \mathbb{R}^4$. Additionally, let C be the set defined as

$$C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 \geq 9\}$$

and the bounded linear operator $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ defined by $A(x) = Mx$ for all $x \in \mathbb{R}^4$, where

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Assume that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by, for all $y = (y_1, y_2)^T \in \mathbb{R}^2$

$$T(y) = \begin{cases} (y_1, y_2)^T & \text{if } y_1 \leq 0, \\ (-2y_1, y_2)^T & \text{if } y_1 > 0. \end{cases}$$

Then T is $\dfrac{1}{3}$-demicontractive and $\mathrm{Fix}(T) = (-\infty, 0] \times \mathbb{R}$.

Consider the mapping $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by $S(x) = x$ for all $x \in \mathbb{R}^4$. This mapping S is strongly monotone with $\eta = 1$ and Lipschitz continuous with $\kappa = 1$ on $\mathbb{R}^4$. In this situation, the problem (1.4) becomes the problem of finding the minimum-norm solution of the SVIFPP.
The solution set Ω_{SVIFPP} of the SVIFPP is given by

$$\begin{aligned} \Omega_{\mathrm{SVIFPP}} &= \{(x_1, x_2, x_3, x_4)^T \in \mathrm{Sol}(C, F) : A(x_1, x_2, x_3, x_4) \in \mathrm{Fix}(T)\} \\ &= \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 = 9, x_1 + x_3 + x_4 \leq 0\}. \end{aligned}$$

and the minimum-norm solution x^* of the SVIFPP is $x^* = \left(\frac{1}{2}, -\frac{1}{4}, 0, -\frac{1}{2}\right)^T$.

We now provide a comparison between Algorithm 3.1 and Algorithm 1 in [14]. Given that the exact solution of the problem is $x^* = \left(\frac{1}{2}, -\frac{1}{4}, 0, -\frac{1}{2}\right)^T$, we using $\|x^n - x^*\| \leq \varepsilon$ as the stopping condition. Both algorithms use the same initial point x^0, obtained by randomly generating values within the interval $[-10, 10]$. The parameters for each algorithm are chosen as follows:

- Algorithm 3.1: $\mu_0 = 2$, $\mu = 0.1$, $\rho_n = 1 - 10^{-2}$ and $\varepsilon_n = \dfrac{1}{n+2}$.
- Algorithm 1 in [14]: $\delta_n = \dfrac{n+1}{500n + 510}$, $\mu_n = \dfrac{n+1}{600n + 605}$ and $\varepsilon_n = \dfrac{1}{n+2}$.

Table 1. A comparison of Algorithm 3.1 and Algorithm 1 in [14] using various tolerances ε and the stopping criterion $\|x^n - x^*\| \leq \varepsilon$

	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$	
	Iter(n)	CPU time(s)	Iter(n)	CPU time(s)
Algorithm 3.1	4274	0.8088	89639	9.6047
Algorithm 1 in [14]	28470	2.1334	295407	18.5596

Table 1 shows that our Algorithm 3.1 outperforms Algorithm 1 in [14] in terms of both the number of iterations and CPU time.

Example 4.2. We consider the mapping $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $S(x) = (4x_1 + 16, 4x_2 - 4, 4x_3 + 3)^T$ for all $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. It is straightforward to verify that S is both strongly monotone and Lipschitz continuous on $\mathbb{R}^3$. Define the sets $C = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 - x_2 + 2x_3 = 4\}$, $Q = \{(u_1, u_2)^T \in \mathbb{R}^2 : 3u_1 - u_2 = 10\}$ and let the bounded linear operator $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $A(x) = Mx$, where

$$M = \begin{pmatrix} 1 & -4 & 2 \\ 2 & -9 & -4 \end{pmatrix}.$$

The solution set Ω_{SFP} of the SFP is given by

$$\begin{aligned} \Omega_{\mathrm{SFP}} &= \begin{cases} x_1 - x_2 + 2x_3 = 4 \\ 3(x_1 - 4x_2 + 2x_3) - (2x_1 - 9x_2 - 4x_3) = 10 \end{cases} \\ &= \begin{cases} x_1 - x_2 + 2x_3 = 4 \\ x_1 - 3x_2 + 10x_3 = 10, \end{cases} \end{aligned}$$

which can be expressed in parametric form as:

$$\Omega_{\mathrm{SFP}} = \big\{(2t+1, 4t-3, t)^T : t \in \mathbb{R}\big\}.$$

Assume that $x^* = (2t^* + 1, 4t^* - 3, t^*)^T \in \Omega_{\text{SFP}}$ satisfies the variational inequality

$$\langle S(x^*), x - x^*\rangle \geq 0 \ \ \forall x \in \Omega_{\text{SFP}}.$$

Given that $S(x^*) = (8t^* + 20, 16t^* - 16, 4t^* + 3)$, $x - x^* = (2t - 2t^*, 4t - 4t^*, t - t^*)^T$, the inequality becomes

$$(8t^* + 20)(2t - 2t^*) + (16t^* - 16)(4t - 4t^*) + (4t^* + 3)(t - t^*) \geq 0 \ \ \forall t \in \mathbb{R}.$$

This expression simplifies to $21(4t^* - 1)(t - t^*) \geq 0$ for all $t \in \mathbb{R}$. This inequality holds if and only if $t^* = \dfrac{1}{4}$. Therefore, the unique solution to the variational inequality problem (3.36) is $x^* = \left(\dfrac{3}{2}, -2, \dfrac{1}{4}\right)^T$.

We select an initial point $x^0 \in \mathbb{R}^3$, where each component of x^0 is randomly generated within the closed interval $[-10, 10]$. With $\varepsilon_n = \dfrac{1}{n+2}$ and the stopping criterion $\|x^n - x^*\| \leq \varepsilon$, we compute approximate solutions to the exact solution $x^* = \left(\dfrac{3}{2}, -2, \dfrac{1}{4}\right)^T$ for various tolerance levels ε, as presented in Table 2.

Table 2. Approximate solutions corresponding to various tolerance levels ε, obtained using Algorithm 3.2 with the stopping criterion $\|x^n - x^*\| \leq \varepsilon$

ε	Iter(n)	CPU time(s)	x^n
$\varepsilon = 10^{-2}$	19392	1.2879	$(1.491653, -1.996977, 0.254602)^T$
$\varepsilon = 10^{-3}$	194115	10.8199	$(1.499165, -1.999698, 0.250460)^T$
$\varepsilon = 10^{-4}$	1941349	109.1283	$(1.499917, -1.999970, 0.250046)^T$

Example 4.3. We consider the set $C \subset \mathbb{R}^3$ defined by

$$C = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : 2x_1 - x_2 + 5x_3 \geq 6\}.$$

Next, define the mapping $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $F(x) = (\sin\|x\| + 6)f^0$ for all $x \in \mathbb{R}^3$, where $f^0 = (2, -1, 5)^T \in \mathbb{R}^3$. It is easy to verify that F is pseudomonotone and Lipschitz continuous on $\mathbb{R}^3$. Furthermore, the solution set $\text{Sol}(C, F)$ of the variational inequality problem $VIP(C, F)$ is given by

$$\text{Sol}(C, F) = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : 2x_1 - x_2 + 5x_3 = 6\}.$$

Now, consider the mapping $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $S(x) = x$ for all $x \in \mathbb{R}^3$. This mapping is strongly monotone with modulus $\eta = 1$ and Lipschitz continuous with constant $L = 1$ on $\mathbb{R}^3$. In this setting, Problem (3.37) reduces to finding the minimum-norm solution of the variational inequality problem $VIP(C,F)$. The resulting minimum-norm solution is given by $x^* = P_{\mathrm{Sol}(C,F)}(0) = (0.4, -0.2, 1)^T$.

We select an initial point $x^0 \in \mathbb{R}^3$, where each component of x^0 is randomly generated within the closed interval $[-10, 10]$. With parameters $\mu_0 = 4$, $\mu = 0.7$, $\varepsilon_n = \dfrac{1}{n+2}$ in Algorithm 3.3 and using the stopping criterion $\|x^{n+1} - x^n\| \leq \varepsilon$. With the tolerance $\varepsilon = 10^{-9}$, an approximate solution is obtained after 84027 iterations (with time 6.109 seconds), given by

$$x^{84027} = (0.400055, -0.199997, 0.999936)^T,$$

which serves as a good approximation to the exact solution $x^* = (0.4, -0.2, 1)^T$.

5. Conclusion

We propose a new algorithm for solving the strongly monotone variational inequality problem over the solution set of split variational inequality and fixed point problem in real Hilbert spaces. By placing suitable conditions on the parameters, we prove a strong convergence theorem for the algorithm, which avoids the need to compute or estimate the norms of the bounded linear operators. Importantly, the algorithm does not require prior knowledge of the Lipschitz or strongly monotone constants of the mappings. Additionally, we derive several corollaries from our main result and demonstrate the algorithm's performance with a basic numerical example.

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Tran Viet Anh
Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology
Hanoi
Vietnam
tranvietanh@outlook.com; tvanh@ptit.edu.vn