A new hybrid cutting-projection algorithm for equilibrium and fixed point problems

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Dedicated to Professor Do Van Luu on the occasion of his 80th birthday

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Abstract. The paper proposes a novel hybrid cutting-projection method for solving equilibrium problems and fixed point problems. By constructing specially cutting-halfspaces, the method only requires to solve a strongly convex optimization program at each iteration without the extra-steps as extragradient methods. The strongly convergence theorem is established and some numerical examples are presented to illustrate its convergence.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) for the bifunction f on C is stated as follows:

(EP) $\qquad$ Find $\quad x^* \in C \quad$ such that $\quad f(x^*, y) \geq 0, \ \forall y \in C.$

The solution set of EP for f on C is denoted by $EP(f, C)$. EP is also well-known as Ky Fan inequality [9]. The EP is very general in the sense that it

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includes many mathematical models as variational inequalities, optimization problems, fixed point problems, Nash equilibrium point problems, complementarity problems, operator equations, see, for instance [3, 6, 7, 19, 20, 21]. A special case for the bifunction $f(x,y) = \langle A(x), y-x\rangle$, where $A : C \to H$ is an operator, then EP becomes a variational inequality problem (VIP):

$$\text{(VIP)} \qquad \text{Find} \quad x^* \in C \quad \text{such that} \quad \langle A(x^*), y-x^*\rangle \geq 0, \ \forall y \in C.$$

One of the most popular methods for solving EPs is the proximal point method (PPM), in which solution approximations are computed via the resolvent of bifunction [6]. The PPM was first introduced by Martinet [18] for variational inequalities, and then it was extended for finding zero points of maximal monotone operators by Rockafellar [26]. This method was further extended to Ky Fan inequalities by Konnov [15] for monotone or weakly monotone bifunctions.

In recent years, the extragradient method [16] has been widely studied and extended to EPs. Quoc et al. [24] introduced the following extragradient algorithm for EPs in Euclidean space,

$$\begin{cases} y_n = \arg\min_{y\in C} \left\{\lambda f(x_n,y) + \frac{1}{2}\|x_n - y\|^2\right\}, \\ x_{n+1} = \arg\min_{y\in C} \left\{\lambda f(y_n,y) + \frac{1}{2}\|x_n - y\|^2\right\}. \end{cases} \tag{1.1}$$

Then, this algorithm was further extended to Hilbert spaces; see, for example, [2, 8, 23, 29, 30]. Here, it is interested in finding a common solution of an equilibrium problem and a fixed point problem. Let $S : C \to C$ be a mapping and $F(S)$ denotes the set of fixed points of S. For finding a common element of the solution set $EP(f,C)$ and the fixed point set $F(S)$ of a nonexpansive mapping S, Anh [2] introduced the following hybrid algorithm in Hilbert spaces

$$\begin{cases} y_n = \arg\min_{y\in C} \left\{\lambda f(x_n,y) + \frac{1}{2}\|x_n - y\|^2\right\}, \\ z_n = \arg\min_{y\in C} \left\{\lambda f(y_n,y) + \frac{1}{2}\|x_n - y\|^2\right\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)S(w_n), \\ x_{n+1} = P_{\Omega_n}(x_0), \end{cases} \tag{1.2}$$

where $\Omega_n = C_n \cap Q_n$ and $C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}$ and $Q_n = \{z \in C : \langle x_0 - x_n, z - x_n\rangle \leq 0\}$. The author proved the sequence $\{x_n\}$ generated by method (1.2) converges strongly to $P_{EP(f,C)\cap F(S)}(x_0)$ under the hypotheses of the pseudomonotonicity and Lipschitz-type condition of bifunction f. Observe that in the algorithms (1.1) and (1.2), two optimization programs onto the feasible set C need to be solved at each iteration. This seems to be costly and and can affect to the efficiency of used methods if the structure of the bifunction f and the constrained set C are complex or in huge-scale problems.

In this paper, motivated and inspired by the results in [13, 14, 17, 27], we introduce a new projection method, which is simple, elegant and has low complexity, for finding a common solution of a monotone EP and a fixed point problem of a nonexpansive mapping $S : C \to C$. The proposed method is described as follows:

Algorithm 1.1. *(The hybrid algorithm without the extra-steps).*

$$\begin{cases} y_{n+1} = \underset{y \in C}{\arg\min}\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ z_{n+1} = \alpha_n y_{n+1} + (1 - \alpha_n) S y_{n+1}, \\ x_{n+1} = P_{\Omega_n}(x_0), \end{cases}$$

where $\Omega_n = C_n \cap Q_n$ and C_n, Q_n are two specially constructed half-spaces (see Algorithm 3.1 in Section 3 below). In this algorithm we do not use the PPM and the resolvent of equilibrium bifunction [6, 15]. Contrary to extragradient methods [2, 8, 23, 24, 29, 30], in our proposed algorithm, only one optimization program needs to be solved at each iterative step without any extra-step. Moreover, note that in the process (1.2), the final projection $x_{n+1} = P_{\Omega_n}(x_0)$, where $\Omega_n = C_n \cap Q_n$ still depends on the constrained set C due to the definition of C_n and Q_n, while Ω_n in Algorithm 1.1 is only the intersection of two halfspaces, so the projection x_{n+1} can be expressed by an explicit formula [6, 28]. So, the proposed algorithm has lower complexity, and is simpler and more elegant.

The remainders of the paper is organized as follows: Section 2 reviews several definitions and results for further use. Section 3 deals with analyzing the convergence of the proposed algorithm and presenting some applications to other problems. Finally, in Section 4 we give two numerical examples to illustrate the convergence of the new algorithm.

2. Preliminaries

In this section, we recall some definitions and preliminary results used in this paper. A mapping $S : C \to H$ is said to be nonexpansive if $\|S(x)-S(y)\| \leq \|x - y\|, \forall x, y \in C$. The set of fixed points of S is denoted by $F(S)$. We have the following properties of a nonexpansive mapping, see [11] for more details.

Lemma 2.1. *Assume that $S : C \to H$ is a nonexpansive mapping. If S has a fixed point, then*

i. $F(S)$ is closed convex subset of C;

ii. $I - S$ *is demiclosed, i.e., whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I - S)x_n\}$ *strongly converges to some* y *, it follows that* $(I - S)x = y$*.*

Next, we present some concepts of the monotonicity of a bifunction and an operator (see, for instance [4, 8, 20]).

Definition 2.1. *A bifunction* $f : C \times C \to \mathbb{R}$ *is said to be*

i. strongly monotone on C *if there exists a constant* $\gamma > 0$ *such that*

$$f(x, y) + f(y, x) \le -\gamma \|x - y\|^2, \quad \forall x, y \in C;$$

ii. monotone on C *if*

$$f(x, y) + f(y, x) \le 0, \quad \forall x, y \in C;$$

iii. pseudomonotone on C *if*

$$f(x, y) \ge 0 \Rightarrow f(y, x) \le 0 \ , \quad \forall x, y \in C.$$

From the definitions above, it is clear that $i. \Rightarrow ii. \Rightarrow iii.$

A bifunction $f : C \times C \to \mathbb{R}$ is called to satisfy a Lipschitz-type condition on C, if there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \ge f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2, \quad \forall x, y, z \in C; \tag{2.1}$$

Definition 2.2. *An operator* $A : C \to H$ *is said to be*

i. monotone on C *if* $\langle A(x) - A(y), x - y\rangle \ge 0, \quad \forall x, y \in C$*;*

ii. pseudomonotone on C *if*

$$\langle A(x), y - x\rangle \ge 0 \Rightarrow \langle A(y), y - x\rangle \ge 0, \quad \forall x, y \in C;$$

iii. α *- inverse strongly monotone on* C *if there exists* $\alpha > 0$ *such that*

$$\langle A(x) - A(y), x - y\rangle \ge \alpha \|A(x) - A(y)\|^2, \quad \forall x, y \in C;$$

iv. L *- Lipschitz continuous on* C *if there exists* $L > 0$ *such that*

$$\|A(x) - A(y)\| \le L\|x - y\|, \quad \forall x, y \in C.$$

Remark that if $A : C \to H$ is a (strongly) monotone operator, then the bifunction $f(x, y) = \langle Ax, y - x\rangle$ is (strongly) monotone. If A is L-Lipschitz continuous, then f satisfies the Lipschitz-type condition with $c_1 = \frac{L\mu}{2}$ and $c_2 = \frac{L}{2\mu}$ for any $\mu > 0$.

For solving problem EP, we assume that the bifunction f satisfies the following conditions:

(A1). f is monotone on C and $f(x,x) = 0, \forall x \in C$;

(A2). f satisfies Lipschitz-type condition on C;

(A3). f is sequentially weakly continuous on $C \times C$;

(A4). $f(x,.)$ is convex and subdifferentiable on C for every fixed $x \in C$.

It is easy to show that under the assumptions (A1), (A3), (A4), the solution set $EP(f,C)$ of EP is closed and convex [4, 6]. Therefore, from Lemma 2.1, the solution set $EP(f,C) \cap F(S)$ is closed and convex. In this paper, we assume that $EP(f,C) \cap F(S)$ is nonempty.

The metric projection $P_C : H \to C$ is defined by $P_C x = \arg\min\{\|y - x\| : y \in C\}$. Since C is nonempty, closed and convex, $P_C(x)$ exists and is unique. It is also known that P_C has the following characteristic properties.

Lemma 2.2. [10] *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

i. for all $x \in C, y \in H$,

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2; \tag{2.2}$$

ii. $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \forall y \in C. \tag{2.3}$$

The subdifferential of a function $g : C \to \mathbb{R}$ at x is defined by

$$\partial g(x) = \{w \in H : g(y) - g(x) \geq \langle w, y - x \rangle, \forall y \in C\}.$$

We recall that the normal cone of C at $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, y - x \rangle \leq 0, \forall y \in C\}.$$

Definition 2.3. *(Weakly lower semicontinuity). A function $\varphi : H \to \mathbb{R}$ is called weakly lower semicontinuous at $x \in H$ if any sequence $\{x_n\}$ in H converges weakly to x then*

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n).$$

It is well-known that the functional $\varphi(x) = \|x\|^2$ is convex and weakly lower semicontinuous. Any Hilbert space has the Kadec-Klee property [11], i.e., if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ then $x_n \to x$ as $n \to \infty$. We need the following lemmas for analyzing the convergence of Algorithm 1.1.

Lemma 2.3. *Let C be a convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then, x^* is a solution to the following convex optimization problem*

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

Proof. This is an infinite version of Theorem 27.4 in [25] and is similarly proved.

Lemma 2.4. *[17] Let $\{M_n\}, \{N_n\}, \{P_n\}$ be nonnegative real sequences, $\alpha, \beta \in \mathbb{R}$ and for all $n \geq 0$ the following inequality holds*

$$M_n \leq N_n + \beta P_n - \alpha P_{n+1}.$$

If $\Sigma_{n=0}^{\infty} N_n < +\infty$ and $\alpha > \beta \geq 0$ then $\lim\limits_{n\to\infty} M_n = 0$.

3. Convergence Analysis

In this section, we rewrite our algorithm in more details and analyze its convergence.

Algorithm 3.1. ***Initialization.*** *Chose $x_0 = x_1 \in H, y_0 = y_1 \in C$ and set $C_0 = Q_0 = H$. The parameters λ, k and $\{\alpha_n\}$ satisfy the following conditions:*

a. $0 < \lambda < \frac{1}{2(c_1+c_2)}, k > \frac{1}{1-2\lambda(c_1+c_2)}$;

b. $\alpha_n \in [0, a]$ *for some* $a \in (0, 1)$.

Step 1. *Compute y_{n+1} and z_{n+1} by*

$$\begin{cases} y_{n+1} = \arg\min\limits_{y \in C}\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ z_{n+1} = \alpha_n y_{n+1} + (1 - \alpha_n) S y_{n+1}. \end{cases}$$

Step 2. *Compute $x_{n+1} = P_{\Omega_n}(x_0)$, where $\Omega_n = C_n \cap Q_n$,*

$$\begin{cases} C_n = \{z \in H : \|w_{n+1} - z\|^2 \leq \|x_n - z\|^2 + \epsilon_n\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \leq 0\}, \end{cases}$$

and $w_{n+1} = \underset{t=y_{n+1},z_{n+1}}{\arg\max} \{\|t - x_n\|\}$, *and*

$$\epsilon_n = k\|x_n - x_{n-1}\|^2 + 2\lambda c_2\|y_n - y_{n-1}\|^2 - (1 - \frac{1}{k} - 2\lambda c_1)\|y_{n+1} - y_n\|^2.$$

Set $n = n + 1$ *and go back* ***Step 1.***

Remark 3.1. *In fact,* w_{n+1} *in Algorithm 3.1 equals to either* y_{n+1} *or* z_{n+1}*, i.e.,* $w_{n+1} = y_{n+1}$ *if* $\|y_{n+1} - x_n\| \geq \|z_{n+1} - x_n\|$ *and* $w_{n+1} = z_{n+1}$ *otherwise.*

Lemma 3.1. *Let* $\{x_n\}, \{y_n\}$ *be the sequences generated by Algorithm 3.1. Then, there holds the following relation for all* $n \geq 0$*,*

$$\langle y_{n+1} - x_n, y - y_{n+1}\rangle \geq \lambda\left(f(y_n, y_{n+1}) - f(y_n, y)\right), \forall y \in C.$$

Proof. Lemma 2.3 and the definition of y_{n+1} imply that

$$0 \in \partial_2\left(\lambda f(y_n, y) + \tfrac{1}{2}\|x_n - y\|^2\right)(y_{n+1}) + N_C(y_{n+1}).$$

Hence, there exist $w \in \partial_2 f(y_n, y_{n+1}) = \partial f(y_n, .)(y_{n+1})$ and $\overline{w} \in N_C(y_{n+1})$ such that $\lambda w + y_{n+1} - x_n + \overline{w} = 0$. Thus, for all $y \in C$, we have

$$\begin{aligned}\langle y_{n+1} - x_n,\, y - y_{n+1}\rangle &= \lambda\langle w,\, y_{n+1} - y\rangle + \langle \bar{w},\, y_{n+1} - y\rangle \\ &\geq \lambda\langle w,\, y_{n+1} - y\rangle\end{aligned}$$

because of the definition of N_C. By $w \in \partial_2 f(y_n, y_{n+1})$,

$$f(y_n, y) - f(y_n, y_{n+1}) \geq \langle w, y - y_{n+1}\rangle, \forall y \in C.$$

From the last two inequalities, we obtain the desired conclusion. Lemma 3.1 is proved. ■

The following lemma plays an important role in proving the convergence of Algorithm 3.1.

Lemma 3.2. *Assume that* $x^* \in EP(f, C) \cap F(S)$*. Let* $\{x_n\}, \{w_n\}$ *be the sequences generated by Algorithm 3.1. Then, there holds the following relation for all* $n \geq 0$

$$\|w_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \epsilon_n.$$

Proof. From $x^* \in F(S)$, the definition of z_{n+1}, the convexity of $\|.\|^2$ and the nonexpansiveness of S,

$$\begin{aligned}\|z_{n+1} - x^*\|^2 &= \|\alpha_n(y_{n+1} - x^* + (1 - \alpha_n)(Sy_{n+1} - x^*)\|^2 \\ &\leq \alpha_n\|y_{n+1} - x^*\|^2 + (1 - \alpha_n)\|Sy_{n+1} - Sx^*\|^2 \\ &\leq \alpha_n\|y_{n+1} - x^*\|^2 + (1 - \alpha_n)\|y_{n+1} - x^*\|^2 \\ &= \|y_{n+1} - x^*\|^2. \qquad (3.1)\end{aligned}$$

From Lemma 3.1, by replacing $n+1$ by n, we have

$$\langle y_n - x_{n-1}, y - y_n \rangle \geq \lambda(f(y_{n-1}, y_n) - f(y_{n-1}, y)), \forall y \in C. \tag{3.2}$$

Substituting $y = y_{n+1}$ into (3.2) and a straightforward computation yields

$$\lambda(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geq \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle . \tag{3.3}$$

Lemma 3.1 with $y = x^*$ leads to

$$\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \lambda(f(y_n, y_{n+1}) - f(y_n, x^*)). \tag{3.4}$$

Since $x^* \in EP(f, C)$ and $y_n \in C, f(x^*, y_n) \geq 0$. Thus, from the monotonicity of f, we have $f(y_n, x^*) \leq 0$. This together with (3.4) implies that

$$\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \lambda f(y_n, y_{n+1}). \tag{3.5}$$

By the Lipschitz-type continuity of f,

$$f(y_{n-1}, y_n) + f(y_n, y_{n+1}) \geq f(y_{n-1}, y_{n+1}) - c_1\|y_{n-1} - y_n\|^2 - c_2\|y_n - y_{n+1}\|^2.$$

Thus,

$$f(y_n, y_{n+1}) \geq f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - c_1\|y_{n-1} - y_n\|^2 - c_2\|y_n - y_{n+1}\|^2. \tag{3.6}$$

The relations (3.5) and (3.6) lead to

$$\begin{aligned}\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq& \lambda\{f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)\}\\ &- \lambda c_1\|y_{n-1} - y_n\|^2 - \lambda c_2\|y_n - y_{n+1}\|^2.\end{aligned}$$

Combining this and the relation (3.3), one gets

$$\begin{aligned}\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq& \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle - \lambda c_1\|y_{n-1} - y_n\|^2\\ &- \lambda c_2\|y_n - y_{n+1}\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&2\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle - 2\langle y_n - x_{n-1}, y_n - x_{n+1} \rangle\\ &\qquad \geq 2\lambda c_1\|y_{n-1} - y_n\|^2 - 2\lambda c_2\|y_n - y_{n+1}\|^2.\end{aligned} \tag{3.7}$$

We have the following fact

$$\begin{aligned}2\langle y_{n+1} - x_n,\ x^* - y_{n+1} \rangle =& \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - y_{n+1}\|^2\\ =& \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - x_{n-1}\|^2\\ &- 2\langle x_n - x_{n-1},\ x_{n-1} - y_{n+1} \rangle - \|x_{n-1} - y_{n+1}\|^2\\ =& \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - x_{n-1}\|^2\\ &- 2\langle x_n - x_{n-1}, x_{n-1} - y_{n+1} \rangle - \|x_{n-1} - y_n\|^2\\ &- 2\langle x_{n-1} - y_n, y_n - y_{n+1} \rangle - \|y_n - y_{n+1}\|^2.\end{aligned} \tag{3.8}$$

By the triangle, Cauchy-Schwarz and Cauchy inequalities,

$$\begin{aligned}
&-2\langle x_n - x_{n-1},\ x_{n-1} - y_{n+1}\rangle \\
&\leq 2\|x_n - x_{n-1}\|\|x_{n-1} - y_{n+1}\| \\
&\leq 2\|x_n - x_{n-1}\|\|x_{n-1} - y_n\| + 2\|x_n - x_{n-1}\|\|y_n - y_{n+1}\| \\
&\leq \|x_n - x_{n-1}\|^2 + \|x_{n-1} - y_n\|^2 + k\|x_n - x_{n-1}\|^2 + \frac{1}{k}\|y_n - y_{n+1}\|^2.
\end{aligned}$$

This together with (3.8) implies that

$$\begin{aligned}
2\langle y_{n+1} - x_n,\ x^* - y_{n+1}\rangle \leq & \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 + k\|x_n - x_{n-1}\|^2 \\
&+ 2\langle y_n - x_{n-1},\ y_n - y_{n+1}\rangle + \left(\frac{1}{k} - 1\right)\|y_n - y_{n+1}\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&2\langle y_{n+1} - x_n, x^* - y_{n+1}\rangle - 2\langle y_n - x_{n-1}, y_n - y_{n+1}\rangle \leq \|x_n - x^*\|^2 \\
&\qquad - \|y_{n+1} - x^*\|^2 + k\|x_n - x_{n-1}\|^2 + \left(\frac{1}{k} - 1\right)\|y_n - y_{n+1}\|^2. \qquad (3.9)
\end{aligned}$$

Combining (3.7) and (3.9) we obtain

$$\begin{aligned}
-2\lambda c_1\|y_{n-1} - y_n\|^2 - 2\lambda c_2\|y_n - y_{n+1}\|^2 \leq & \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 \\
&+ k\|x_n - x_{n-1}\|^2 + \left(\frac{1}{k} - 1\right)\|y_n - y_{n+1}\|^2.
\end{aligned}$$

Thus, from the definition of ϵ_n,

$$\begin{aligned}
\|y_{n+1} - x^*\|^2 \leq & \|x_n - x^*\|^2 + k\|x_n - x_{n-1}\|^2 \\
&+ 2\lambda c_1\|y_{n-1} - y_n\|^2 - \left(1 - \frac{1}{k} - 2\lambda c_2\right)\|y_n - y_{n+1}\|^2 \\
= & \|x_n - x^*\|^2 + \epsilon_n. \qquad (3.10)
\end{aligned}$$

From (3.1) and (3.10), we obtain

$$\|z_{n+1} - x^*\|^2 \leq \|y_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \epsilon_n.$$

Thus, from the definition of w_{n+1}, we obtain the desired conclusion. Lemma 3.2 is proved. ■

We have the following main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the bifunction f satisfies all conditions* (A1)-(A4) *and $S : C \to C$ is a nonexpansive mapping. In addition, the solution set $EP(f, C) \cap F(S)$ is nonempty. Then, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ generated by Algorithm 3.1 converge strongly to $P_{EP(f,C)\cap F(S)}(x_0)$.*

Proof. We divide the proof of Theorem 1 into three steps.
Claim 1. $EP(f,C)\cap F(S)\subset\Omega_n$ for all $n\geq 0$. Indeed, Lemma 3.2 and the definition of C_n ensure that $EP(f,C)\cap F(S)\subset C_n$ for all $n\geq 0$. We have $EP(f,C)\cap F(S)\subset H=\Omega_0$. Suppose that $EP(f,C)\cap F(S)\subset\Omega_n$ for some $n\geq 0$. From $x_{n+1}=P_{\Omega_n}(x_0)$ and Lemma 2.2 *ii.*, we see that $\langle z-x_{n+1},x_0-x_{n+1}\rangle\leq 0$ for all $z\in\Omega_n$. Thus, $\langle z-x_{n+1},x_0-x_{n+1}\rangle\leq 0$ for all $z\in EP(f,C)\cap F(S)$. Hence, from the definition of Q_{n+1}, $EP(f,C)\cap F(S)\subset Q_{n+1}$ or $EP(f,C)\cap F(S)\subset\Omega_{n+1}$. By the induction, $EP(f,C)\cap F(S)\subset\Omega_n$ for all $n\geq 0$. Since $EP(f,C)\cap F(S)$ is nonempty, so Ω_n is. Therefore, $P_{\Omega_n}(x_0)$ and $P_{EP(f,C)\cap F(S)}(x_0)$ are well-defined.
Claim 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|z_n-x_n\|=\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$ and $\lim_{n\to\infty}\|y_n-Sy_n\|=0$. Indeed, from the definition of Q_n and Lemma 2.2 *ii.*, $x_n=P_{Q_n}(x_0)$. Thus, from Lemma 2.2 *i.*, we have

$$\|z-x_n\|^2\leq\|z-x_0\|^2-\|x_n-x_0\|^2,\quad\forall z\in Q_n.\tag{3.11}$$

Substituting $z=x^\dagger:=P_{EP(f,C)\cap F(S)}(x_0)\in Q_n$ into 3.11, one has

$$\|x^\dagger-x_0\|^2-\|x_n-x_0\|^2\geq\|x^\dagger-x_n\|^2\geq 0.\tag{3.12}$$

Thus, the sequence $\{\|x_n-x_0\|\}$ and $\{x_n\}$ are bounded. The relation 3.11 with $z=x_{n+1}\in Q_n$ leads to

$$0\leq\|x_{n+1}-x_n\|^2\leq\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2.\tag{3.13}$$

This implies that $\{\|x_n-x_0\|\}$ is non-decreasing. Hence, there exists the limit of $\{\|x_n-x_0\|\}$. By (3.13),

$$\sum_{n=1}^{K}\|x_{n+1}-x_n\|^2\leq\|x_{K+1}-x_0\|^2-\|x_1-x_0\|^2,\quad\forall K\geq 1.$$

Passing the limit in the last inequality as $K\to\infty$, we obtain

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2<+\infty.\tag{3.14}$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.15}$$

From the definition of C_n and $x_{n+1}\in C_n$,

$$\|w_{n+1}-x_{n+1}\|^2\leq\|x_n-x_{n+1}\|^2+\epsilon_n.\tag{3.16}$$

Set $M_n = \|w_{n+1} - x_{n+1}\|^2$, $N_n = \|x_n - x_{n+1}\|^2 + k\|x_n - x_{n-1}\|^2$, $P_n = \|y_n - y_{n-1}\|^2$, $\beta = 2\lambda c_2$, and $\alpha = 1 - \frac{1}{k} - 2\lambda c_1$. From the definition of ϵ_n, $\epsilon_n = k\|x_n - x_{n-1}\|^2 + \beta P_n - \alpha P_{n+1}$. Thus, from (3.16),

$$M_n \leq N_n + \beta P_n - \alpha P_{n+1}. \tag{3.17}$$

From the hypotheses of λ, k and (3.14), we see that $\alpha > \beta \geq 0$ and $\sum_{n=1}^{\infty} N_n < +\infty$. Lemma 2.4 and (3.17) imply that $M_n \to 0$, or $\|w_{n+1} - x_{n+1}\| \to 0$. Thus, $\|w_{n+1} - x_n\| \to 0$ due to the relation (3.15) and the triangle inequality. Hence, from the definition of w_{n+1}, we obtain

$$\lim_{n\to\infty} \|z_{n+1} - x_n\| = \lim_{n\to\infty} \|y_{n+1} - x_n\| = 0.$$

These together with (3.15) and the triangle inequality imply that

$$\lim_{n\to\infty} \|z_{n+1} - x_{n+1}\| = \lim_{n\to\infty} \|y_{n+1} - x_{n+1}\| = \lim_{n\to\infty} \|z_{n+1} - y_{n+1}\| = 0. \tag{3.18}$$

By the triangle inequality,

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\|.$$

From the last inequality, (3.15) and (3.18),

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.19}$$

From the definition of z_{n+1} we have

$$\|z_{n+1} - y_{n+1}\| = (1 - \alpha_n)\|y_{n+1} - Sy_{n+1}\| \geq (1 - \alpha)\|y_{n+1} - Sy_{n+1}\|.$$

This together with (3.18) and $1 - \alpha > 0$ implies that $\lim_{n\to\infty} \|y_{n+1} - Sy_{n+1}\| = 0$.

Claim 3. $x_n \to x^\dagger = P_{EP(f,C)\cap F(S)}(x_0)$ as $n \to \infty$. Assume that p is any weak cluster point of $\{x_n\}$. Without loss of generality, we can write $x_n \rightharpoonup p$ as $n \to \infty$. Since $\|y_n - y_{n+1}\| \to 0$, $y_n \rightharpoonup p$. Now, we show that $p \in EP(f, C) \cap F(S)$. Indeed, from Claim 2 and the demiclosedness of S we obtain $p \in F(S)$. By Lemma 3.1, we get

$$\lambda\big(f(y_n, y) - f(y_n, y_{n+1})\big) \geq \langle x_n - y_{n+1}, y - y_{n+1}\rangle, \quad \forall y \in C. \tag{3.20}$$

Passing to the limit in (3.20) as $n \to \infty$ and using Claim 2, the boundedness of $\{y_n\}$ and $\lambda > 0$, we obtain

$$f(p, y) \geq 0 \quad \text{for all } y \in C.$$

Hence, $p \in EP(f, C)$. Thus, $p \in EP(f, C) \cap F(S)$. From the inequality (3.12), we get $\|x_n - x_0\| \leq \|x^\dagger - x_0\|$, where $x^\dagger = P_{EP(f,C)\cap F(S)}(x_0)$. By the weak lower semicontinuity of the norm $\|\cdot\|$ and $x_n \rightharpoonup p$, we have

$$\|p - x_0\| \leq \lim_{n\to\infty} \inf \|x_n - x_0\| \leq \lim_{n\to\infty} \sup \|x_n - x_0\| \leq \|x^\dagger - x_0\|.$$

By the definition of $x^\dagger$, $p = x^\dagger$ and $\lim_{n\to\infty} \|x_n - x_0\| = \|x^\dagger - x_0\|$. Thus, $\lim_{n\to\infty} \|x_n\| = \|x^\dagger\|$. By the Kadec–Klee property of the Hilbert space H, we have $x_n \to x^\dagger = P_{EP(f,C)\cap F(S)}(x_0)$ as $n \to \infty$. From Claim 2, we also see that $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{EP(f,C)\cap F(S)}(x_0)$. Theorem 3.1 is proved. ■

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the bifunction f satisfies all conditions* (A1)–(A4). *In addition the solution set $EP(f,C)$ is nonempty. Let $\{x_n\}, \{y_n\}$ be two sequences generated by the following manner: $x_0 = x_1 \in H, y_0 = y_1 \in C$ and*

$$\begin{cases} y_{n+1} = \arg\min_{y\in C} \left\{ \lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \right\}, \\ C_n = \{z \in H : \|y_{n+1} - z\|^2 \le \|x_n - z\|^2 + \epsilon_n\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases}$$

where ϵ_n, λ, k are defined as in Algorithm 3.1. Then, the sequences $\{x_n\}, \{y_n\}$ converge strongly to $P_{EP(f,C)}(x_0)$.

Proof. Set $S = I$ (the identity operator in H), from Algorithm 3.1 we have $z_{n+1} = y_{n+1} = w_{n+1}$. Thus, Corollary 3.1 is directly followed from Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $A : C \to H$ is a weakly-strongly continuous, monotone and L-Lipschitz continuous operator and $S : C \to C$ is a nonexpansive mapping such that the solution set $VI(A,C) \cap F(S)$ is nonempty, where $VI(A,C)$ is the solution set of VIP. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by the following manner: $x_0 = x_1 \in H, y_0 = y_1 \in C$ and*

$$\begin{cases} y_{n+1} = P_C(x_n - \lambda A(y_n)), \\ z_{n+1} = \alpha_n y_{n+1} + (1-\alpha_n) S y_{n+1}, \\ C_n = \{z \in H : \|w_{n+1} - z\|^2 \le \|x_n - z\|^2 + \epsilon_n\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{29}$$

where $w_{n+1}, \epsilon_n, \lambda, k$ are defined as in Algorithm 3.1 with $c_1 = c_2 = L/2$. Then, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to $P_{VI(A,C)\cap F(S)}(x_0)$.

Proof. Set $f(x,y) = \langle A(x), y - x\rangle$ for all $x, y \in C$. It is clear that f satisfies the conditions (A1), (A3), (A4) automatically. Now, we show that the condition (A2) holds for the bifunction f. Indeed, from the L-Lipschitz continuity of A, we have

$$\begin{aligned} f(x,y)+f(y,z)-f(x,z) &= \langle A(x)-A(y), y-z\rangle \\ &\geq -\|A(x)-A(y)\|\|y-z\| \\ &\geq -L\|x-y\|\|y-z\| \\ &\geq -\frac{L}{2}\|x-y\|^2-\frac{L}{2}\|y-z\|^2. \end{aligned}$$

This implies that f satisfies the condition (A2) with $c_1 = c_2 = L/2$. According to Algorithm 3.1, we have

$$\begin{aligned} y_{n+1} &= \arg\min_{y\in C}\left\{\langle \lambda A(y_n), y-y_n\rangle + \frac{1}{2}\|x_n-y\|^2\right\} \\ &= \arg\min_{y\in C}\left\{\frac{1}{2}\|y-(x_n-\lambda A(y_n))\|^2 - \frac{\lambda^2}{2}\|A(y_n)\|^2 - \lambda\langle A(y_n), y_n-x_n\rangle\right\} \\ &= \arg\min_{y\in C}\left\{\frac{1}{2}\|y-(x_n-\lambda A(y_n))\|^2\right\} \\ &= P_C(x_n-\lambda A(y_n)). \end{aligned}$$

Hence, Corollary 3.2 is directly followed from Theorem 3.1. ■

Remark 3.2. *(i) From the proofs of Lemma 3.2 and Theorem 3.1, we see that Theorem 3.1, Corollaries 3.1 and 3.2 remain true if the monotonicity is replaced by the pseudomonotonicity.*

(ii) Corollary 3.2 can be considered as an improvement of the results in [5, 22] in the sense that we only need to find a projection onto the constrained set C at each iteration.

(iii) The set Ω_n in Step 2 of Algorithm 3.1 can be replaced by $\Omega_n = C_n^1 \cap C_n^2 \cap Q_n$, where C_n^1, C_n^2 are two halfspaces defined by

$$\begin{aligned} C_n^1 &= \left\{z\in H : \|z_{n+1}-z\|^2 \leq \|y_{n+1}-z\|^2\right\}, \\ C_n^2 &= \left\{z\in H : \|y_{n+1}-z\|^2 \leq \|x_n-z\|^2+\epsilon_n\right\}. \end{aligned}$$

Remark 3.3. *We can generalize Algorithm 3.1 for finding a common solution of problem EP and a finite family of fixed point problems for nonexpansive mappings $\{S_j\}_{j=1}^N$. The algorithm is designed as follows:*

Algorithm 3.2. ***Initialization.*** *Chose $x_0 = x_1 \in H, y_0 = y_1 \in C$ and set $C_0 = Q_0 = H$. The parameters λ, k and $\{\alpha_n\}$ satisfy the following conditions:*

a. $0 < \lambda < \frac{1}{2(c_1+c_2)}, k > \frac{1}{1-2\lambda(c_1+c_2)}$;

b. $\alpha_n \in [0, a]$ *for some* $a \in (0, 1)$.

Step 1. *Solve a strongly convex optimization program*

$$y_{n+1} = \arg\min_{y\in C}\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}.$$

Step 2. *Compute* z^j_{n+1} *for each* $j = 1, 2, \cdots, N$ *in parallel*

$$z^j_{n+1} = \alpha_n y_{n+1} + (1 - \alpha_n)S_j y_{n+1},\ j = 1, 2, \cdots, N.$$

Step 3. *Find* w_{n+1} *such that*

$$w_{n+1} = \arg\max\{\|y_{n+1} - x_n\|, \|z^j_{n+1} - x_n\| : j = 1, 2, \cdots, N\}.$$

Step 4. *Compute* $x_{n+1} = P_{\Omega_n}(x_0)$*, where* $\Omega_n = C_n \cap Q_n$*,*

$$\begin{cases} C_n = \{z \in H : \|w_{n+1} - z\|^2 \leq \|x_n - z\|^2 + \epsilon_n\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \leq 0\}, \end{cases}$$

and

$$\epsilon_n = k\|x_n - x_{n-1}\|^2 + 2\lambda c_2\|y_n - y_{n-1}\|^2 - (1 - \frac{1}{k} - 2\lambda c_1)\|y_{n+1} - y_n\|^2.$$

Set $n = n + 1$ *and go back* ***Step 1.***

In Step 2 of Algorithm 3.2, the intermediate approximations z^j_{n+1} can be computed in parallel. Among the approximations y_{n+1}, z^j_{n+1}, $j = 1, 2, \cdots, N$, we find the furthest element from x_n, denoted by w_{n+1}. Using this element to construct the half-space C_n. This technique is developed from the papers [1, 12]. We have the following result which is proved similarly to Theorem 3.1.

Theorem 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Assume that the bifunction* f *satisfies all conditions* (A1)-(A4) *and* $S_j : C \to C$, $j = 1, 2, \cdots, N$ *are nonexpansive mappings. In addition, the solution set* $\Omega = EP(f, C) \cap \left(\bigcap_{j=1}^N F(S_j)\right)$ *is nonempty. Then, the sequences* $\{x_n\}, \{y_n\}, \{z_n\}$ *generated by Algorithm 3.2 converge strongly to* $P_\Omega(x_0)$*.*

4. A Numerical Illustration

In this section, we consider two numerical examples to illustrate the convergence of Algorithm 3.1. The bifunction $f : C \times C \to \mathbb{R}$ which comes from the Nash-Cournot equilibrium model in [24, 30] is defined by

$$f(x, y) = \langle Px + Qy + q, y - x\rangle,$$

where $q \in \mathbb{R}^n$, $P, Q \in \mathbb{R}^{n\times n}$ are two matrices of order n such that Q is symmetric, positive semidefinite and $Q - P$ is negative semidefinite. By [24, Lemma 6.2], f is monotone and Lipschitz-type continuous with $c_1 = c_2 = \frac{1}{2}\|P - Q\|$. In two numerical experiments below we chose $\lambda = \frac{1}{5c_1}, k = 6, x_1 = x_0 \in \mathbb{R}^n$ and y_0, y_1 are the zero vector. All convex quadratic optimization programs are solved by the MATLAB Optimization Toolbox. The algorithm is performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB.

Example 1. We consider the feasible set C defined by

$$C = \left\{x \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i \geq 1,\ 0 \leq x_i \leq 1,\ i = 1, \ldots, 3\right\}$$

and S is the identity operator I. This example is tested with $q = (1; -2; 3)^T$ and

$$P = \begin{pmatrix} 3.1 & 2 & 0 \\ 2 & 3.6 & 0 \\ 0 & 0 & 3.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 1.6 & 1 & 0 \\ 1 & 1.6 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}.$$

The iterate x_{n+1} is expressed by the explicit formula in [28]. In this case, the solution set $EP(f, C) \cap F(S) = EP(f, C)$ is not known. Thus, the stopping criterion used in this experiment is $\|w_{n+1} - x_n\| \leq \text{TOL} = 0.0001$. Table 1 shows the numbers of iterates (Iter.), time for execution of Algorithm 3.1 in second (CPU in sec.), and approximation solutions x_n to EP for choosing different starting points.

Table 1. Results for given starting points in *Example 1.*

x_0	Iter.	CPU in sec.	x_n
(1; 3; 1)	377	9.18	(0.0000004; 0.9806232; 0.0194736)
(-3; 4; 1)	220	4.96	(0.0000000; 0.9806290; 0.0194844)
(3; -2; 1)	480	15.53	(0.0000004; 0.9806289; 0.0194885)

Example 2. We consider the constrained set C as a box by

$$C = \left\{x \in \mathbb{R}^3 : 0 \leq x_i \leq 1,\ i = 1, \ldots, 3\right\}.$$

Two matrices P, Q are defined as in *Example 1*, q is the zero vector. Let $C_i, i = 1, 2, 3$ be three halfspaces such that $C \cap C_1 \cap C_2 \cap C_3 = \emptyset$. For each $x \in \mathbb{R}^3$, set

$$\Phi(x) := \frac{1}{3}\sum_{i=1}^{3} \min_{z \in C_i} \|x - z\|^2 = \frac{1}{3}\sum_{i=1}^{3} d^2(x, C_i), \quad \text{and} \quad C_\Phi := \arg\min_{x \in C} \Phi(x).$$

Define $S : C \to C$ by

$$S := P_C\left(\frac{1}{3}\sum_{i=1}^{3} P_{C_i}\right).$$

Since the projection is nonexpansive, S is nonexpansive and $F(S) = C_\Phi$ (see, [31, Proposition 4.2]). In this example, we chose $C_1 = \{x \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 \le -6\}$, $C_2 = \{x \in \mathbb{R}^3 : 5x_1 + 4x_2 + 3x_3 \le -12\}$, and $C_3 = \{x \in \mathbb{R}^3 : 2x_1 + x_2 + x_3 \le -4\}$. It is easy to show that $EP(f, C) \cap F(S) = \{0\}$. For each starting point x_0 then the sequence x_n generated by Algorithm 1 converges strongly to $x^\dagger := P_{EP(f,C)\cap F(S)}(x_0) = 0$. The termination criterion is $\|x_n - x^\dagger\| \le \text{TOL} = 0.001$. The results are shown in Table 2 for choosing different starting points and parameters α_n.

Table 2. Results for different starting points and parameters in *Example 2*.

x_0	$\alpha_n = \frac{n-1}{2(n+1)}$		$\alpha_n = 10^{-n}$		$\alpha_n = \frac{1}{\log_{10}(n+1)}$	
	Iter.	CPU in sec.	Iter.	CPU in sec.	Iter.	CPU in sec.
(1; 3; 1)	23	1.54	26	1.75	67	4.45
(-3; 4; 1)	47	3.07	20	1.34	81	5.46
(3; -2; 1)	29	1.96	17	1.03	47	2.40
(-2; 3; -1)	7	0.40	6	0.26	18	1.18

The study of the numerical experiments here is preliminary and it is obvious that EPs and fixed point problems depend on the structure of the constrained set C, the bifunction f, and the mapping S. However, the results in Tables 1 and 2 have illustrated the convergence of our proposed algorithm and we also see that the number of iterative step and time for execution of the algorithm depend on the starting point x_0 and the parameter α_n.

5. Conclusions

The paper has proposed a hybrid projection method for finding a common solution of an equilibrium problem and fixed point problems. The main advantage of the method over the extragradient methods is that, at each step, only one optimization problem needs to be solved. This comes from choosing a special intermediate approximation to construct half-spaces. The strong convergence of the method is established.

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References

[1] **Anh, P.K. and Hieu, D.V.,** *Parallel hybrid iterative methods for variational inequalities, equilibrium problems, and common fixed point problems*, Vietnam Journal of Mathematics, **44(2)** (2016), 351-374.

[2] **Anh, P.N.,** *A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems*, The Bulletin of the Malaysian Mathematical Society, **1(1)** (2013), 107-116.

[3] **Bigi, G., Castellani, M., Pappalardo, M. and Passacantando, M.,** Nonlinear programming techniques for equilibria, Springer, 2019.

[4] **Blum, E. and Oettli, W.,** *From optimization and variational inequalities to equilibrium problems*, Mathematics Student, **63** (1994) 123-145.

[5] **Censor, Y., Gibali, A. and Reich, S.,** *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optimization Methods and Software, **26(4-5)** (2011), 827-845.

[6] **Combettes, P.L., and Hirstoaga, S.A.,** *Equilibrium programming in Hilbert spaces*, Journal of Nonlinear and Convex Analysis, **6(1)** (2005), 117-136.

[7] **Daniele, P., Giannessi, F. and Maugeri, A.,** Equilibrium problems and variational models, Kluwer Academic Publishers, 2003.

[8] **Dinh, B.V., Hung, P.G. and Muu, L.D.,** *Bilevel optimization as a regularization approach to pseudomonotone equilibrium problems*, Numerical Functional Analysis and Optimization, **35(5)** (2014), 539-563.

[9] **Fan, K. and Shisha, O.,** *A minimax inequality and applications*, Inequalities III, Academic Press New York, (1972), (103-113).

[10] **Goebel, K. and Reich, S.,** *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, (1984).

[11] **Goebel, K. and Kirk, W.A.,** *Topics in Metric Fixed Point Theory*, Cambridge University Press, (1990).

[12] **Hieu, D.V.,** *A parallel hybrid method for equilibrium problems, variational inequalities and nonexpansive mappings in Hilbert space*, Journal of the Korean Mathematical Society, **52(2)** (2015), 373-388.

[13] **Hieu, D.V.,** *A new shrinking gradient-like projection method for equilibrium problems*, Optimization, **66(12)** (2017), 2291-2307.

[14] **Hieu, D.V.,** *An extension of hybrid method without extrapolation step to equilibrium problems*, Journal of Industrial and Management Optimization, **13(4)** (2017), 1723-1741.

[15] **Konnov, I.V.,** Combined relaxation methods for variational inequalities, Springer, (2001).

[16] **Korpelevich, G.M.,** *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody, **12** (1976), 747-756.

[17] **Malitsky, Y.V. and Semenov, V.V.,** *A hybrid method without extrapolation step for solving variational inequality problems.* Journal of Global Optimization, **61(1)** (2015), 193-202.

[18] **Martinet, B.,** *Régularisation d'inéquations variationnelles par approximations successives*, Mathematical Modelling and Numerical Analysis, **3** (1970), 154-159.

[19] **Mordukhovich, B., Panicucci, B., Pappalardo, M. and Passacantando, M.,** *Hybrid proximal methods for equilibrium problems*, Optimization Letters, **6** (2012), 1535-1550.

[20] **Muu, L.D. and Oettli, W.,** *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Analysis, **18(12)** (1992), 1159-1166.

[21] **Muu, L.D., Thanh, L.X. and Strodiot, J.J.,** *A survey on solution approaches for the equilibrium problem defined by the Nikaido-Isoda-Fan inequality*, Mathematics and Mathematical Sciences, **2(8)** (2023), 67-85.

[22] **Nadezhkina, N. and Takahashi, W.,** *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM Journal on Optimization, **16(4)** (2006), 1230-1241.

[23] **Nguyen, T.T.V., Strodiot, J.J. and Nguyen, V.H.,** *Hybrid methods for solving simultaneously an equilibrium problem and countably many fixed point problems in a Hilbert space*, Journal of Optimization Theory and Applications, **160(3)** (2014), 809–831.

[24] **Quoc, T.D., Muu, L.D. and Hien, N.V.,** *Extragradient algorithms extended to equilibrium problems*, Optimization, **57(6)** (2008), 749-776.

[25] **Rockafellar, R.T.,** Convex Analysis, Princeton University Press, (1970).

[26] **Rockafellar, R.T.,** *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, **14(5)** (1976), 877–898.

[27] **Solodov, M.V. and Svaiter, B.F.,** *A new projection method for variational inequality problems*, SIAM Journal on Control and Optimization, **37(3)** (1999), 765-776.

[28] **Solodov, M.V. and Svaiter, B.F.,** *Forcing strong convergence of proximal point iterations in Hilbert space*, Mathematical Programming, **87(1)** (2000), 189-202.

[29] **Strodiot, J.J., Nguyen, V.H. and Vuong, P.,** *Strong convergence of two hybrid extragradient methods for solving equilibrium and fixed point problems*, Vietnam Journal of Mathematics, **40(2)** (2012), 371-389.

[30] **Strodiot, J.J., Nguyen, T.T.V. and Nguyen, V.H.,** *A new class of hybrid extragradient algorithms for solving quasi-equilibrium problems*, Journal of Global Optimization, **56(2)** (2013), 373-397.

[31] **Yamada, I.**, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*. In Butnariu, D., Censor, Y., Reich, S. (eds.) Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications, Elsevier, Amsterdam, (2001), 473–504.

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