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Convergence in capacity for Cegrell classes on complex varieties in domains of $\mathbb{C}^n$

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Abstract. Let V be a complex variety embedded in a bounded domain $D \subset \mathbb{C}^n$. Assuming that the singularities of V are not too severe−for example, if V is locally irreducible−we prove a convergence in capacity for the complex Monge-Ampère operator acting on plurisubharmonic functions in the Cegrell class on V.

1. Introduction

Let V be a connected complex variety of pure dimension k contained in a bounded domain $D \subset \mathbb{C}^n$, where $n \geq 2$ and $1 \leq k \leq n$. We denote by V_{reg} the set of regular points of V; that is, V_{reg} is the largest (possibly disconnected) complex manifold of dimension k contained in V. The singular locus of V is then defined by

$$V_{\text{sig}} := V \setminus V_{\text{reg}}.$$

We are also interested in the local irreducibility of V. More precisely, V is said to be *locally irreducible* at a point $a \in V$ if there exists a fundamental system of open neighborhoods $\{U_j\}$ of a such that each intersection $U_j \cap V$ is irreducible as a complex analytic set in U_j (see [8], p. 55).

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Let $V_{\rm red}$ denote the set of points in V where V is *not* locally irreducible; that is, $a \in V_{\rm red}$ if there exists an open neighborhood $U \ni a$ such that $U \cap V$ is reducible in U. It is then evident that $V_{\rm red}$ is a local (though not necessarily closed) complex subvariety of V.

For instance, consider the Whitney umbrella variety

$$V^1 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 - z_2^2 z_3 = 0\}.$$

Let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^3$; then the reducible locus of $V^1 \cap \mathbb{B}$ is given by

$$(V^1 \cap \mathbb{B})_{\rm red} = \{(0, 0, z_3) : 0 < |z_3| < 1\},$$

while $V^1 \cap \mathbb{B}$ is nonetheless irreducible at the origin.

In this work, we are particularly concerned with the case where V is locally irreducible at every point$-$that is, $V_{\rm red} = \emptyset$. This assumption is essential in our study, as it enables the construction of plurisubharmonic functions via the upper semicontinuous regularization of families of plurisubharmonic functions that are locally bounded from above on V.

Recall that $u : V \to [-\infty, \infty), u \not\equiv -\infty$ on any irreducible component of V, is said to be plurisubharmonic (resp. strictly plurisubharmonic, $\mathcal{C}^\infty$ smooth strictly plurisubharmonic) if u is locally the restriction (on V) of plurisubharmonic (resp. strictly plurisubharmonic, $\mathcal{C}^\infty$ smooth strictly plurisubharmonic) functions on open subsets of D. A fundamental result of Fornaess and Narasimhan (cf. Theorem 5.3.1 in [13]) asserts that an upper semicontinuous function $u : V \to \mathbb{R} \cup [-\infty, \infty)$ which is not identically $-\infty$ on any irreducible component of V, is plurisubharmonic if and only if for every holomorphic map $\theta : \Delta \to V$, where Δ is the unit disk in $\mathbb{C}$, we have $u \circ \theta$ is subharmonic on Δ. This powerful result implies immediately the nontrivial facts that plurisubharmonicity is preserved under local uniform convergence. We write $\mathrm{PSH}(V)$ for the set of plurisubharmonic functions on V and $\mathrm{PSH}^-(V)$ denotes the subset of negative plurisubharmonic functions on V. The complex variety V is said to be *hyperconvex* if there exists $\rho \in \mathrm{PSH}^-(V) \cap L^\infty(V)$ such that $\{z \in V : \rho(z) < c\}$ is relatively compact in V for all $c < 0$. According to Theorem 1 in [16] we can always find a *continuous* negative plurisubharmonic exhaustion function on a complex hyperconvex variety. In the case of domains in $\mathbb{C}^n$, such a result was established earlier in [2]. Since $-1/\rho \in \mathrm{PSH}(V)$ for every $\rho \in \mathrm{PSH}^-(V)$, we infer that V is Stein since it admits a strictly plurisubharmonic exhaustion function, namely $-1/\rho(z) + |z|^2$.

Next, we turn to the complex Monge-Ampère operator for locally bounded plurisubharmonic functions on V. First, we note that by Proposition 1.8 in [9], each $\psi \in \mathrm{PSH}(V)$ (not necessarily locally bounded) is locally integrable on V with respect to the Lebesgue measure. Thus $dd^c\psi$ is a closed positive $(1, 1)$

current on V_{reg}. Here we use the standard notation $d := \partial + \bar{\partial}, d^c := i(\bar{\partial} - \partial)$. According to Bedford in [1] (see also [9]), the complex Monge-Ampère operator

$$(dd^c)^k : \text{PSH}(V) \cap L^{\infty}_{\text{loc}}(V) \to M_{k,k}(V),$$

where $M_{k,k}(V)$ denotes Radon measures on V, may be defined in the usual way on the regular locus V_{reg} of V (cf. [3], [4]), and it extends trivially "by zero" through the singular locus V_{sig}, i.e., for Borel sets $E \subset V$

$$\int\limits_E (dd^c u)^k := \int\limits_{E \cap V_{\text{reg}}} (dd^c u)^k, \ \forall u \in \text{PSH}(V) \cap L^{\infty}_{\text{loc}}(V). \tag{1.1}$$

Following the approach given in [5] and [6], in [10] we investigate the largest possible class $\mathcal{E}(V)$ of $\text{PSH}^-(V)$ on which the complex Monge-Ampère operator can be reasonably defined. There are at least two technical difficulties in the process of generalizing Cegrell's machinery from the case of domains in $\mathbb{C}^n$ to the case of complex varieties V. Firstly, semi-local smoothing of plurisubharmonic functions by taking convolutions with approximation of identities does not work on V, and secondly the upper-regularization of a family of plurisubharmonic functions on V may fail to be plurisubharmonic if V is not locally irreducible.

The aim of this note is to establish a convergence-in-capacity property for the Monge-Ampère operator on suitably chosen complex varieties V. In the setting of hyperconvex domains in $\mathbb{C}^n$, such a result was first proven in [17] for locally bounded plurisubharmonic functions, and later extended in full generality in [7] (see also [14] for a simpler proof). However, due to the technical challenges outlined earlier, we are compelled to substantially modify the existing methods in order to achieve the desired approximation result.

2. Preliminaries

We begin by recalling several fundamental notions from pluripotential theory on complex varieties which will be of central relevance to our subsequent analysis. For more comprehensive accounts, the reader is referred to the standard references [1] and [9].

Throughout this paper, the symbol V will denote a complex variety of pure dimension $1 \leq k \leq n$ contained in some bounded domain $D \subset \mathbb{C}^n$.

A subset $X \subset V$ is said to be *(locally) pluripolar* if, for every point $a \in V$, there exists a neighborhood U of a and a function $v \in \text{PSH}(U)$ such that

$v|_{X\cap U} = -\infty$. For instance, the singular locus $V_{\rm sig}$, being a proper closed complex subvariety of V, is (locally) pluripolar in V.

According to Theorem 5.3 in [1], every pluripolar subset X of a Stein variety V admits a plurisubharmonic function $u \in \mathrm{PSH}(V)$ satisfying $u|_X \equiv -\infty$. This result generalizes a classical theorem of Josefson, which treats the special case where V is a domain in $\mathbb{C}^n$.

To determine the pluripolarity of subsets in V, we follow the approach of [1] (see also [4] for the case of open subsets in $\mathbb{C}^n$), and define the capacity of a Borel subset E of an open set $\Omega \subset V$ by

$$C(E,\Omega) := \sup\left\{\int_E (dd^c u)^k : u \in \mathrm{PSH}(\Omega),\ -1 \le u < 0\right\}. \tag{2.1}$$

In the special case where $\Omega = V$, we shall simply write $C(E)$ in place of $C(E,V)$.

From definition (2.1), it follows immediately that the singular locus $V_{\rm sig}$ has zero capacity; that is, $C(V_{\rm sig}\cap\Omega,\Omega) = 0$ for every open set $\Omega \subset V$.

The following result, due to Bedford (Lemma 3.1 in [1]), strengthens this observation by asserting that $V_{\rm sig}$ has *outer* capacity zero.

Lemma 2.1. *Let $\Omega \subset V$ be open. Then for every $\varepsilon > 0$, there exists an open neighborhood U of $V_{\rm sig}$ in Ω such that $C(U,\Omega) < \varepsilon$.*

Lemma 2.1 ensures that for any function $u \in \mathrm{PSH}(V)\cap L^\infty_{\rm loc}(V)$, the Monge-Ampère operator $(dd^c u)^k$ defines a Radon measure on V that assigns zero mass to pluripolar subsets of V. In particular, $C(E) = 0$ for every such pluripolar set E. Moreover, as observed in [1], all local properties of the Monge-Ampère operator established in [4] for locally bounded plurisubharmonic functions extend naturally to V. For example, the operator $(dd^c)^k$ is continuous under monotone convergence of locally uniformly bounded sequences in $\mathrm{PSH}(V)$ (i.e., the Bedford–Taylor monotone convergence theorem holds on V), and every function $u \in \mathrm{PSH}(V)$ is quasi-continuous, meaning that it is continuous outside open sets of arbitrarily small capacity. Additionally, the Chern–Levine–Nirenberg inequality, which provides upper estimates for the Monge–Ampère mass of $(dd^c u)^k$ in terms of the sup-norm of u, remains valid for locally bounded plurisubharmonic functions on V.

We next recall a notion of convergence that plays a central role in our analysis.

Definition 2.1. *A sequence of measurable functions $\{f_j\}$ on V is said to converge* locally in capacity *to a measurable function f if, for every relatively compact Borel subset $E \subset V$ and every $\varepsilon > 0$, we have*

$$\lim_{j\to\infty} C\left(\{z \in E : |f_j(z) - f(z)| > \varepsilon\}\right) = 0.$$

This mode of convergence is referred to as *locally quasi-uniform* in [1]. In the case where V is a domain in $\mathbb{C}^n$, it has been studied in detail in [17]. Using the quasi-continuity of plurisubharmonic functions on V and an application of Dini's lemma, one readily sees that monotone convergence of plurisubharmonic functions implies convergence in capacity.

Definition 2.2. *The hyperconvex variety V is said to be mildly singular if it satisfies the following conditions:*

(a) V is contained in a connected complex variety V' of pure dimension k in some domain $D', D \subset D'$ such that $\overline{V_{red}} \cap \partial V \subset V'$;

(b) There exists a negative (not necessarily continuous) plurisubharmonic exhaustion function ρ on V such that ρ extends to a bounded plurisubharmonic function on V'.

Moreover, V is called very mildly singular if V_{red} is empty or relatively compact in V.

Remark 2.1. (a) The following implications are straightforward: locally irreducible $\Rightarrow$ very mildly singular $\Rightarrow$ mildly singular.

(b) On the other hand, it is also clear that $V^1 \cap D$ is mildly singular but not very mildly singular, where $V^1 := \{(z_1, z_2, z_3) \in {}^3 : z_1^2 - z_2^2 z_3 = 0\}$ and

$$D := \{(z_1, z_2, z_3) : |z_1|^2 + |z_2|^2 < \text{Re } z_3, |z_3| < 1\}.$$

A main use of this notion is the following technical result yielding a nontrivial element of $\mathcal{E}_0(V)$ for a mildly singular V (see Proposition 3.3 in [10]).

Proposition 2.1. *Let V be a mildly singular hyperconvex variety then for each open relatively compact subset U of V we can find $\rho' \in \mathcal{E}_0(V)$ such that $dd^c \rho' = \omega$ on $U \cap V_{\text{reg}}$.*

Remark 2.2. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^2$ and $\{a_j\}$ be a discrete sequence in Δ. Set $V^2 := \bigcup\limits_{j \geq 1} \{(z_1, z_2) \in \mathbb{B} : z_1(z_2 - a_j) = 0\}$. Since there are infinitely many branches of V^2 clustered at some boundary point of ∂V^2 we infer that V^2 is not mildly singular. Nevertheless, for a suitably chosen $\{a_j\}$ the variety V^2 given above admits an element in $\mathcal{E}_0(V_2)$. For details, we refer the reader to Remark 3.4 in [10].

The lemma below is essentially proved in [6]. It illustrates the fact that $\mathcal{E}_0$ may play the role of test functions in pluripotential theory.

Lemma 2.2. *Let $\varphi \in \mathcal{C}_0^\infty(D)$ and U be a relatively compact open subset of V that contains the support K of $\varphi|_V$. Then for every $\lambda > 0$ there exist $u_1, u_2 \in \mathcal{E}_0(V) \cap \mathcal{C}(V)$ such that:*

(a) $\varphi = u_1 - u_2$ on V;

(b) $dd^c u_1 \geq \lambda\omega, dd^c u_2 \geq \lambda\omega$ on $U \cap V_{\text{reg}}$.

Now, for mildly singular variety we may define the following basic Cegrell class.

Definition 2.3. *Let V be a mildly singular complex variety. We define*

$$\mathcal{E}_0(V) := \Big\{u \in \mathrm{PSH}^-(V) \cap L^\infty(V) : \lim_{z\to\partial V} u(z) = 0, \int\limits_V (dd^c u)^k < \infty\Big\}.$$

Following [6] we are now able to define a class of negative plurisubharmonic functions on V on which the complex Monge-Ampère operator may be well defined.

Definition 2.4. *We say that $u \in \mathcal{E}(V)$ if for every $z_0 \in V$ there exists a relatively compact open neighborhood U of z_0 in V, a decreasing sequence $\{h_j\} \subset \mathcal{E}_0(V)$ such that $h_j \downarrow u$ on U and*

$$\sup_j \int\limits_V (dd^c h_j)^k < \infty.$$

Further, by an abuse of language, we say that $u \in \mathcal{F}(V)$ if the above open set U can be chosen to equal to V.

A major tool is the following integration by part formula (see Proposition 2.10 in [10]),

Lemma 2.3. *Let $u, v, v_1, ..., v_{k-1} \in L^\infty_{\mathrm{loc}}(V) \cap PSH(V)$ be such that $\lim\limits_{z\to\partial V} u(z) = \lim\limits_{z\to\partial V} v(z) = 0$ and that*

$$\int_V dd^c u \wedge T < \infty, \int_V dd^c v \wedge T < \infty,$$

where $T := dd^c v_1 \wedge \cdots \wedge dd^c v_{k-1}$. Then we have:

$$\int\limits_V (-u) dd^c v \wedge T = \int_V du \wedge d^c v \wedge T < \infty.$$

The following estimate of Monge-Ampère currents is taken from Lemma 3.9 in [10]. In the case of domains in $\mathbb{C}^n$, this fact was proved in [15].

Lemma 2.4. *Let V be a mildly singular hyperconvex variety, $u_1, ..., u_k \in \mathcal{E}_0(V)$ and E be a Borel relatively compact subset of V. Set*

$$T := dd^c u_1 \wedge \cdots \wedge dd^c u_k.$$

Then we have the following estimate

$$\int\limits_E T \leq \|u_1\|_V \Big(\int\limits_V (dd^c u_2)^k\Big)^{\frac{1}{k}} \cdots \Big(\int\limits_V (dd^c u_k)^k\Big)^{\frac{1}{k}} C(E)^{\frac{1}{k}}.$$

3. Convergence in capacity

We start with the following approximation result which is perhaps of some interest in its own.

Proposition 3.1. *Suppose that V is a hyperconvex variety (in some bounded domain in $\mathbb{C}^n$). Then the following assertions hold true:*

(a) There exists a negative $\mathcal{C}^\infty-$smooth strictly plurisubharmonic exhaustion function for V.

(b) For $u \in \mathrm{PSH}^-(V)$, there exists a sequence $\{v_j\}$ of bounded $\mathcal{C}^\infty-$smooth strictly plurisubharmonic function on V such that $v_j \downarrow u$ on V.

Proof. (a) By the assumption, there exists $\rho \in PSH^-(V) \cap \mathcal{C}(\overline{V})$ such that $\rho|_{\partial V} = 0$. Choose $M > 0$ so large such that $\varphi(z) := |z|^2 - M < 0$ on V. Set $\tilde{\rho} := -2\sqrt{-\varphi\rho}$. Fix $z_0 \in V$. then there exists a small neighborhood U of z_0 in D such that ρ extends to a plurisubharmonic function (still denoted by ρ) on U. By a direct computation as in [2], p. 728, we can check that $\tilde{\rho}$ is *strictly* plurisubharmonic on U. It follows that $\tilde{\rho}|_V$ is a negative continuous plurisubharmonic exhaustion function for V. Now we let $f(z) := \min\{-\rho(z), \mathrm{dist}\,(z, \partial V)\}$. Then f is a positive continuous function on V that tends to 0 at ∂V. Since V is Stein, we may apply a classical approximation theorem of Richberg to get a $\mathcal{C}^\infty-$smooth strictly plurisubharmonic function $\hat{\rho}$ on V such that $\tilde{\rho} < \hat{\rho} < \tilde{\rho} + f$ on V. In particular, $\hat{\rho} \in PSH^-(V)$ is an exhaustion function for V.

(b) By Theorem A in [10], there exists a sequence $u_j \in \mathrm{PSH}^-(V) \cap \mathcal{C}(\overline{V})$ such that $u_j|_{\partial V} = 0$ and $u_j \downarrow u$ on V. After rescaling we may assume that $\hat{\rho} > -1$ on V. Consider $\hat{u}_j := u_j + (\hat{\rho}+1)/j$. Then $\hat{u}_j \downarrow u$ on V. Moreover, $\hat{u}_j$ is bounded, continuous and *strictly* plurisubharmonic on V. Since $\hat{u}_j > \hat{u}_{j+1}$, using Richberg's approximation theorem again, we can find a smooth $\mathcal{C}^\infty-$strictly plurisubharmonic function v_j on V such that $\hat{u}_j > v_j > \hat{u}_{j+1}$. It follows easily that $v_j \downarrow u$ on V.

We now turn to the main result of this note concerning the convergence in capacity of the Monge-Ampère operator. Analogous results in the setting of plurisubharmonic functions on domains in $\mathbb{C}^n$ were previously established in [17] and subsequently refined in [7]. ∎

Theorem 3.2. *Let V be a mildly singular hyperconvex variety and $u, v \in \mathcal{E}(V)$. Let $\mathcal{E}(V) \ni \{u_j\} \to u$ in capacity on V as $j \to \infty$. Assume that $u_j \geq v$ for each j. Then $(dd^c u_j)^k$ converges weakly to $(dd^c u)^k$ on V.*

Our proof follows the elegant strategy developed in [14]. We begin with a set of auxiliary estimates that may also be of independent interest.

Lemma 3.1. *Let $u, v, h \in \mathcal{F}(V)$ and $s > 0$. Then the following assertions hold true:*

(a) If $u \geq v$ then

$$\int_{\{v \leq -s\}} -v(dd^c u)^{k-1} \wedge dd^c h \leq 2^{k-1} \int_{\{v \leq -\frac{s}{2^{k-1}}\}} -v(dd^c v)^{k-1} \wedge dd^c h.$$

(b) If h is bounded from below then

$$\lim_{s \to -\infty} \int_{\{v \leq -s\}} -v(dd^c u)^{k-1} \wedge dd^c h = 0$$

uniformly in $u \in \mathcal{F}(V)$ with $u \geq v$.

Proof. (a) We use an integration by parts technique devised in [14]. More precisely, for $s > 0$ we set $v_s := \max\{v, -s\}$. Then $v_s \in \mathcal{F}(V)$, so using integration by parts (Lemma 2.3) we obtain

$$\begin{aligned}
\int_{\{v \leq -s\}} -v(dd^c u)^{k-1} \wedge dd^c h &\leq 2 \int_V (v_{(s/2)} - v)(dd^c u)^{k-1} \wedge dd^c h \\
&\leq 2 \int_V v(dd^c v_{(s/2)} - dd^c v) \wedge (dd^c u)^{k-2} \wedge dd^c h \\
&= 2 \int_{\{v \leq -s/2\}} u(dd^c v_{(s/2)} - dd^c v) \wedge (dd^c u)^{k-2} \wedge dd^c h \\
&\leq 2 \int_{\{v \leq -s/2\}} -u dd^c v \wedge (dd^c u)^{k-2} \wedge dd^c h \\
&\leq 2 \int_{\{u \leq -s/2\}} -v dd^c v \wedge (dd^c u)^{k-2} \wedge dd^c h.
\end{aligned}$$

Here the first inequality follows from the simple fact that $-v \leq 2(v_{(s/2)} - v)$ on $\{v \leq -s\}$ and the third equality is a consequence of the identity

$$dd^c v_{(s/2)} \wedge (dd^c u)^{k-2} \wedge dd^c h = dd^c v \wedge (dd^c u)^{k-2} \wedge dd^c h$$

on the Borel set $\{v > -s/2\}$, which may be proved by approximating v from above by continuous elements in $\mathcal{E}_0(V)$ as in the proof of Theorem 4.1 (a) in [15]. Continuing in this fashion $(k-2)$ times, we finally reach the desired estimate.

(b) In view of (a), it suffices to check that

$$\lim_{s\to\infty}\int\limits_{\{v\leq -s\}} -v(dd^cv)^{k-1}\wedge dd^ch = 0. \tag{3.1}$$

For this, we note the following estimate that was already contained in the proof of (a)

$$\int\limits_{\{v\leq -s\}} -v(dd^cv)^{k-1}\wedge dd^ch \leq 2\int_V (v_{(s/2)}-v)(dd^cv)^{k-1}\wedge dd^ch.$$

Since $-v_s\uparrow -v$ on V and since

$$\int\limits_V -v(dd^cv)^{k-1}\wedge dd^ch = \int\limits_V -h(dd^cv)^k \leq \|h\|_V\int\limits_V (dd^cv)^k < \infty,$$

by Lebesgue monotone convergence theorem we obtain (3.1). The proof is thereby completed. ■

Lemma 3.2. *Let $u,u',v\in\mathcal{F}(V)$ with $u\geq v, u'\geq v$. Let φ be a smooth function on V such that $\varphi=0$ outside a compact subset K of V. Then there exist a positive constant A_k depend only on k and $h\in\mathcal{E}_0(V)\cap\mathcal{C}(V)$ which is determined by φ such that for each Borel subset E of K and $s>0$ we have*

$$\begin{aligned}|\int_V\varphi(dd^cu)^k-\int_V\varphi(dd^cu')^k| \leq A_k\Big[\|u-u'\|_E[\int\limits_V(dd^ch)^k+\int_V(dd^cv)^k]+\\
+\int\limits_{\{v\leq -s\}}(-v)(dd^cv)^{k-1}\wedge dd^ch\\
+s\|h\|_V C(K\setminus E)^{\frac{1}{k}}\Big(\int_V(dd^cv)^k\Big)^{\frac{k-1}{k}}\Big].\end{aligned}$$

Proof. First, by Lemma 2.2, we may write $\varphi=h_1-h_2$ with $h_1,h_2\in\mathcal{E}_0(V)$. Set

$$T:=\sum_{j=0}^{k-1}(dd^cu)^j\wedge(dd^cu')^{k-j-1}.$$

Then, using integration by parts formula (Lemma 2.3) we obtain

$$\begin{aligned}|\int_V\varphi(dd^cu)^k-\int_V\varphi(dd^cu')^k| &= |\int_V(u-u')dd^c\varphi\wedge T|\\
&\leq |\int_K(u-u')dd^ch_1\wedge T|+|\int_K(u-u')dd^ch_2\wedge T|.\end{aligned}$$

Now we have

$$(3.2)\quad \Big|\int_K (u-u')dd^ch_1\wedge T\Big| \le \|u-u'\|_E\int_V dd^ch_1\wedge T+\int_{K\setminus E}(-v)dd^ch_1\wedge T.$$

From now on, by $A_{k,1}, A_{k,2},\cdots$ we mean positive constants that depends only on k. To estimate the first term on the right hand side of (3.2), we note that by the comparison principle

$$\max\{\int_V (dd^cu)^k, \int_V (dd^cu')^k\} \le \int_V (dd^cv)^k.$$

Hence using Lemma 4.3 (b) in [10] we obtain

$$(3.3)\quad \int_V dd^ch_1\wedge T\le A_{k,1}\big[\int_V (dd^ch_1)^k+\int_V (dd^cv)^k\big].$$

For the second one, we first set $s':=2^{k-1}s$. Then one obtains

$$(3.4)\quad \int_{K\setminus E}(-v)dd^ch_1\wedge T\le \int_{\{v\le -s'\}}(-v)dd^ch_1\wedge T+s'\int_{K\setminus E}dd^ch_1\wedge T.$$

Set $u'':=\frac{u+u'}{2}\ge v$. Then using Lemma 3.1 we obtain

$$(3.5)\quad \begin{aligned}\int_{\{v\le -s'\}}(-v)dd^ch_1\wedge T&\le A_{k,2}\int_{\{v\le -s'\}}(-v)dd^ch_1\wedge (dd^cu'')^{k-1}\\ &\le A_{k,3}\int_{\{v\le -s\}}(-v)(dd^cv)^{k-1}\wedge dd^ch_1.\end{aligned}$$

Set

$$A_{k,4}:=\max\{A_{k,1},A_{k,3},2^{k-1}\}.$$

Plugging (3.3), (3.4) and (3.5) together into (3.2) we obtain

$$\begin{aligned}\Big|\int_K (u-u')dd^ch_1\wedge T\Big| &\le A_{k,4}\Big[\|u-u'\|_E[\int_V (dd^ch_1)^k+\int_V (dd^cv)^k]\\ &\quad+\int_{\{v\le -s\}}(-v)(dd^cv)^{k-1}\wedge dd^ch_1+s\int_{K\setminus E}dd^ch_1\wedge T\Big].\end{aligned}$$

By the same reasoning we get

$$\begin{aligned}\Big|\int_K (u-u')dd^ch_2\wedge T\Big| &\le A_{k,5}\Big[\|u-u'\|_E[\int_V (dd^ch_2)^k+\int_V (dd^cv)^k]\\ &\quad+\int_{\{v\le -s\}}(-v)(dd^cv)^{k-1}\wedge dd^ch_2+s\int_{K\setminus E}dd^ch_2\wedge T\Big].\end{aligned}$$

Set $h := h_1 + h_2 \in \mathcal{E}_0(V)$. We obtain

$$|\int_V \varphi(dd^c u)^k - \int_V \varphi(dd^c u')^k| \le A_{k,6}\Big[\|u-u'\|_E[\int\limits_V (dd^c h)^k + \int\limits_V (dd^c v)^k] + \int\limits_{\{v\le -s\}} (-v)(dd^c v)^{k-1}\wedge dd^c h + s\int\limits_{K\setminus E} dd^c h\wedge T\Big].$$

Finally, by Lemma 2.4 we have

$$\int\limits_{K\setminus E} dd^c h\wedge T \le A_{k,7}\|h\|_V C(K\setminus E)^{\frac{1}{k}}\Big(\int\limits_V (dd^c v)^k\Big)^{\frac{k-1}{k}}.$$

Combining the last two estimates we finish the proof of the lemma. ■

Proof. (Proof of Theorem 3.2) Since the problem is local, we may assume that u_j, u and v are all in the class $\mathcal{F}(V)$. Fix a smooth function φ on D with compact support. We must show

$$\lim_{j\to\infty}\int\limits_V \varphi(dd^c u_j)^k = \int\limits_V \varphi(dd^c u)^k.$$

According to Lemma 3.2, there exist $A_k > 0$ depend only on k and $h \in \mathcal{E}_0(V)\cap \mathcal{C}(V)$ which is determined by φ such that for each Borel subset E of $K := \operatorname{supp}\varphi$ and $s > 0$ we have

$$|\int_V \varphi(dd^c u_j)^k - \int_V \varphi(dd^c u)^k| \le A_k\Big[\|u_j-u\|_E[\int\limits_V (dd^c h)^k + \int_V (dd^c v)^k] + \int\limits_{\{v\le -s\}} (-v)(dd^c v)^{k-1}\wedge dd^c h + s\|h\|_V C(K\setminus E)^{\frac{1}{k}}\Big(\int_V (dd^c v)^k\Big)^{\frac{k-1}{k}}\Big].$$

We reach the desired conclusion by using the assumption that $u_j \to u$ in capacity on V and Lemma 3.1 (b). ■

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