Images of the Singer transfers and their possibility to be injective

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Abstract. This article is an attempt to investigate the possibility to be injective of the Singer transfer $\mathrm{Tr}_s^M : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes M_*) \to \mathrm{Ext}^s_{\mathscr{A}}(\Sigma^{-s}M, \mathbb{F}_2)$ for M being the $\mathscr{A}$-modules $\mathbb{F}_2 = \widetilde{H}^*S^0$ or $\widetilde{H}^*\mathbb{RP}^\infty$. The existence of a positive stem critical element of $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ is equivalent to the existence of a positive stem critical element of $\mathrm{Ext}^{s+1,t+1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ in the image of the transfer Tr_{s+1}. If the existences happen, then $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ and Tr_{s+1} are not injective. We show that the critical element $\widehat{Ph_2}$ is not in the image of the fourth transfer, $\mathrm{Tr}_4^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_4} P(H_*\mathbb{V}_4 \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-4} \to \mathrm{Ext}^{4,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$. Singer's conjecture is still open, as we have not known any critical element, which is in the image of the transfer.

1. Recollections on the Singer transfers and related topics

We sketch briefly the Singer transfer, which is the subject of this article.

Let $\mathscr{A}$ be the mod 2 Steenrod algebra. Singer defined in [10] the algebraic transfer for an $\mathscr{A}$-module M:

$$\mathrm{Tr}_s^M : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes M_*) \to \mathrm{Ext}^s_{\mathscr{A}}(\Sigma^{-s}M, \mathbb{F}_2),$$

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where $\mathbb{V}_s$ denotes an elementary abelian 2-group of rank s, and $H_*\mathbb{V}_s$ is the mod 2 homology of a classifying space $B\mathbb{V}_s$ of $\mathbb{V}_s$, while M_* is the dual of the $\mathscr{A}$-module M, and $P(H_*\mathbb{V}_s \otimes M_*)$ denotes the primitive part of $H_*\mathbb{V}_s \otimes M_*$ under the action of $\mathscr{A}$. The Singer transfer is a useful tool in the study of $\mathrm{Ext}^s_{\mathscr{A}}(M, \mathbb{F}_2)$ by means of the Peterson hit problem and invariant theory.

Let $M = \widetilde{H}^*X$ be the reduced cohomology of X. Here X is a pointed CW-complex, whose mod 2 homology H_*X is finitely generated in each degree. Then the Singer transfer for $\widetilde{H}^*X$ is also called the Singer transfer for X, that is $\mathrm{Tr}^X_s := \mathrm{Tr}^{\widetilde{H}^*X}_s$. It is a remarkable tool to study $\mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*X, \mathbb{F}_2)$. The interest in studying $\mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*X, \mathbb{F}_2)$ is that this forms the E_2-page of the Adams spectral sequence converging to the 2-completion of the stable homotopy groups $\pi^S_*(X)$.

Let $t : \mathbb{RP}^\infty \to S^0$ be any map that induces an isomorphism in the first stable homotopy group π^S_1. The so-called algebraic Kahn-Priddy homomorphism $t_* : \mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s+1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ is its "coboundary" one. This is an useful manner to attack the cohomology of the Steenrod algebra $\mathscr{A}$. The reason why the Kahn-Priddy map and particularly the infinite real projective space are taken into account is that this homomorphism is an epimorphism in positive stems and further it lowers the cohomology degree by relating $\mathrm{Ext}^{s+1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ to $\mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$.

It should be noted that the Singer map

$$\mathrm{Ext}^{i,j}_{\mathscr{A}}(H^*\mathbb{V}_s, \mathbb{F}_2) \to \mathrm{Ext}^{i+s,j+s}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2),$$

which becomes the Singer transfer for $i = 0$, is absolutely not the Kahn-Priddy map even for $s = 1$. The subtlety comes from the fact that the Kahn-Priddy map works with reduced cohomology, while the Singer map works with cohomology.

In this article, we follow all the notations of our preceding paper [4].

J. P. May proved in [8] that: As $\mathscr{A}$ is a cocommutative Hopf algebra, if M is a coalgebra in the category of $\mathscr{A}$-modules and N is an algebra in this category, then there exist Steenrod operations

$$Sq^i : \mathrm{Ext}^{s,t}_{\mathscr{A}}(M, N) \to \mathrm{Ext}^{s+i,2t}_{\mathscr{A}}(M, N).$$

In particular, for $i = 0$, $Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(M, N) \to \mathrm{Ext}^{s,2t}_{\mathscr{A}}(M, N)$.

By checking on bi-grading, $Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s,2t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ and $Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,2t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ can not commute with each other through the algebraic Kahn-Priddy homomorphism

$$t_* : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s+1,t+1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

Indeed, on the one hand t_*Sq^0 sends the bi-degree (s,t) to $(s+1, 2t+1)$, on the other hand Sq^0t_* sends the bi-degree (s,t) to $(s+1, 2(t+1))$.

Similarly, $Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s,2t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ and the Kameko one $Sq^0 : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-s} \to \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{2(t-s)+s+1}$ can not commute with each other through the Singer transfer $\mathrm{Tr}^{\mathbb{RP}^\infty}_s : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-s} \to \mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ because of bi-grading reason.

W. H. Lin defined in [7, p. 469] a map which is also denoted

$$Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s,2t+1}_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$$

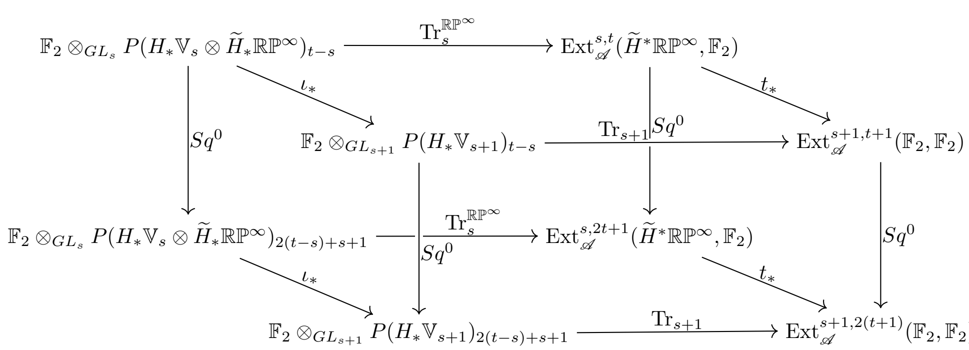
by ambiguity of notation. Remarkably, this Sq^0 commutes with the classical squaring operation $Sq^0 : \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,2t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic Kahn-Priddy morphism, and also commutes with the Kameko one through the Singer transfer $\mathrm{Tr}^{\mathbb{RP}^\infty}_s$ (see [4, Prop. 4.1] or Proposition 1.1 below).

The operation Sq^0 defined by Lin on $\mathrm{Ext}^*_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ should have been called Lin's Sq^0. Note that, Lin's Sq^0 is not May's Sq^0. In the article we only use Lin's Sq^0 on $\mathrm{Ext}^*_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$. So, this is simply called Sq^0 for abbreviation.

We start with a commutative diagram and the concept of critical elements given in [2] and [4].

In the following diagram, the horizontal arrows are the Singer transfers, the two vertical right arrows are the squaring operations and the two vertical left arrows are the Kameko squaring ones, while t_* denotes the algebraic Kahn-Priddy morphism, and ι_* is the homomorphism induced from the canonical inclusion.

Proposition 1.1. ([4, Prop. 4.1]) *The diagram*



is commutative.

We created the concept of critical elements (in [2], [3], [4]) in order to show that, in general, at most of homological degrees, the Singer transfers Tr_s and $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ are not an isomorphism in infinitely many internal degrees. However, Singer's conjecture, which predicts that the Singer transfers is injective, is still open.

In particular, we defined on [2, page 2] the notion of s-spike as follows: An s-spike number is an one that can be written as $(2^{n_1}-1)+\cdots+(2^{n_s}-1)$, but cannot be written as a sum of less than s terms of the form (2^n-1).

Definition 1.1. ([2, Def. 5.2]) A nonzero element $z \in \mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is called *critical* if

(a) $Sq^0(z)=0$,

(b) $2\mathrm{Stem}(z)+s$ is an s-spike.

By [2, Lemma 3.5], if $\mathrm{Stem}(z)$ is an s-spike, then so is $2\mathrm{Stem}(z)+s$.

Note that $Ph_1 \in \mathrm{Ext}^{5,14}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is not a critical element, since $2\mathrm{Stem}(Ph_1)+5=23$ is a 3-spike but not 5-spike. Actually, $23=(16-1)+(8-1)+(2-1)$, and it is easy to see that 23 cannot be written as a sum of less than 3 terms of the form (2^n-1).

However, $Ph_2 \in \mathrm{Ext}^{5,16}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is critical (see [2, Prop. 5.5]). Indeed,

(a) $Sq^0(Ph_2)=0$, as is well known $\mathrm{Ext}^{5,32}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)=0$ (see e.g. Tangora [11]),

(b) $27=2\mathrm{Stem}(Ph_2)+5$ is a 5-spike (see [2, Lemma 3.3]).

Definition 1.2. ([4, Def. 6.3]) A nonzero element $\widehat{z} \in \mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty,\mathbb{F}_2)$ is called *critical* if

(a) $Sq^0(\widehat{z})=0$,

(b) $2\mathrm{Stem}(\widehat{z})+(s+1)$ is an $(s+1)$-spike,

(c) $t_*(\widehat{z})\neq 0$, where $t_*:\mathrm{Ext}^s_{\mathscr{A}}(\widetilde{H}^*\mathbb{RP}^\infty,\mathbb{F}_2)\to \mathrm{Ext}^{s+1}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is the Kahn-Priddy homomorphism.

2. Results

The motivation for us to be interested in Proposition 2.3 is that the following theorem would probably give a negative answer to Singer's conjecture [10] on the transfer monomorphism. More precisely, we have

Theorem 2.1. *(i) If a critical element* $z \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ *is in the image of the transfer* $\mathrm{Tr}_s : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s)_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$*, then* Tr_s *is not a monomorphism.*

(ii) If a critical element $\widehat{z} \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ *is in the image of the transfer* $\mathrm{Tr}_s^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$*, then* $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ *is not a monomorphism.*

Proof. (i) See Case 2 of [2, Thm. 5.6] and [2, Thm. 5.9].

(ii) We are using the notation of Proposition 1.1.

We prove the following fact: If $\widehat{z} \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ is a critical element, which is in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$, then $z = t_*(\widehat{z}) \in \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$ is also a critical element, which is in the image of the transfer $\mathrm{Tr}_{s+1} : \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_*\mathbb{V}_{s+1})_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$. See Case 2 in Part (ii) of [4, Thm. 6.7].

Indeed, there is an element $\widehat{y} \in \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-s}$ such that $\mathrm{Tr}_s^{\mathbb{RP}^\infty}(\widehat{y}) = \widehat{z}$. Then $z = t_*(\widehat{z}) \in \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$ is nonzero (by (c) of Definition 1.2) and it is a critical element, which is in the image of the transfer $\mathrm{Tr}_{s+1} : \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_*\mathbb{V}_{s+1})_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$. (It should be noted that z satisfies Definition 1.1 with the number $s+1$ taken into account, instead of s.) Actually, we get

(a) $Sq^0(z) = Sq^0 t_*(\widehat{z}) = t_* Sq^0(\widehat{z}) = t_*(0) = 0$,

(b) $2\mathrm{Stem}(z) + (s+1) = 2\mathrm{Stem}(\widehat{z}) + (s+1)$ is an $(s+1)$-spike.

Denoting $y = \iota_*(\widehat{y})$, we have $\mathrm{Tr}_{s+1}(y) = \mathrm{Tr}_{s+1}\iota_*(\widehat{y}) = t_*\mathrm{Tr}_s^{\mathbb{RP}^\infty}(\widehat{y}) = t_*(\widehat{z}) = z$. That is, z is in the image of the transfer Tr_{s+1}. As z is nonzero and $z = \mathrm{Tr}_{s+1}(y)$, it implies that y is nonzero.

Note that, as $\mathrm{Tr}_{s+1}(y) = z$, by definition of the Singer transfer, it implies $\deg(y) = \mathrm{Stem}(z)$. From (b) of Definition 1.2, it concludes that $2\deg(y) + (s+1)$ is an $(s+1)$-spike. So, according to Kameko [5] (see also [2, Cor. 3.8]),

$$Sq^0 : \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_*(\mathbb{V}_{s+1})_{t-s} \to \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_*(\mathbb{V}_{s+1})_{2(t-s)+s+1}$$

is an isomorphism. In particular, from $y \neq 0$ it implies $Sq^0(y) \neq 0$. By the commutativity in Proposition 1.1, we have

$$\iota_* Sq^0(\widehat{y}) = Sq^0 \iota_*(\widehat{y}) = Sq^0(y) \neq 0.$$

Therefore $Sq^0(\widehat{y}) \neq 0$. Also, by the commutativity in Proposition 1.1,

$$\mathrm{Tr}_s^{\mathbb{RP}^\infty} Sq^0(\widehat{y}) = Sq^0 \mathrm{Tr}_s^{\mathbb{RP}^\infty}(\widehat{y}) = Sq^0(\widehat{z}) = 0,$$

(by (a) of Definition 1.2). That is, $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ sends a nonzero element to zero, so it is not a monomorphism.

The theorem is completely proved. ■

Theorem 2.2. *The existence of a positive stem critical element*

$$\widehat{z} \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2)$$

in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ *is equivalent to the existence of a positive stem critical element* $z \in \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$ *in the image of the transfer* Tr_{s+1}. *If the existences happen, then both* $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ *and* Tr_{s+1} *are not injective.*

Proof. From the beginning of the proof for Theorem 2.1, if there exists a critical element $\widehat{z} \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2)$, which is in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_s} P(H_* \mathbb{V}_s \otimes \widetilde{H}_* \mathbb{RP}^\infty)_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2)$, then, by Definitions 1.1 and 1.2, $z = \widehat{z}$ is a critical element, which is in the image of the transfer $\mathrm{Tr}_{s+1} : \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_* \mathbb{V}_{s+1})_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$.

Conversely, suppose there exists a positive stem critical element

$$z \in \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2),$$

which is in the image of the transfer

$$\mathrm{Tr}_{s+1} : \mathbb{F}_2 \otimes_{GL_{s+1}} P(H_* \mathbb{V}_{s+1})_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2).$$

From [4, Thm. 1.1], the algebraic Kahn-Priddy homomorphism

$$t_* : \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{F}_2, \mathbb{F}_2)$$

is an epimorphism from $\mathrm{Im}\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ onto $\mathrm{Im}\mathrm{Tr}_*$ in stem $t - s > 0$. So there exists $\widehat{z} \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2)$ with $t_*(\widehat{z}) = z$, which is in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_s} P(H_* \mathbb{V}_s \otimes \widetilde{H}_* \mathbb{RP}^\infty)_{t-s} \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(\widetilde{H}^* \mathbb{RP}^\infty, \mathbb{F}_2)$. By combination of Definitions 1.1 and 1.2, $\widehat{z}$ is also a critical element.

According to Theorem 2.1, if the existences happen, then both $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ and Tr_{s+1} are not injective. The theorem is proved. ■

The following shows that Theorem 2.1(ii) can not be applied to $\widehat{Ph_2}$. Therefore, Singer's conjecture is still open, as we have not known any critical element, which is in the image of the transfer.

Proposition 2.3. *The elements $\widehat{Ph_1}$ and $\widehat{Ph_2}$ are not in the image of the fourth transfer for $\mathbb{RP}^\infty$: $\mathrm{Tr}_4^{\mathbb{RP}^\infty}: \mathbb{F}_2\otimes_{GL_4} P(H_*\mathbb{V}_4\otimes\widetilde{H}_*\mathbb{RP}^\infty)_{t-4} \to \mathrm{Ext}_{\mathscr{A}}^{4,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$.*

Proof. According to Singer [10, Prop. 13.3],

$$(\mathbb{F}_2 \otimes_{\mathscr{A}} H^*(\mathbb{V}_5))_9^{GL_5} = 0.$$

By duality, Ph_1 is not in the image of the fifth transfer for S^0:

$$\mathrm{Tr}_5 : \mathbb{F}_2 \otimes_{GL_5} P(H_*\mathbb{V}_5)_9 \to \mathrm{Ext}_{\mathscr{A}}^{5,14}(\mathbb{F}_2, \mathbb{F}_2),$$

as the domain is zero in degree 9.

From Quỳnh [9, Prop. 1.3],

$$(\mathbb{F}_2 \otimes_{\mathscr{A}} H^*(\mathbb{V}_5))_{11}^{GL_5} = 0.$$

Passing to the duality, Ph_2 is not in the image of the fifth transfer for S^0:

$$\mathrm{Tr}_5 : \mathbb{F}_2 \otimes_{GL_5} P(H_*\mathbb{V}_5)_{11} \to \mathrm{Ext}_{\mathscr{A}}^{5,16}(\mathbb{F}_2, \mathbb{F}_2),$$

as the domain is zero in degree 11.

Lin [7] and Chen [1] constructed elements $\widehat{Ph_1}$ and $\widehat{Ph_2}$ in $\mathrm{Ext}_{\mathscr{A}}^{4,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ for respectively $t = 14$ and $t = 16$, whose behaviors are given by the algebraic Kahn-Priddy homomorphism

$$t_*(\widehat{Ph_i}) = Ph_i, \quad (i = 1, 2).$$

Now we show that $\widehat{Ph_i}$ are not in the image of the fourth transfer for $\mathbb{RP}^\infty$

$$\mathrm{Tr}_4^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_4} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-4} \to \mathrm{Ext}_{\mathscr{A}}^{4,t}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2),$$

for respectively $t = 13$ and $t = 15$. Suppose the contrary that $\widehat{Ph_i}$ is in the image of the transfer. That is, there exists $\widehat{z_i} \in P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{t-4}$ such that $\mathrm{Tr}_4^{\mathbb{RP}^\infty}(\widehat{z_i}) = \widehat{Ph_i}$ Since the commutativity of the diagram in [4, Lemma 4.6], we have

$$\begin{aligned} \mathrm{Tr}_5\iota_*(\widehat{z_i}) &= t_*\mathrm{Tr}_4^{\mathbb{RP}^\infty}(\widehat{z_i}) \\ &= t_*(\widehat{Ph_i}) = Ph_i. \end{aligned}$$

So, it concludes that Ph_i is in the image of the fifth transfer for S^0. This contradicts to the result by Singer for $i = 1$ or by Quỳnh for $i = 2$. This contradiction rejects the contrary hypothesis. The proposition is proved. ■

Remark 2.4. In [4, Prop. 6.5], we actually prove that the set $\{h_{n_1} \cdots h_{n_k} \widehat{Ph_2}\}$ contains infinitely many critical elements, where h_n denotes the well-known Adams element. However, so far we have not known whether there is any element of the form $h_{n_1} \cdots h_{n_k} \widehat{Ph_2}$, which is in the image of the transfer $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$. Therefore, Singer's conjecture on the injectivity of $\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ and Tr_{s+1} is still open.

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