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Random dynamical systems generated by nonautonomous stochastic differential equations driven by fractional Brownian motions

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Abstract. In this paper, we prove that a non-autonomous stochastic differential equation generates a continuous random dynamical system. The flow then possesses a random pullback attractor under the dissipativity condition(s) of the drift and smallness of diffusion part.

1. Introduction

This work is a follow up part of [7], [14] to study the the asymptotic qualitative behavior of the differential equation

$$dy_t = f(t, y_t)dt + g(t, y_t)dB^H_t, t \in \mathbb{R},\ y_0 \in \mathbb{R}^d. \tag{1.1}$$

in which B^H is a fractional Brownian motion with Hurst parameter H bigger than $\frac{1}{2}$; f and g are some continuous functions on $\mathbb{R} \times \mathbb{R}^d$.

When dealing with qualitative properties of (1.1), one important problem is the generation of *random dynamical system*, RDS in short ([1]). The concept of RDS is a combining idea of randomness and dynamical system. Theory

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of RDS is the frame work to study the system's asymptotic for instance the random attractors, random manifolds, Lyapunov spectrum,...In general cases, when f, g are functions of $(t, y) \in \mathbb{R} \times \mathbb{R}^d$, the system generates a stochastic two-parameter flow $X(t_0, t, y_0, \omega)$ by mean of its Cauchy operators [4], the flow induces a random dynamical system (RDS) in case $f,\ g$ are time independent.

In [22, 23], a nonautonomous ordinary differential equations $dy(t) = f(t, y_t)dt$ is considered. By introducing the space "hull" of f, the solution can be viewed as a dynamical system. Motivated by these results, we establish conditions on $f,\ g$ to construct appropriate spaces for f, g which admit needed probability structures. The flow is then defined on the product spaces and possesses group property. Equation (1.1) then generates a RDS in the sense of Bebutov flow [22].

One another topic in this paper is study the existence of random pullback attractor of the system, see for instant [5] or [8], [10] for recent results established for stochastic differential equations driven by Hölder noises. We show in Section 3 that the generated RDS possesses a random pullback random attractor under dissipative assumption of f and point out that the attractor is singleton if dissipativity is strict and g is small in some sense.

2. Preliminaries

We briefly recall some notions used in the sequence.

- Let $\mathcal{C}([a,b], \mathbb{R}^r)$, $r \geq 1$, denote the space of all continuous paths $x : [a,b] \to \mathbb{R}^r$ equipped with supremum norm $\|\cdot\|_{\infty,[a,b]}$ given by $\|x\|_{\infty,[a,b]} = \sup_{t\in[a,b]} |x_t|$.

- For $0 < \alpha < 1$, let x is a Hölder continuous function with exponent α on $[a,b]$. The semi norm $\alpha-$ Hölder of x is defined as

$$\|x\|_{\alpha-\mathrm{Hol},[a,b]} = \sup_{a\leq s<t\leq b} \frac{|x_t - x_s|}{(t-s)^\alpha}.$$

- For given $p \geq 1$, denote by $\mathcal{C}^{p-\mathrm{var}}([a,b], \mathbb{R}^r) \subset \mathcal{C}([a,b], \mathbb{R}^r)$ the space consists of all continuous paths x of finite $p-$variation, i.e.

$$\|x\|_{p-\mathrm{var},[a,b]} := \left(\sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{i=1}^{n} |x_{t_{i+1}} - x_{t_i}|^p \right)^{1/p} < \infty.$$

The $p-$variation norm of x is defined by

$$\|x\|_{p-\mathrm{var},[a,b]} \quad := \quad |x_a| + \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} .$$

Then $(\mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^r), \|\cdot\|_{p-\mathrm{var},[a,b]})$ is a (nonseparable) Banach space [11, Theorem 5.25, p. 92].

Young integral

Assume $y \in \mathcal{C}^{q-\mathrm{var}}([a,b],\mathbb{R}^{d\times m})$ and $x \in \mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^m)$ with $\frac{1}{p}+\frac{1}{q}>1$, the Young integral $\int_a^b y_t dx_t$ is defined as the limitation of the Darboux sum

$$\int_a^b y_t dx_t := \lim_{|\Pi|\to 0} \sum_{t_i\in\Pi} y_{t_i}(x_{t_{i+1}} - x_{t_i}),$$

where the limit is taken over all the finite partitions $\Pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ with $|\Pi| := \max_i |t_{i+1} - t_i|$ (see [24]). The integral satisfies ([11, Theorem 6.8, p. 116])

$$\Big| \int_a^b y_u dx_u - y_a(x_b - x_a) \Big| \leq (1 - 2^{1-\frac{1}{p}-\frac{1}{q}})^{-1} \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} .$$

Fractional Brownian motions

A m-dimensional fractional Brownian motion index H, $B^H = (B^H_t),\ t\in\mathbb{R}$, is a vector consists of m independent one dimensional fractional Brownian motions index H which are centered continuous Gaussian processes with covariance function

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),\ s,t\in\mathbb{R}.$$

For each $p\geq 1$ denote by $\mathcal{C}^{0,p-\mathrm{var}}([a,b],\mathbb{R}^m)$ the closure of set of smooth paths in $\mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^m)$ and Ω the spaces of all continuous functions $\omega:\mathbb{R}\to\mathbb{R}^m$ vanish at 0 such that the restriction of ω on $[a,b]$ is in $\mathcal{C}^{0,p-\mathrm{var}}([a,b],\mathbb{R}^m)$ for all $[a,b]$. Then Ω is a separable metric space with the metric (see [2])

$$d(\omega^1,\omega^2) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|\omega^1-\omega^2\|_{p\text{-var},[-n,n]}}{1+\|\omega^1-\omega^2\|_{p\text{-var},[-n,n]}}. \tag{2.1}$$

Follow [13], one can construct a canonical space for B^H on Ω for some $p > 1/H$ with Borel $\sigma-$algebra $\mathcal{F}$ and the law $\mathbb{P}$ of B^H. It is proved in [13] that together with Wiener shift (θ_t) defined as

$$\theta_t(\omega)(\cdot) := \omega(t+\cdot) - \omega(t), \quad \omega\in\Omega,$$

the space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))$ forms an ergodic dynamical system. From now on, we always work on the canonical space of B^H. We keep the old notation B^H and identify $B^H_\cdot(\omega) = \omega(\cdot), \omega \in \Omega$. Moreover, since we consider the case $H > 1/2$, p can be choosen in $(1/H, 2)$, the integral w.r.t. B^H can be defined by Young sense [24].

Finally, recall from [15, Proposition 2.1] that there exists random variable $\xi(\omega)$ and $\kappa > 0$ satisfying $\mathbb{E}e^{\kappa\xi^2} < \infty$ such that for some constant D, for almost all ω

$$\left|\!\left|\!\left| B^H_\cdot(\omega) \right|\!\right|\!\right|_{p-\text{var},[0,1]} \leq D\xi(\omega).$$

It follows that for all $k > 0$, $\mathbb{E} \left|\!\left|\!\left| B^H_\cdot(\omega) \right|\!\right|\!\right|^k_{p-\text{var},[0,1]} < \infty$.

3. Generation of random dynamical system

3.1. Bebutov flow

In this section we show that (1.1) generates a random dynamical system (RDS) in an extended space. A RDS on $\mathbb{R}^d$ over a metric dynamical system (see for instant [1]) $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\theta^*_t))$ is a measurable mapping

$$\varphi : \mathbb{R}^+ \times \mathbb{R}^d \times \Omega^* \to \mathbb{R}^d, (t, x, \omega) \mapsto \varphi(t, \omega)x$$

satisfying
(i) $\varphi(0, \omega) = Id$ for all $\omega \in \Omega^*$,
(ii) $\varphi(t + s, \omega) = \varphi(t, \theta^*_s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega^*$.

If, in addition, $x \mapsto \varphi(t, \omega)x$ is continuous for all t, ω then φ is called continuous.

Recall from [22] that on $\mathcal{C} := \mathcal{C}(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ the shift mapping $S = (S_t)_{t\in\mathbb{R}}$ is defined as

$$S_t h = S(t, h) =: h_t, \ \ \forall h \in \mathcal{C},$$

h_t is called a translate of h given by $h_t(s, x) = h(t + s, x), \ (s, x) \in \mathbb{R} \times \mathbb{R}^d$.

Observe that if y is a solution to

$$dy_t = f(t, y_t)dt + g(t, y_t)d\omega_t, t \in \mathbb{R}, \ y_0 \in \mathbb{R}^d, \tag{3.1}$$

where ω is a realization of B^H, then

$$
\begin{aligned}
(3.2)\quad & y_{s+t} \\
= & \int_0^{s+t} f(u,y_u)du + \int_0^{s+t} g(u,y_u)d\omega_u \\
= & \int_0^{s} f(u,y_u)du + \int_0^{s} g(u,y_u)d\omega_u + \int_s^{s+t} f(u,y_u)du + \int_s^{s+t} g(u,y_u)d\omega_u \\
= & y_s + \int_0^{t} S_s f(u,y_{s+u})du + \int_0^{t} S_s g(u,y_{s+u})d\theta_s\omega_u.
\end{aligned}
$$

Then $y_{s+\cdot}$ is the solution of (3.1) with coefficients $S_sf,\ S_sg$. This suggested using Krylov-Bogoliubov theorem [18, Chapter VI, §9] to construct probability structures on hull of f and g in appropriate metric spaces. To do this we consider (1.1) under the conditions as follows.

Assumptions

($\mathbf{H}_1$) $f(t,x)$ is uniformly continuous on $\mathbb{R}\times K$ for each K compact in $\mathbb{R}^d$, and there exists $C_f,\ f_0>0$ such that for all $x,y\in\mathbb{R}^d$, $s,t\in\mathbb{R}$

$$
\begin{cases}
(i)\ |f(t,x)-f(t,y)|\le C_f|x-y|,\\
(ii)\ |f(t,0)|\le f_0.
\end{cases}
$$

($\mathbf{H}_2$) $g(t,x)$ is bounded by $\|g\|_\infty$ and differentiable in x with $\partial_x g$ being locally Lipschitz in x uniformly in t. Moreover, there exists $C_g>0$ and $\beta\in(1-1/p,1)$ such that the following properties hold for all $x,y\in\mathbb{R}^d$, $s,t\in\mathbb{R}$

$$
\begin{cases}
(i)\ |g(t,x)-g(t,y)|\le C_g|x-y|,\\
(ii)\ |g(t,x)-g(s,x)|+\|\partial_x g(t,x)-\partial_x g(s,x)\|\le C_g|t-s|^\beta.
\end{cases}
$$

Under these conditions, system (1.1) possesses a unique solution $y_t=y(t,x_0,\omega)$, $t\in\mathbb{R}$ for each realization ω of B^H. Moreover, for all $[a,b]\subset\mathbb{R}$,

$$
(3.3)\qquad \|y\|_{p-\mathrm{var},[a,b]}\le M(b-a)\,[|y_a|+1]\,\Lambda(\omega,[a,b])
$$

where M is a constant depend on $b-a$ and $\Lambda(\omega,[a,b])$ is a polynomial of $|\!|\!|\omega|\!|\!|_{p-\mathrm{var},[a,b]}$ (see [4],[8]).

3.1.1. Hull of *f*

In a similar manner of (2.1), define the metric d_0 in $\mathcal{C}$ - space of all continuous functions on $\mathbb{R}$ by replacing the $p-$variation norm $\|\cdot\|_{p-\mathrm{var},[a,b]}$ by supreme norm $\|\cdot\|_{\infty,[a,b]}$. For given f, the hull of f, denoted by $\mathcal{H}^f_{d_0}$ the closure of the sets $\{S_\tau f|\tau\in\mathbb{R}\}$ in $(\mathcal{C},d_0)$,

$$
\mathcal{H}^f_{d_0}:=\overline{\{S_\tau f|\tau\in\mathbb{R}\}}^{(\mathcal{C},d_0)}.
$$

According to [22, Theorem 1, 14] S defines a dynamical system on $\mathcal{C}$. Moreover, by the assumptions, f is bounded and uniformly continuous on $\mathbb{R}\times K$ for each K compact in $\mathbb{R}^d$, $\mathcal{H}^f_{d_0}$ is compact in $\mathcal{C}$. We derive required properties for $\mathcal{H}^f_{d_0}$.

Note that, similar results apply for $\mathcal{C}^{1,0} = (\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d),\rho)$-the space of continuous functions h with $\partial_x h\in\mathcal{C}$ with metric

$$\rho(h,k) = d_0(h,k) + d_0(\partial_x h,\partial_x k). \tag{3.4}$$

3.1.2. Hull of g

Next, we construct similar space for g. Firstly, consider the subspace $\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})\subset\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$ containing functions h which is of local $\alpha-$Hölder w.r.t. t for each $x\in\mathbb{R}^d$ and moreover for each compact set K in $\mathbb{R}^d$

$$\sup_{x\in K}\left\|\left|h(\cdot,x)\right\|\right|_{\alpha-\mathrm{Hol},[a,b]}<\infty,\ \forall[a,b]\subset\mathbb{R}^d.$$

We consider the following metric on $\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$ which is denoted by d_1

$$d_1(h^1,h^2) := \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|h^1-h^2\|_{\alpha,1,0;K_n}}{1+\|h^1-h^2\|_{\alpha,1,0;K_n}}, \tag{3.5}$$

where

$$\begin{aligned}
\|h^1-h^2\|_{\alpha,1,0;K^1\times K^2} &:= \|h^1-h^2\|_{1,0;K^1\times K^2} + \left\|\left|h^1-h^2\right\|\right|_{\alpha,K^1\times K^2}\\
\|h^1-h^2\|_{1,0;K^1\times K^2} &:= \sup_{K^1\times K^2}|h^1-h^2| + \sup_{K^1\times K^2}\|\partial_x h^1-\partial_x h^2\|\\
\left\|\left|h^1-h^2\right\|\right|_{\alpha,K^1\times K^2} &:= \sup_{x\in K^2}\left\|\left|h^1(\cdot,x)-h^2(\cdot,x)\right\|\right|_{\alpha-\mathrm{Hol},K^1}
\end{aligned}$$

with K^1,K^2 are compact sets in $\mathbb{R}$, $\mathbb{R}^d$ respectively.

Proposition 3.1. $(\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m}),d_1)$ *is a complete metric space.*

Proof. See in the Appendix.

Next, we fix $1-\frac{1}{p}<\beta_0<\beta$, denoted by $(\mathcal{C}^{\beta_0;1,0},d_1)$ the space $(\mathcal{C}^{\beta_0;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m}),d_1)$. Put $\mathcal{H}^g_{d_1}$ the closure of $\{S_\tau g|\tau\in\mathbb{R}\}$ in $\mathcal{C}^{\beta_0;1,0}$, i.e.

$$\mathcal{H}^g_{d_1} := \overline{\{S_\tau g|\tau\in\mathbb{R}\}}^{(\mathcal{C}^{\beta_0;1,0},d_1)}.$$

The similar results hold for hull of g as stated below.

Lemma 3.1. *All* $g^*\in\mathcal{H}^g_{d_1}$ *satisfies* $(\mathbf{H_2})$ *and moreover,* $\mathcal{H}^g_{d_1}$ *is a compact set in* $(\mathcal{C}^{\beta_0;1,0},d_1)$*.*

Proof. See in the Appendix.

Since $\mathcal{C}^{\beta_0;1,0}$ is not separable, in the following we directly prove that S defines a dynamical system on $\mathcal{H}^g_{d_1}$.

Lemma 3.2. *S defines a dynamical system on $\mathcal{H}^g_{d_1}$.*

Proof. Due to [22, Theorem 12], S defined a dynamical system on $\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$. We just need to check that for fixed $(t_0,h^0)\in\mathbb{R}\times\mathcal{H}^g_{d_1}$, if $t\in\mathbb{R}, h\in\mathcal{H}^g_{d_1}$ such that $|t-t_0|, d_1(h,h^0)\to 0$ then $|||h_t(\cdot,x)-h^0_{t_0}(\cdot,x)|||_{\beta^0-\mathrm{Hol},[a,b]\times K}\to 0$ for each a,b, each K compact in $\mathbb{R}^d$. Namely, by choosing appropriate $[a',b']$ we have

$$\begin{aligned}
& |||h_t-h^0_{t_0}|||_{\beta^0-\mathrm{Hol},[a,b]\times K}\\
\leq\ & |||h_t-h^0_t|||_{\beta^0-\mathrm{Hol},[a,b]\times K}+|||h^0_t-h^0_{t_0}|||_{\beta^0-\mathrm{Hol},[a,b]\times K}\\
\leq\ & |||h-h^0|||_{\beta^0-\mathrm{Hol},[a',b']\times K}+2\,|||h^0|||^{\beta_0/\beta}_{\beta-\mathrm{Hol},[a',b']\times K}\cdot\|h^0_t-h^0_{t_0}\|^{1-\beta_0/\beta}_{\infty,[a,b]\times K}\\
& \to 0,\ \text{ as } |t-t_0|\to 0,\ d_1(h,h^0)\to 0.
\end{aligned}$$

This shows the continuity of S on $\mathcal{H}^g_{d_1}$. Since $\mathcal{H}^g_{d_1}$ is compact, S is measurable w.r.t. the $\sigma-$algebra generated by d_1. The proof is completed.

■

3.2. Generation of RDS

Since $\mathcal{H}^f_{d_0},\mathcal{H}^g_{d_1}$ are compact sets with appropriate metrics constructed above, we deduce from Krylov-Bogoliubov theorem [18, Chapter VI, §9] that there are probability measures $\mathbb{P}^f,\mathbb{P}^g$ on measurable space $(\mathcal{H}^f_{d_0},\mathcal{B}^f),(\mathcal{H}^g_{d_1},\mathcal{B}^g)$ with Borel $\sigma-$algebras $\mathcal{B}^f,\mathcal{B}^g$, that are invariant under the shifts mapping S. Denote by $\bar\Omega$ the Catersian product $\mathcal{H}^f_{d_0}\times\mathcal{H}^g_{d_1}\times\Omega$ with the product Borel $\sigma-$field denoted by $\bar{\mathcal{B}}$ and the product measure $\bar{\mathbb{P}}=\mathbb{P}^f\times\mathbb{P}^g\times\mathbb{P}$ and consider the product dynamical system $\bar\theta:\mathbb{R}\times\bar\Omega\to\bar\Omega$ given by

$$\bar\theta(t,\tilde f,\tilde g,\omega)=(S_t\tilde f,S_t\tilde g,\theta_t\omega),\ (\tilde f,\tilde g,\omega)\in\bar\Omega.$$

It is evident that $(\bar\Omega,\bar{\mathcal{B}},\bar{\mathbb{P}},\bar\theta)$ forms a metric dynamical system.

Proposition 3.2. *For each $\bar\omega=(\bar f,\bar g,\omega)\in\bar\Omega$, equation*

$$dy_t=\bar f(t,y_t)dt+\bar g(t,y_t)d\omega_t,\ y_0\in\mathbb{R}^d,\ t\in\mathbb{R}^+ \qquad (3.6)$$

possesses a unique solution $y(t,y_0,\bar\omega)$. The solution is continuous w.r.t. the initial condition y_0 and satisfies (3.3).

Proof. It is easy to check that all elements in $\mathcal{H}^f_{d_0}$ satisfies $(\mathbf{H_1})$. As stated in Lemma 3.1, $\bar{g}$ satisfies $(\mathbf{H_2})$. The statement is evident due to [4].

■

Theorem 3.3. *System*

$$dy_t = f(t, y_t)dt + g(t, y_t)dB^H_t \tag{3.7}$$

generates a continuous random dynamical system over $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{P}, \bar{\theta})$.

Proof. For each $\bar{\omega} = (\bar{f}, \bar{g}, \omega) \in \bar{\Omega}$, consider (3.6). Define

$$\Phi^* : \mathbb{R}^+ \times \mathbb{R}^d \times \bar{\Omega} \to \mathbb{R}^d$$

where $\Phi^*(t, \bar{\omega})y_0$ is the value of the of the solution of (3.6) at the time $t \in \mathbb{R}^+$ with the initial time $s = 0$ and initial value y_0, i.e. $y(t, y_0, \bar{\omega})$. From (3.2), Φ^* satisfies cocycle property

$$\Phi^*(t + s, \bar{\omega})y_0 = \Phi^*(t, \bar{\theta}_s\bar{\omega}) \circ \Phi^*(s, \bar{\omega})y_0.$$

Next, to complete the proof we prove that the solution is continuous w.r.t. $\bar{\omega}$ as an element in the product of separable metric spaces $\mathcal{H}^f_{d_0}, \mathcal{H}^g_{d_1}, \Omega$. The measurability of the solution is obtained thank to [3, Lemma III. 14]. Namely, we fix t, x_0 and $[0, T]$ contains t and consider $\bar{\omega}^1 = (f^1, g^1, \omega^1)$, $\bar{\omega}^2 = (f^2, g^2, \omega^2)$ in $\bar{\Omega}$. Put $y^1_t := y(t, y_0, \bar{\omega}^1), y^2_t := y(t, y_0, \bar{\omega}^2)$ then we have

$$\begin{aligned} y^1_t &= x_0 + \int_0^t f^1(s, y^1_s)ds + \int_0^t g^1(s, y^1_s)d\omega^1_s, \\ y^2_t &= x_0 + \int_0^t f^2(s, y^2_s)ds + \int_0^t g^2(s, y^2_s)d\omega^2_s. \end{aligned}$$

Therefore, $z_t := y^1_t - y^2_t$ satisfies the equation

$$\begin{aligned} z_t &= y^1_t - y^2_t \\ &= \int_0^t [f^1(s, y^1_s) - f^2(s, y^2_s)]ds + \int_0^t [g^1(s, y^1_s)d\omega^1_s - \int_0^t g^2(s, y^2_s)]d\omega^2_s \\ &\quad + \int_0^t [f^2(s, y^1_s) - f^2(s, y^2_s)]ds + \int_0^t [f^1(s, y^1_s) - f^2(s, y^1_s)]ds \\ &\quad + \int_0^t g^1(s, y^1_s)d(\omega^1_s - \omega^2_s) + \int_0^t [g^1(s, y^1_s) - g^2(s, y^1_s)]d\omega^2_s \\ &\quad + \int_0^t [g^2(s, y^1_s) - g^2(s, y^2_s)]d\omega^2_s. \end{aligned}$$

Fixing $\bar{\omega}^1$, due to (3.3) one can find R depends on $\bar{\omega}^1$ such that $\|y(\cdot,y_0,\bar{\omega})\|_{p-\text{var},[0,T]} \leq R$ for all $\bar{\omega}$ lies in the neighbor of $\bar{\omega}^1$ of radius 1. We choose a upper bound for the norms of f^i, g^i, ω^i on $\bar{K} := [0,T]\times \bar{B}(0,R)$ and reuse the notation R for convenient. We will show that z is near 0 when $\|f^1-f^2\|_{\infty,\bar{K}}$, $\|g^1-g^2\|_{\infty,\bar{K}}$, $\|\partial_x g^1 - \partial_x g^2\|_{\infty,\bar{K}}$, and $\left\|\!\left|g^1-g^2\right|\!\right\|_{\beta_0,\bar{K}}$ less than ε small enough.

For $0\leq u<v\in[0,T]$ and $q:=1/\beta$

$$\begin{aligned}|z_u-z_v| &= \left|\int_u^v [f^2(s,y^1_s)-f^2(s,y^2_s)]ds\right| + \left|\int_u^v [f^1(s,y^1_s)-f^2(s,y^1_s)]ds\right|\\ &\quad + \left|\int_u^v [g^2(s,y^1_s)-g^2(s,y^2_s)]d\omega^2_s\right| + \left|\int_u^v g^1(s,y^1_s)d(\omega^1_s-\omega^2_s)\right|\\ &\quad + \left|\int_u^v [g^1(s,y^1_s)-g^2(s,y^1_s)]d\omega^2_s\right|\end{aligned}$$

in which

$$\begin{aligned}\left|\int_u^v [f^2(s,y^1_s)-f^2(s,y^2_s)]ds\right| &\leq C_f\int_u^v |z_s|ds,\\ \left|\int_u^v [g^2(s,y^1_s)-g^2(s,y^2_s)]d\omega^2_s\right| &\leq DC_g(1+\left\|\!\left|y^1\right|\!\right\|_{p-\text{var},[u,v]}+\left\|\!\left|y^2\right|\!\right\|_{p-\text{var},[u,v]})\times\\ &\qquad\times \left\|\!\left|\omega^2\right|\!\right\|_{p-\text{var},[u,v]}\|z\|_{q-\text{var},[u,v]}\end{aligned}$$

where the final estimate due to [4]. And

$$\begin{aligned}\left|\int_u^v [f^1(s,y^1_s)-f^2(s,y^1_s)]ds\right| &\leq \|f^1-f^2\|_{\infty,\bar{K}}(v-u),\\ \left|\int_u^v g^1(s,y^1_s)d(\omega^1_s-\omega^2_s)\right| &\leq D\left\|\!\left|\omega^1-\omega^2\right|\!\right\|_{p-\text{var},[u,v]}\left[\|y^1\|_{q-\text{var},[u,v]}+(v-u)^\beta+1\right],\\ \left|\int_u^v [g^1(s,y^1_s)-g^2(s,y^1_s)]d\omega^2_s\right| &\leq D\left\|\!\left|\omega^2\right|\!\right\|_{p-\text{var},[u,v]}\left[\|g^1-g^2\|_{\infty,\bar{K}}+\left\|\!\left|g^1-g^2\right|\!\right\|_{q-\text{var},[u,v]}\right],\\ &\leq D\left\|\!\left|\omega^2\right|\!\right\|_{p-\text{var},[u,v]}\Big[\|g^1-g^2\|_{\infty,\bar{K}}\\ &\quad+\left\|\!\left|g^1-g^2\right|\!\right\|_{\beta_0,\bar{K}}(v-u)^\beta+\|\partial_x g^1-\partial_x g^2\|_{\infty,\bar{K}}\left\|\!\left|y^1\right|\!\right\|_{p-\text{var},[u,v]}\Big].\end{aligned}$$

In the final estimate we use the mean value theorem namely for $s,t\in[u,v]$

$$\begin{aligned}&|g^1(t,y^1_t)-g^2(t,y^1_t)-g^1(s,y^1_s)+g^2(s,y^1_s)|\\ &\leq |(g^1-g^2)(t,y^1_t)-|(g^1-g^2)(s,y^1_t)|+|(g^1-g^2)(s,y^1_t)-|(g^1-g^2)(s,y^1_s)|\\ &\leq \left\|\!\left|g^1-g^2\right|\!\right\|_{\beta_0,\bar{K}}(t-s)^{\beta_0}+\|\partial_x g^1-\partial_x g^2\|_{\infty,\bar{K}}|y^1_t-y^1_s|.\end{aligned}$$

Therefore

$$\left\|\!\left|z\right|\!\right\|_{q-\text{var},[u,v]}\leq D\left(\int_u^v |z_s|ds+\|z\|_{q,[u,v]}+A^{1/q}_{u,v}\right)$$

where D is a constant depending on R and A is a control function defined by

$$A_{u,v}^{1/q} := \varepsilon(v-u) + \left\|\omega^1 - \omega^2\right\|_{p-\mathrm{var},[u,v]} + \varepsilon \left\|\omega^2\right\|_{p-\mathrm{var},[u,v]}.$$

Apply Lemma 4.1, since $z_0 = 0$ we obtain

$$\|z\|_{q-\mathrm{var},[0,T]} \leq D(\|z_0\| + \varepsilon) = D\varepsilon \to 0 \ \ as \ \ \varepsilon \to 0.$$

This completes the proof. ■

4. Random attractors

In what follows we recall the notion of the (global) random attractor. For a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, a set $\mathcal{M} \subset \mathbb{R}^d \times \Omega^*$ with closed $\omega-$section $\mathcal{M}(\omega) = \{x \in \mathbb{R}^d | (\omega, x) \in \mathcal{M}\}$ is called random set if the map $\omega \mapsto d(x, \mathcal{M}(\omega))$ is measurable for every $x \in \mathbb{R}^d$, where d is the Hausdorff semi-distance.

We work with the universe $\hat{\mathcal{D}}$- the family of *tempered* random sets $\hat{D}(\omega)$, i.e $\hat{D}(\omega)$ is contained in a ball $B(0, r(\omega))$ a.s., where the radius $r(\omega)$ is a tempered random variable, namely satisfies

$$\lim_{t\to\pm\infty} \frac{1}{t} \log^+ r(\theta_t^*\omega) = 0, \quad \text{a.s.} \tag{4.1}$$

Let φ be a continuous random dynamical system on $\mathbb{R}^d$ over a metric dynamical system $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\theta_t^*))$. A random subset $\mathcal{A}$ is called invariant, if

$$\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t^*\omega) \ \ \forall t \in \mathbb{R}^+, \ a.s \ \ \omega \in \Omega^*.$$

It is called a *pullback random attractor* in $\hat{\mathcal{D}}$ if it is compact, invariant and attracts any $\hat{D} \in \hat{\mathcal{D}}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}^*\omega)\hat{D}(\theta_{-t}^*\omega)|\mathcal{A}(\omega)) = 0, \ \ \forall \hat{D} \in \hat{\mathcal{D}}, \ a.s. \ \ \omega \in \Omega^*. \tag{4.2}$$

A random set $\mathcal{B} \in \hat{\mathcal{D}}$ is called *pullback absorbing* in the universe $\hat{\mathcal{D}}$ if $\mathcal{B}$ absorbs all sets in $\hat{\mathcal{D}}$, i.e. for any $\hat{D} \in \hat{\mathcal{D}}$, there exists a time $t_0 = t_0(\omega, \hat{\mathcal{D}})$ such that

$$\varphi(t, \theta_{-t}^*\omega)\hat{D}(\theta_{-t}^*\omega) \subset \mathcal{B}(\omega), \text{ for all } t \geq t_0. \tag{4.3}$$

If there exists pullback absorbing set for φ, then it is proved that

$$\mathcal{A}(\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \theta_{-t}^*\omega)\mathcal{B}(\theta_{-t}^*\omega)}. \tag{4.4}$$

is the random pullback attractor of φ. Moreover, it is unique in $\hat{\mathcal{D}}$ ([21]).

In the following, we assume that f is uniform dissipative ([6]), i.e. there exist $c, d > 0$ such that for all $t \in \mathbb{R},\ x \in \mathbb{R}^d$

$$\langle x, f(t,x)\rangle \leq c - d\|x\|^2. \tag{4.5}$$

We will prove that the RDS generated by (3.7) possesses a random attractor. The technique is followed from [8]. Here we sketch some main details.

Theorem 4.1. *In addition to* $(\mathbf{H_1}), (\mathbf{H_2})$ *if* f *satisfies* (4.5)*, then RDS generated by system* (3.7) *possesses a random pullback attractor almost sure.*

Proof. Step 1: First, fix $\bar{\omega} = (\bar{f}, \bar{g}, \omega) \in \bar{\Omega}$, $[a,b] \subset \mathbb{R}^+$. We consider the corresponding ordinary differential equation

$$\dot{\mu_t} = \bar{f}(t, \mu_t),\ t \in [a,b],\ \mu_a = y_a. \tag{4.6}$$

where y is a solution of (3.6) on $[a,b]$.

Since f is dissipative,

$$\begin{aligned} \|\mu\|_{\infty,[a,b]} &\leq |\mu_a| + L, \\ \|\!|\mu|\!\|_{p-\mathrm{var},[a,b]} &\leq L\left(|\mu_a| + 1\right)(b-a) \end{aligned}$$

where L is a constant.

Define $k_t = y_t - \mu_t, t \in [a,b]$. Since k satisfies the equation

$$dk_t = d(y_t - \mu_t) = [\bar{f}(t, \mu_t + k_t) - \bar{f}(t, \mu_t)]dt + \bar{g}(t, \mu_t + k_t)d\omega_t$$

we have

$$k_t - k_s = \int_s^t \Big[\bar{f}(u, k_u + \mu_u) - \bar{f}(u, \mu_u)\Big]du + \int_s^t \bar{g}(u, k_u + \mu_u)d\omega_u.$$

It follows from $(\mathbf{H}_2)$ and the boundedness of g that

$$\begin{aligned} |k_t - k_s| \ \leq\ & \int_s^t C_f |k_u| du + \|\bar{g}\|_\infty \|\!|\omega|\!\|_{p-\mathrm{var},[s,t]} \\ & + K \|\!|\omega|\!\|_{p-\mathrm{var},[s,t]} \|\!|\bar{g}(\cdot, k_\cdot + \mu_\cdot)|\!\|_{q-\mathrm{var},[s,t]}, \end{aligned} \tag{4.7}$$

where $q = 1/\beta$, $K = (1 - 2^{1-1/p-1/q})^{-1}$. Since

$$\begin{aligned} & |\bar{g}(t, k_t + \mu_t) - \bar{g}(s, k_s + \mu_s)| \\ & \quad \leq\ |\bar{g}(t, k_t + \mu_t) - \bar{g}(t, k_s + \mu_s)| + |\bar{g}(t, k_s + \mu_s) - \bar{g}(s, k_s + \mu_s)| \\ & \quad \leq\ C_g |k_t - k_s| + C_g |\mu_t - \mu_s| + C_g (t-s)^\beta \\ & \quad \leq\ C_g |k_t - k_s| + M(1 + |\mu_a|^\beta)(t-s)^\beta, \qquad \forall a \leq s < t \leq b \end{aligned}$$

where $M = M(r)$ depends on $r = b - a$, we have

$$\left\|\bar{g}(\cdot, k_\cdot + \mu_\cdot)\right\|_{q-\text{var},[s,t]} \leq C_g \left\|k\right\|_{p-\text{var},[s,t]} + M(1 + |\mu_a|^\beta)(t-s)^\beta,$$

with a note that $q\beta \geq 1$ and $q \geq p$. Then

$$\begin{aligned} |k_t - k_s| &\leq \Big[\|g\|_\infty + KM(1 + |\mu_a|^\beta)\Big] \left\|\omega\right\|_{p-\text{var},[s,t]} + \int_s^t C_f |k_u| du \\ &\quad + KC_g \left\|\omega\right\|_{p-\text{var},[s,t]} \left\|k\right\|_{p-\text{var},[s,t]}. \end{aligned}$$

Using Lemma 4.1 and Young inequality for product

$$\begin{aligned} \|k\|_{\infty,[a,b]} &\leq e^{2C_f r}\Big[|k_a| + M(1 + |\mu_a|^\beta) \left\|\omega\right\|_{p-\text{var},[a,b]} (1 + \left\|\omega\right\|^p_{p-\text{var},[a,b]})\Big] \\ &\leq M(1 + |\mu_a|^\beta) \left\|\omega\right\|_{p-\text{var},[a,b]} (1 + \left\|\omega\right\|^p_{p-\text{var},[a,b]}) \\ &\leq \varepsilon |y_a| + \Lambda(\omega, [a,b]), \end{aligned} \tag{4.8}$$

where $\varepsilon > 0$ is chosen later and $\Lambda(\omega, [a,b])$ is a general polynomial of $\left\|\omega\right\|_{p-\text{var},[a,b]}$.

Step 2: Next, we estimate the solution of (3.6) by discretization.
By assumption of f, it can be seen that all $\tilde{f} \in \mathcal{H}^f_{d_0}$ satisfy (4.5). For each n, consider(4.6) with $[a,b]$ is replaced by $[n-1, n]$. By known result of (4.6) under condition (4.5), there exists $\eta \in (0,1), L > 0$ such that

$$|\mu_n| \leq \eta^* |y_n| + L.$$

Now in (4.8), we choose $0 < \varepsilon < 1 - \eta^*$ and $\eta = \eta^* + \varepsilon \in (0,1)$. Then,

$$\begin{aligned} |y_n| &\leq |k_n| + |\mu_n| \\ &\leq \eta |y_{n-1}| + \Lambda(\omega, [n-1, n]). \end{aligned}$$

Therefore,

$$\begin{aligned} |y_n| &\leq \eta |y_{n-1}| + \Lambda(\omega, [n-1, n]) \\ &\leq \eta^n |y_0| + \sum_{j=1}^n \eta^j \Lambda(\omega, [n-1-j, n-j]). \end{aligned} \tag{4.9}$$

Define $R(\bar{\omega}) := \sum_{j \geq 0} \eta^j \Lambda(\omega, [-j, -j+1])$, then as n large enough

$$|y(n, y_0, \theta_{-n}\bar{\omega})| \leq 1 + R(\bar{\omega}).$$

Step 3: Finally, we prove the existence of an absorbing set.

Using (3.3) the value of solution at arbitrary time is evaluated similarly. Namely, there exists a tempered random variable $\tilde{R}(\bar{\omega})$ (see [8]) such that

$$|y(t, y_0, \bar{\theta}_{-t}\bar{\omega})| \leq 1 + \tilde{R}(\bar{\omega})$$

as t large enough. It shows the existence of the absorbing set $\mathcal{B}(\bar{\omega}) = \bar{B}(0, \tilde{R}(\bar{\omega}))$. The proof of this step relies on the ergodicity of canonical space $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and ergodic Birkhoff theorem.

Note that $\mathbb{E} \left\| B^H \right\|^m_{p-\mathrm{var},[0,1]} < \infty$ for all $m \in \mathbb{N}$. This deduces that $\Lambda(\omega)$ and then $\tilde{R}(\omega)$ is also integrable. Moreover, in (4.9), one can evaluate $|y_n|^m$ for any $m > 0$ and choose $\tilde{R}$ to be integrable at arbitrary order m.

The existence of random pullback attractor $\mathcal{A}(\bar{\omega})$ for Φ^* is proved.

■

Theorem 4.2. *If we assume f satisfies uniform one-sided dissipative condition*

$$\langle x - y, f(t,x) - f(t,y) \rangle \le -L|x-y|^2, \ \forall t, x, y$$

for some $L > 0$. Then there exists $\epsilon > 0$ such that if $C_g < \epsilon$ the attractor is singleton.

Proof. Let y^1, y^2 be two solutions of (3.6) where the initial conditions lie in $\bar{B}(0, R)$. Put $\bar{y} = y^2 - y^1$ then

$$d\bar{y}_t = [\bar{f}(t, \bar{y}_t + y^1_t) - \bar{f}(t, y^1_t)]dt + [\bar{g}(t, y^2_t) - \bar{g}(t, y^1_t)]d\omega_t.$$

Once again, we consider the pure dt equation

$$d\bar{\mu}_t = [\bar{f}(t, \bar{\mu}_t + y^1_t) - \bar{f}(t, y^1_t)]dt, \quad \bar{\mu}_0 = \bar{y}_0.$$

By assumption of f, there exists $\eta \in (0,1)$ such that

$$|\bar{\mu}_1| \le \eta |\bar{\mu}_0|.$$

Now, put $z = \bar{y} - \bar{\mu}$, we have

$$dz_t = [\bar{f}(t, \bar{y}_t + y^1_t) - \bar{f}(t, \bar{\mu}_t)]dt + [\bar{g}(t, y^2_t) - \bar{g}(t, y^1_t)]d\omega_t.$$

Computation leads to

$$|z_t - z_s| \le \int_s^t C_f |z_u| du + DC_g \left\| \omega \right\|_{p-\mathrm{var},[s,t]} . \|z + \bar{\mu}\|_{p-\mathrm{var},[s,t]} (1 + \left\| y^1 \right\|_{p-\mathrm{var},[s,t]}). \tag{4.10}$$

By (3.3), for all $s, t \in [0,1]$

$$|z_t - z_s| \le \int_s^t C_f |z_u| du + DRC_g \left\| \omega \right\|_{p-\mathrm{var},[s,t]} \Lambda(\omega, [0,1]) . \|z + \bar{\mu}\|_{p-\mathrm{var},[s,t]},$$

then using Lemma 4.1,

$$\|z\|_{p-\mathrm{var},[0,1]} \le DR|\bar{y}_0| C_g e^{RC_g \Lambda(\omega,[0,1])},$$

where $\Lambda(\omega,[a,b])$ is a general polynomial of $\left\|\omega\right\|_{p-\mathrm{var},[a,b]}$. We arrive at

$$|\bar{y}_1| \leq \eta|\bar{y}_0|\left[1+DRC_g e^{RC_g\Lambda(\omega,[0,1])}\right]. \tag{4.11}$$

The rest of the proof is followed step by step to [8, Theorem 3.11].

■

Appendix

Proof of Proposition 3.1

Proof. That d_1 is a metric on $\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d\times m)$ is evident due to the seminorm properties of the Hölder norm. We only need to prove the completeness. Let h^n be a Cauchy sequence in $\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$. Since $(\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d\times m),\rho)$ is complete, there exists a subsequence, which we still use the notation h^n, converges to h in $\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$, i.e.

$$\lim_{n\to\infty}\rho(h^n,h)=0.$$

We will prove that for each K^1,K^2 compact sets in $\mathbb{R},\mathbb{R}^d$, $\left\|h^n-h\right\|_{\alpha,K^1\times K^2}\to 0$ as $n\to\infty$. Fix $K\subset\mathbb{R}^d$ compact, we have for each $[a,b]\subset\mathbb{R}$ there exist a constant M such that

$$\sup_n\sup_{x\in K}\left\|h^n(\cdot,x)\right\|_{\alpha-\mathrm{Hol},[a,b]}\leq M.$$

For each $x\in K$

$$|h(t,x)-h(s,x)|=\lim_{n\to\infty}|h^n(t,x)-h^n(s,x)|\leq M|t-s|^\alpha,$$

this implies that $\sup_{x\in K}\left\|h(\cdot,x)\right\|_{\alpha-\mathrm{Hol},[a,b]}<\infty$ or $h\in\mathcal{C}^{\alpha;1,0}(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^{d\times m})$.

Now to complete the proof we show that h^n converges to h, in $\alpha-$Hölder norm on each K compact in $\mathbb{R}^d$. For each $s<t\in[a,b]$, $x\in K$

$$\begin{aligned}\frac{|(h^n-h)(t,x)-(h^n-h)(s,x)|}{|t-s|^\alpha} &= \lim_{m\to\infty}\frac{|(h^n-h^m)(t,x)-(h^n-h^m)(s,x)|}{|t-s|^\alpha}\\ &\leq \lim_{m\to\infty}\sup_{x\in K}\sup_{a\leq v<u\leq b}\frac{|(h^n-h^m)(u,x)-(h^n-h^m)(v,x)|}{|u-v|^\alpha}\\ &\leq \lim_{m\to\infty}\left\|h^n-h^m\right\|_{\alpha,[a,b]\times K},\end{aligned}$$

which implies

$$|||h^n - h|||_{\alpha,[a,b]\times K} \leq \lim_{m\to\infty} |||h^n - h^m|||_{\alpha,[a,b]\times K} \to 0, \text{ as } n \to \infty.$$

The proof is completed.

Proof of Lemma 3.1

Proof.

It can be seen from the assumptions of g that g together with $\partial_x g$ satisfies the condition boundedness and equicontinuous on $\mathbb{R} \times K$ for each K compact in $\mathbb{R}^d$.

Due to [22, Theorem 16] $\mathcal{H}^g_{d_1}$ is compact in $(\mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^{d\times m}), \rho)$. Hence, for $g^* \in \mathcal{H}^g_{d_1}$, $\partial_x g^*$ exists and is continuous. Moreover, and there exists t_n such that $\lim_{n\to\infty} d_1(g^*, g_{t_n}) = 0$.

It is evident that g^* is bounded by $\|g\|_\infty$, and

$$\begin{aligned}
|g^*(t,x) - g^*(t,y)| &= \lim_{n\to\infty} |g_{t_n}(t,x) - g_{t_n}(t,y)| \\
&= \lim_{n\to\infty} |g(t_n + t, x) - g(t_n + t, y)| \leq C_g|x - y|, \\
|g^*(t,x) - g^*(s,x)| &+ \|\partial_x g^*(t,x) - \partial_x g^*(s,x)\| \\
&= \lim_{n\to\infty} |g_{t_n}(t,x) - g_{t_n}(s,x)| + \|\partial_x g_{t_n}(t,x) - \partial_x g_{t_n}(s,x)\| \\
&\leq C_g|t - s|^\beta.
\end{aligned}$$

That $\partial_x g^*(t,x)$ is local Lipschitz in x uniformly in t is also obvious. The first statement is proved.

For the second one, since $\mathcal{H}^g_{d_1}$ is compact in $\mathcal{C}^{1,0}$, from a sequence $h^n \in \mathcal{H}^g_{d_1}$ there exists a subsequence h^{n_k} that converges (in ρ) to $h \in \mathcal{C}^{1,0}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^{d\times m})$. One may choose the subsequence in the form g_{t_n}. Applying the above arguments for $g^* = h$ and the sequence g_{t_n} we have $h \in \mathcal{C}^{\beta;1,0}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^{d\times m})$. Moreover, $|||h^{n_k}|||_{\beta,K^1\times K^2}$, $|||h|||_{\beta,K^1\times K^2}$ are less than C_g for K^1, K^2 are compact sets in $\mathbb{R}, \mathbb{R}^d$ respectively.

Finally, put $h_k = h^{n_k} - h$. Since $\beta_0 < \beta$, for $s,t \in K^1, x \in K^2$

$$\begin{aligned}
\frac{|h_k(t,x) - h_k(s,x)|}{|t-s|^{\beta_0}} &= \left(\frac{|h_k(t,x) - h_k(s,x)|}{|t-s|^\beta}\right)^{\frac{\beta_0}{\beta}} .|h_k(t,x) - h_k(s,x)|^{1-\frac{\beta_0}{\beta}} \\
&\leq |||h_k|||^{\frac{\beta_0}{\beta}}_{\beta,K^1\times K^2} \left(|h_k(t,x)| + |h_k(s,x)|\right)^{1-\frac{\beta_0}{\beta}}, \text{ hence} \\
|||h_k|||_{\beta_0,K^1\times K^2} &\leq 4C_g^{\frac{\beta_0}{\beta}} \|h_k\|^{1-\frac{\beta_0}{\beta}}_{\infty,K^1\times K^2} \to 0 \ \ as \ \ k \to \infty.
\end{aligned}$$

To sum up, h^{n_k} converges to h in d_1. The proof is completed.

■

Lemma 4.1 (Gronwall-type Lemma). *For $q \geq p$ so that $\frac{1}{p} + \frac{1}{q} > 1$, if y satisfies the following condition*

$$|y_t - y_s| \leq \hat{A}_{s,t}^{1/q} + a_1 \int_s^t |y_u| du + \|\omega\|_{p,[s,t]} (a_2 |y_s| + a_3 \|y\|_{q-\text{var},[s,t]})$$

for all $s \leq t \in [a,b]$, where a_1, a_2, a_3 are positive real constants, $\hat{A}$ is a control function on $\{(s,t) | a \leq s \leq t \leq b\}$, then

$$\|y\|_{p,[a,b]} \leq \left[|y_a| + 2\hat{A}_{a,b}^{1/q} N_{[a,b]} \right] e^{2a_1(b-a) + \kappa N_{[a,b]}} N_{[a,b]}^{\frac{p-1}{p}}(\omega)$$

with $\kappa = \log \frac{a_3/a_2+2}{a_3/a_2+1}$, and

$$N_{[a,b]} \leq D(1 + \|\omega\|_{p,[a,b]}^p)$$

for D depends on a_i. If $a_2 = 0$ one may take $\kappa = 0$.

Proof. See [8, Theorem 2.4].

■

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