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Global well-posedness of pseudo almost periodic mild solutions for a Keller-Segel system of parabolic-parabolic type

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Abstract. We study the global existence and uniqueness of pseudo almost periodic mild solutions for the parabolic-parabolic Keller-Segel systems on the real hyperbolic spaces $\mathbb{H}^n(\mathbb{R})$, where $n \geqslant 2$. First, we use the dispersive estimates of the scalar heat semigroup to estabilish the well-posedness of bounded mild solutions for the corresponding linear systems. Then, we prove the existence and uniquess of pseudo almost periodic mild solutions by proving a Massara-type principle. Finally, the well-posedness of such solutions for semilinear systems are obtained by employing fixed point arguments.

1. Introduction

We consider the following parabolic-parabolic (P-P) Keller–Segel system on the real hyperbolic space $\mathbb{H}^n$ ($n \geqslant 2$) and on the whole line time-axis, i.e., $t \in \mathbb{R}$:

$$\begin{cases} u_t &= \Delta u - \nabla \cdot (u \nabla v) + g(t) \qquad (t,x) \in \mathbb{R} \times \mathbb{H}^n, \\ v_t &= \Delta v - \gamma v + \kappa u + h(t) \qquad (t,x) \in \mathbb{R} \times \mathbb{H}^n, \end{cases} \tag{1.1}$$

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where the functions $g(t,x)$ and $h(t,x)$ are given and the parameters $\gamma \geqslant 0$ and $\kappa \geqslant 0$ denote the decay and production rate of the attractant, respectively. The unkown of system (1.1) are $u(t,\cdot)$ (scalar function) representing the density of cells and $v(t,\cdot)$ (scalar function) which is the concentration of the chemoattractant.

If the function $h = 0$ and the concentration of the chemoattractant is independent of time, the system (1.1) simplifies to a form known as the parabolic-elliptic Keller-Segel system (also called patlak-Keller-Segel system), which is given by (see [25] for the model on $\mathbb{H}^2$):

$$(1.2) \qquad \begin{cases} \quad u_t &= \Delta u - \nabla \cdot (u \nabla v) + g(t) \qquad (t,x) \in \mathbb{R} \times \mathbb{H}^n, \\ -\Delta v + \gamma v &= \kappa u \qquad\qquad\qquad\qquad\quad (t,x) \in \mathbb{R} \times \mathbb{H}^n. \end{cases}$$

The chemotaxis model, initially introduced in Euclidean space by Keller and Segel [18], describes the aggregation of biological species. Specifically, this model captures the behavior of organisms moving toward regions with higher concentrations of food molecules or chemicals secreted by the organisms themselves. For further details on the well-posedness and long-term behavior of solutions for system (1.2), we refer readers to the relevant works in [2, 6, 8, 10, 22, 23, 12, 13, 15] and the references therein.

We briefly review previous studies on Keller–Segel (P–P) systems and related models. Winkler [42] obtained L^p–L^q dispersive and smoothing estimates for the Neumann heat semigroup, proving exponential stability of mild solutions for $n \geqslant 3$, and later studied finite-time blow-up in higher dimensions [44]. Cao [5] extended these results by deriving smallness conditions in optimal Lebesgue spaces, ensuring global boundedness and large-time convergence for $n \geqslant 2$. Hao et al. [14] established global classical solutions for the Keller–Segel–Navier–Stokes system with matrix-valued sensitivity. Jiang proved global stability in critical spaces [16] and for homogeneous steady states in scaling-invariant spaces [17]. Other important contributions include [43, 45], and time-periodic solutions were addressed in [20, 31]. For the whole space $\mathbb{R}^n$ $(n \geqslant 2)$, well-posedness and asymptotic behavior have been studied extensively; see [3, 4, 7, 32, 41, 29] and references therein.

In non-Euclidean geometries, the curvature of Riemannian manifolds crucially affects the phase space and asymptotic behavior of solutions to Keller–Segel systems and other PDEs, in contrast to the flat Euclidean case. Hence, studying chemotaxis on curved spaces, such as hyperbolic manifolds, is of particular interest for understanding organism motion under curvature effects. On positively curved domains, Montaru [26] analyzed the Patlak–Keller–Segel system on a ball $B \subset \mathbb{R}^n$ and proved exponential convergence of radial solutions to the steady state in the subcritical case $M = \int_B u_0, \mathrm{dVol}_B < 8\pi$. More recently, Ahmedou et al. [1] showed blow-up for steady states on compact Riemann surfaces. For negatively curved manifolds, Pierfelice and Maheux [25] established global well-posedness on $\mathbb{H}^2$ with any L^1 initial data under $M < 8\pi$, and blow-up for $M > 8\pi$. Furthermore, Xuan et al. [37, 38] proved well-posedness and exponential stability (including decay) of periodic and asymptotically almost periodic mild solutions on $\mathbb{H}^n$ $(n \geqslant 2)$.

The well-posedness of periodic and generalized mild solutions—such as almost periodic, asymptotically almost periodic, and pseudo almost periodic ones—has been widely studied for parabolic and hyperbolic equations (see [9] and references therein). For chemotaxis models, substantial progress has been achieved in the parabolic-elliptic Keller–Segel setting: periodic mild solutions [37], almost periodic [39], asymptotically almost periodic [38], and pseudo almost periodic solutions [33]. More recently, the parabolic–parabolic Keller–Segel system on the real hyperbolic space $\mathbb{H}^n$ was shown to admit well-posed and exponentially stable periodic and almost periodic mild solutions [34].

In this paper, extending [34, 40], we establish the existence and uniqueness of pseudo almost periodic mild solutions to system (1.1). Our approach proceeds as follows: we first prove well-posedness of bounded mild solutions to the associated linear system using L^p–L^q dispersive and smoothing estimates for the scalar heat semigroup (Lemma 3.1). Next, we establish a Massera-type principle, ensuring pseudo almost periodic mild solutions of the linear system for given pseudo almost periodic inputs. Finally, by combining the linear results with fixed-point arguments, we obtain existence and uniqueness of pseudo almost periodic mild solutions for the semilinear system (1.1).

Our paper is organized as follows. Section 2 introduces the Keller–Segel (P–P) system on real hyperbolic spaces, recalls the L^p–L^q dispersive and smoothing estimates for the scalar heat semigroup, and defines the functional spaces for mild solutions. Section 3 presents the proofs of well-posedness for pseudo almost periodic mild solutions of both linear and semilinear systems.

2. Parabolic-parabolic Keller-Segel systems on hyperbolic spaces

Let $(\mathbb{H}^n, g) = (\mathbb{H}^n(\mathbb{R}), g)$ be a real hyperbolic manifold, where $n \geqslant 2$ is the dimension, endowed with a Riemannian metric g. This space is realized via a hyperboloid in $\mathbb{R}^{n+1}$ by considering the upper sheet

$$\left\{(x_0, x_1, ..., x_n) \in \mathbb{R}^{d+1};\ \ x_0 \geq 1 \text{ and } x_0^2 - x_1^2 - x_2^2 ... - x_n^2 = 1\,\right\}$$

where the metric is given by $dg = -dx_0^2 + dx_1^2 + ... + dx_n^2$.

In geodesic polar coordinates, the hyperbolic manifold $(\mathbb{H}^n, g)$ can be described as

$$\mathbb{H}^n = \left\{(\cosh\tau, \omega\sinh\tau),\ \tau \geq 0, \omega \in \mathbb{S}^{n-1}\right\}$$

with $dg = d\tau^2 + (\sinh\tau)^2 d\omega^2$, where $d\omega^2$ is the canonical metric on the sphere $\mathbb{S}^{n-1}$. In these coordinates, the Laplace-Beltrami operator $\Delta := \Delta_{\mathbb{H}}$ on $\mathbb{H}^n$ can be expressed as

$$\Delta := \Delta_{\mathbb{H}} = \partial_r^2 + (n-1)\coth r\partial_r + \sinh^{-2} r\Delta_{\mathbb{S}^{n-1}}.$$

It is well known that the spectrum of $-\Delta$ is the half-line $\left[\dfrac{(n-1)^2}{4}, \infty\right)$.

For the sake of simplicity, we assume $\gamma = \kappa = 1$, the Keller-Segel (P-P) system (1.1) becomes

$$\left\{\begin{array}{llr} u_t & = \Delta u - \nabla\cdot(u\nabla v) + g(t) & (t,x) \in \mathbb{R}\times\mathbb{H}^n, \\ v_t & = \Delta v - v + u + h(t) & (t,x) \in \mathbb{R}\times\mathbb{H}^n. \end{array}\right. \tag{2.1}$$

By employing the term of matrix, we can rewrite the system (2.1) as follows

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v\end{bmatrix} + \mathcal{A}\begin{bmatrix} u \\ v\end{bmatrix} = -\begin{bmatrix} \nabla\cdot(u\nabla v) \\ 0\end{bmatrix} + F\left(t, \begin{bmatrix} u \\ v\end{bmatrix}\right) \quad (t,x)\in\mathbb{R}\times\mathbb{H}^n, \tag{2.2}$$

where Id denotes identity operator and

$$\mathcal{A} = \begin{bmatrix} -\Delta & 0 \\ 0 & -\Delta + \mathrm{Id}\end{bmatrix} \text{ and } F\left(t, \begin{bmatrix} u \\ v\end{bmatrix}\right) = \begin{bmatrix} 0 \\ u\end{bmatrix} + \begin{bmatrix} g(t) \\ h(t)\end{bmatrix}. \tag{2.3}$$

We recall the dispersive and smoothing estimates of the scalar heat semigroup on the real hyperbolic space $\mathbb{H}^n$ $(n \geqslant 2)$ in the following lemma:

Lemma 2.1. *Let* $\gamma_{p,q} = \frac{\delta_n}{2}\left[\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{8}{q}\left(1 - \frac{1}{p}\right)\right]$. *The following dispersive estimates for scalar heat semigroup hold*

(*i*) *If* $t > 0$, *and* p, q *such that* $1 \leq p \leq q \leq \infty$, *then*

$$\left\|e^{t\Delta}\omega\right\|_{L^q} \leq k_1(1 + t^{-\frac{n}{2}(\frac{1}{p}) - \frac{1}{q}})e^{-t(\gamma_{p,q})}\left\|\omega\right\|_{L^p}, \tag{2.4}$$

for all scalar function $\omega \in L^p(\mathbb{H}^n)$.

(*ii*) *If* $t > 0$, *and* p, q *such that* $1 < p \leq q < \infty$, *then*

$$\left\|\nabla e^{t\Delta} v\right\|_{L^q} \leqslant k_2(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})})e^{-t\left(\frac{\gamma_{q,q} + \gamma_{p,q}}{2}\right)}\left\|v\right\|_{L^p}, \tag{2.5}$$

for all scalar function $v \in L^p(\mathbb{H}^n)$.

(*iii*) *Moreover, the dual version of* (2.5) *holds. In particular, if* $t > 0$*, and* p*,* q *such that* $1 < p \leq q < \infty$*, then*

$$\left\|e^{t\Delta}\nabla \cdot T\right\|_{L^q} \leqslant k_2(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})})e^{-t\left(\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \|T\|_{L^p}\,, \tag{2.6}$$

for all vector field $T \in L^p(\Gamma(\mathbb{H}^n))$*.*

Proof. The proof was given in [28] (see Theorem 4.1, Corollary 4.3 and Remark 4.4), with noting that

$$[h_n(t)]^\alpha \leqslant C\left(t^{-\frac{\alpha n}{2}} + 1\right)\ (\alpha > 0).$$

Now we recall some concepts of generalized functions. For more details we refer the readers to book [9] and references therein. Let X be a Banach space, we denote

$$C_b(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid f \text{ is continuous on } \mathbb{R} \text{ and } \sup_{t\in\mathbb{R}} \|f(t)\|_X < \infty\}$$

which is a Banach space endowed with the norm $\|f\|_{\infty,X} = \|f\|_{C_b(\mathbb{R},X)} := \sup\limits_{t\in\mathbb{R}} \|f(t)\|_X$.

Definition 2.1. *(AP-function) A function* $h \in C_b(\mathbb{R}, X)$ *is called almost periodic function if for each* $\epsilon > 0$*, there exists* $l_\epsilon > 0$ *such that every interval of length* l_ϵ *contains at least a number* T *with the following property*

$$\sup_{t\in\mathbb{R}} \|h(t + T) - h(t)\| < \epsilon.$$

The collection of all almost periodic functions $h : \mathbb{R} \to X$ *will be denoted by* $AP(\mathbb{R}, X)$ *which is a Banach space endowed with the norm* $\|h\|_{AP(\mathbb{R},X)} = \sup\limits_{t\in\mathbb{R}} \|h(t)\|_X$*.*

Definition 2.2. *(PAP-function) A function* $f \in C_b(\mathbb{R}, X)$ *is called pseudo almost periodic if it can be decomposed as* $f = g + \phi$ *where* $g \in AP(\mathbb{R}, X)$ *and* ϕ *is a bounded continuous function with vanishing mean value i.e.*

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \|\phi(t)\|_X dt = 0.$$

We denote the set of all functions with vanishing mean value by $PAP_0(\mathbb{R}, X)$ *and the set of all the pseudo almost periodic (PAP-) functions by* $PAP(\mathbb{R}, X)$*.*

We have that $(PAP(\mathbb{R},X),\|.\|_{\infty,X})$ is a Banach space, where $\|.\|_{\infty,X}$ is the supremum norm. As well as AAP- functional space, we have the following decomposition

$$PAP(\mathbb{R},X) = AP(\mathbb{R},X) \oplus PAP_0(\mathbb{R},X).$$

The notion of pseudo almost periodic function is a generalisation of the periodic and almost periodic functions. Precisely, we have the following inclusions

$$P(\mathbb{R},X) \hookrightarrow AP(\mathbb{R},X) \hookrightarrow PAP(\mathbb{R},X) \hookrightarrow C_b(\mathbb{R},X).$$

where $P(\mathbb{R},X)$ is the space of all continuous and periodic functions from $\mathbb{R}$ to X.

Example. The function $h(t) = \cos t + \cos(\sqrt{2}t)$ is almost periodic but not periodic, $\tilde{h}(t) = \cos t + \cos(\sqrt{2}t) + e^{-|t|}$ is pseudo almost periodic but not almost periodic. Moreover, let X be a Banach space and $g \in X - \{0\}$, we have that $f = hg \in AP(\mathbb{R},X)$ and $\tilde{f} = \tilde{h}g \in PAP(\mathbb{R},X)$.

3. Well-posedness of pseudo almost periodic solutions

For a given vector $\begin{bmatrix}\omega\\ \zeta\end{bmatrix}$ we study the following inhomogeneous linear equation corresponding to origin equation (2.2):

$$\frac{\partial}{\partial t}\begin{bmatrix}u\\ v\end{bmatrix} + \mathcal{A}\begin{bmatrix}u\\ v\end{bmatrix} = -\begin{bmatrix}\nabla\cdot(\omega\nabla\zeta)\\ 0\end{bmatrix} + \begin{bmatrix}g(t)\\ u(t)+h(t)\end{bmatrix} \quad (t,x)\in\mathbb{R}\times\mathbb{H}^n. \tag{3.1}$$

By Duhamel's principle we define the mild solution (u,v) of equation (3.1) as the solution of following integral equations

$$u(t) = -\int_{-\infty}^{t} e^{(t-s)\Delta}\nabla\cdot(\omega\nabla\zeta)(s)ds + \int_{-\infty}^{t} e^{(t-s)\Delta}g(s)ds \tag{3.2}$$

and

$$v(t) = \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}u(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}h(s)ds. \tag{3.3}$$

In the matrix form, these equations are equivalent to

$$\begin{bmatrix}u\\ v\end{bmatrix}(t) = \mathcal{B}\left(\begin{bmatrix}\omega\\ \zeta\end{bmatrix}\right)(s) + \mathcal{F}\left(\begin{bmatrix}u\\ 0\end{bmatrix}\right)(s), \tag{3.4}$$

where

$$\mathcal{B}\left(\begin{bmatrix}\omega\\ \zeta\end{bmatrix}\right)(s) = -\int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix}\nabla\cdot(\omega\nabla\zeta)\\ 0\end{bmatrix}(s)ds \tag{3.5}$$

and

$$\mathcal{F}\left(\begin{bmatrix}u\\ 0\end{bmatrix}\right)(s) = \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} F\left(s, \begin{bmatrix}u\\ 0\end{bmatrix}\right) ds = \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix}g\\ u+h\end{bmatrix}(s)ds. \tag{3.6}$$

Let $2 \leqslant n$ and $\max\{3, n\} < p$. We consider the well-posedness of system (3.1) in the following Banach space

$$\begin{aligned}\mathcal{X} \quad = \quad & \Big\{(u, v) \in C_b(\mathbb{R}, L^{\frac{p}{2}}(\mathbb{H}^n) \times L^{\frac{p}{2}}(\mathbb{H}^n)),\ \nabla v \in C_b(\mathbb{R}_+, L^p(\mathbb{H}^n))\\ & \text{such that } : \sup_{t\in\mathbb{R}}\left(\|u(t)\|_{L^{\frac{p}{2}}} + \|v(t)\|_{L^{\frac{p}{2}}} + \|\nabla v(t)\|_{L^p}\right) < +\infty\Big\}\end{aligned}$$

endowed with the norm

$$\|(u, v)\|_{\mathcal{X}} = \sup_{t\in\mathbb{R}}\left(\|u(t)\|_{L^{\frac{p}{2}}} + \|v(t)\|_{L^{\frac{p}{2}}} + \|\nabla v(t)\|_{L^p}\right). \tag{3.7}$$

For convenience, we denote also that

$$\left\|\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}\right\| = \|u(t)\|_{L^{\frac{p}{2}}} + \|v(t)\|_{L^{\frac{p}{2}}} + \|\nabla v(t)\|_{L^p},\ t \in \mathbb{R}. \tag{3.8}$$

Remark 3.1. *Note that, the condition on the dimension of phase space $p > max\{3, n\}$ arises due to technical issues that appear in the proofs of the results. In particular, this condition ensures the applicability of Lemma 2.1 and the boundedness of certain improper integrals, which are crucial for the proofs of Lemma 3.1 and Theorem 3.2 below.*

The existence and uniqueness of mild solution for the linear equation (3.1) in the space $\mathcal{X}$ is established in the following lemma.

Lemma 3.1. *Let $2 \leqslant n$ and $\max\{3, n\} < p$. For given functions $\begin{bmatrix}\omega\\ \zeta\end{bmatrix} \in \mathcal{X}$ and $\begin{bmatrix}g\\ h\end{bmatrix} \in C_b(\mathbb{R}, L^{\frac{p}{2}}(\mathbb{H}^n) \times L^{\frac{p}{2}}(\mathbb{H}^n))$, there exists a unique mild solution of linear equation (3.1) satisfying the integral equation (3.4). Moreover, the following boundedness holds*

$$\left\|\begin{bmatrix}u\\ v\end{bmatrix}\right\|_{\mathcal{X}} \leqslant C_1 \left\|\begin{bmatrix}\omega\\ \zeta\end{bmatrix}\right\|_{\mathcal{X}}^2 + C_2 \left\|\begin{bmatrix}g\\ h\end{bmatrix}\right\|_{\infty, L^{\frac{p}{2}}\times L^{\frac{p}{2}}}. \tag{3.9}$$

Proof. Using the dispersive and smoothing estimates (2.4), (2.6) in Lemma 2.1, we have

$$
\begin{aligned}
&\|u(t)\|_{L^{\frac{p}{2}}} \\
&\leqslant \int_{-\infty}^{t} \left\|e^{(t-s)\Delta}\nabla\cdot(\omega\nabla\zeta)(s)\right\|_{L^{\frac{p}{2}}} ds + \int_{-\infty}^{t} \left\|e^{(t-s)\Delta}g(s)\right\|_{L^{\frac{p}{2}}} ds \\
&\leqslant k_2 \int_{-\infty}^{t} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)\frac{\gamma_{p/2,p/2}+\gamma_{p/3,p/2}}{2})} \|\omega(s)\nabla\zeta(s)\|_{L^{\frac{p}{3}}} ds \\
&+k_1 \int_{-\infty}^{t} e^{-(t-s)(\gamma_{p/2,p/2})} \|g(s)\|_{L^{\frac{p}{2}}} ds \\
&\leqslant k_2 \|\omega\|_{\infty,L^{\frac{p}{2}}} \|\nabla\zeta\|_{\infty,L^p} \int_0^{+\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z\frac{\gamma_{p/2,p/2}+\gamma_{p/3,p/2}}{2}} dz \\
&+k_1 \|g\|_{\infty,L^{\frac{p}{2}}} \int_0^{+\infty} e^{-z(\gamma_{p/2,p/2})} dz
\end{aligned}
$$

$$
\leqslant k_2 C \|\omega\|_{\infty,L^{\frac{p}{2}}} \|\nabla\zeta\|_{\infty,L^p} + \frac{k_1}{\gamma_{p/2,p/2}} \|g\|_{\infty,L^{\frac{p}{2}}}, \tag{3.10}
$$

where $C = \left(\frac{2}{\gamma_{p/3,p/3}+\gamma_{p/3,p/2}} + (\frac{\gamma_{p/2,p/2}+\gamma_{p/3,p/2}}{2})^{-\frac{1}{2}+\frac{n}{2p}} \mathbf{\Gamma}\left(\frac{1}{2}-\frac{n}{2p}\right)\right) < +\infty$ since the boundedness of Gamma function $\mathbf{\Gamma}\left(\frac{1}{2}-\frac{n}{2p}\right)$ provided that $p > n$.

By the same way as above we can estimate

$$
\begin{aligned}
\|v(t)\|_{L^{\frac{p}{2}}} &\leqslant \int_{-\infty}^{t} \left\|e^{(t-s)(\Delta-\mathrm{Id})}u(s)\right\|_{L^{\frac{p}{2}}} ds + \int_{-\infty}^{t} \left\|e^{(t-s)(\Delta-\mathrm{Id})}h(s)\right\|_{L^{\frac{p}{2}}} ds \\
&\leqslant k_1 \int_{-\infty}^{t} e^{-(\gamma_{p/2,p/2}+1)(t-s)} \|u(s)\|_{L^{\frac{p}{2}}} ds \\
&\quad +k_1 \int_{-\infty}^{t} e^{-(\gamma_{p/2,p/2}+1)(t-s)} \|h(s)\|_{L^{\frac{p}{2}}} ds \\
&\leqslant \frac{k_1}{\gamma_{p/2,p/2}+1} \left(\|u\|_{\infty,L^{\frac{p}{2}}} + \|h\|_{\infty,L^{\frac{p}{2}}}\right) \\
&\leqslant \frac{k_1 k_2 C}{\gamma_{p/2,p/2}+1} \|\omega\|_{\infty,L^{\frac{p}{2}}} \|\nabla\zeta\|_{\infty,L^p} \\
&\quad + \max\left\{\frac{k_1^2}{\gamma_{p/2,p/2}(\gamma_{p/2,p/2}+1)}, \frac{k_1}{\gamma_{p/2,p/2}+1}\right\} \left\|\begin{bmatrix} g \\ h \end{bmatrix}\right\|_{\infty,L^{\frac{p}{2}}\times L^{\frac{p}{2}}},
\end{aligned} \tag{3.11}
$$

where the last estimate holds by using (3.10).

Moreover, by using the estimates (2.4) and (2.5) we estimate the gradient of $v(t)$ as

$$
\begin{aligned}
&\|\nabla v(t)\|_{L^p} \\
\leqslant\ & \int_{-\infty}^{t} \left\|\nabla e^{(t-s)(\Delta-\mathrm{Id})}u(s)\right\|_{L^p} ds + \int_{-\infty}^{t} \left\|\nabla e^{(t-s)(\Delta-\mathrm{Id})}h(s)\right\|_{L^p} ds
\end{aligned}
$$

$$\begin{aligned}
&\leqslant k_2 \int_{-\infty}^{t} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(\gamma_{p/2,p}+1)(t-s)} \|u(s)\|_{L^{\frac{p}{2}}} ds \\
&\quad +k_2 \int_{-\infty}^{t} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(\gamma_{p/2,p}+1)(t-s)} \|h(s)\|_{L^{\frac{p}{2}}} ds \\
&\leqslant k_2 \left(\frac{1}{\gamma_{p/2,p}+1} + (\gamma_{p/2,p}+1)^{\frac{1}{2}-\frac{n}{2p}} \mathbf{\Gamma}\left(\frac{1}{2}-\frac{n}{2p}\right)\right) \left(\|u\|_{\infty, L^{\frac{p}{2}}} + \|h\|_{\infty, L^{\frac{p}{2}}}\right) \\
&\leqslant k_2 C' \left(\|u\|_{\infty, L^{\frac{p}{2}}} + \|h\|_{\infty, L^{\frac{p}{2}}}\right) \\
&\quad (\text{where } C' = \left(\frac{1}{\gamma_{p/2,p}+1} + (\gamma_{p/2,p}+1)^{-\frac{1}{2}+\frac{n}{2p}} \mathbf{\Gamma}\left(\frac{1}{2}-\frac{n}{2p}\right)\right)) \\
&\leqslant k_2^2 C' C \|\omega\|_{\infty, L^{\frac{p}{2}}} \|\nabla\zeta\|_{\infty, L^p} \\
&\quad + \max\left\{\frac{k_1 k_2 C'}{\gamma_{p/2,p}+1}, k_2 C'\right\} \left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}},
\end{aligned} \tag{3.12}$$

where the last estimate holds by using again (3.10).

Combining inequalities (3.10), (3.11) and (3.12) we obtain the boundedness (3.9) which leads to the existence of mild solution of (3.1). The uniqueness holds clearly. ■

As a direct consequence of Lemma 3.1, we can define the solution operator $S : \mathcal{X} \to \mathcal{X}$ associating with linear equation (3.1) as

$$\begin{aligned}
S : \mathcal{X} &\to \mathcal{X} \\
\begin{bmatrix} \omega \\ \zeta \end{bmatrix} &\mapsto S\left(\begin{bmatrix} \omega \\ \zeta \end{bmatrix}\right),
\end{aligned}$$

where $S(\omega, \zeta)$ is a unique solution of integral equation (3.4), i.e, mild solution of (3.1).

We state and prove the existence and uniqueness of PAP-mild solutions for linear equation (3.1) in the following theorem.

Theorem 3.1. *Let* $2 \leqslant n$ *and* $\max\{3, n\} < p$. *For a given PAP-function* $t \mapsto (\omega, \zeta, g, h)(t)$ *with respect on the norm*

$$\|(\omega, \zeta, g, h)(t)\| = \|\omega(t)\|_{L^{\frac{p}{2}}} + \|\zeta(t)\|_{L^{\frac{p}{2}}} + \|\nabla\zeta(t)\|_{L^p} + \|g(t)|_{L^{\frac{p}{2}}} + \|h(t)\|_{L^{\frac{p}{2}}}, \tag{3.13}$$

there exists a unique PAP-mild solution of linear equation (3.1) *satisfying the integral equation* (3.4).

Proof. This theorem is in fact a Massera-type principle for PAP-mild solutions of parabolic-parabolic Keller-Segel systems (see the similar subjects for parabolic-elliptic Keller-Segel systems in [37, 38, 39]). In particular, we need to prove that the solution operator S preserves the pseudo almost periodic property of functions $\begin{bmatrix}\omega\\ \zeta\end{bmatrix}$ and $\begin{bmatrix}g\\ h\end{bmatrix}$. Indeed, we have

$$S\left(\begin{bmatrix}\omega\\ \zeta\end{bmatrix}\right)=\begin{bmatrix}u\\ v\end{bmatrix} \tag{3.14}$$

which is a unique solution of integral equation (3.1) which means that the functions u and v satisfy equations (3.2) and (3.3), respectively.

By hypothesis, we can decompose

$$\begin{bmatrix}\omega\\ \zeta\\ g\\ h\end{bmatrix}=\begin{bmatrix}\omega_1\\ \zeta_1\\ g_1\\ h_1\end{bmatrix}+\begin{bmatrix}\omega_2\\ \zeta_2\\ g_2\\ h_2\end{bmatrix} \tag{3.15}$$

for $(\omega_1,\zeta_1,g_1,h_1)$ is almost periodic in $C_b(\mathbb{R},L^{\frac{p}{2}}(\Omega)\times L^{\frac{p}{2}}(\mathbb{H}^n))\times L^{\frac{p}{2}}(\mathbb{H}^n))\times L^{\frac{p}{2}}(\mathbb{H}^n)))$ with respect to the norm

$$\left|\left|\left|\begin{bmatrix}\omega_1\\ \zeta_1\\ g_1\\ h_1\end{bmatrix}(t)\right|\right|\right|=\|\omega_1(t)\|_{L^{\frac{p}{2}}}+\|\zeta_1(t)\|_{L^{\frac{p}{2}}}+\|\nabla\zeta_1(t)\|_{L^p}+\|g_1(t)\|_{L^{\frac{p}{2}}}+\|h_1(t)\|_{L^{\frac{p}{2}}},\ t\in\mathbb{R} \tag{3.16}$$

and $(\omega_2,\zeta_2,g_2,h_2)$ satisfying

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\left|\left|\left|\begin{bmatrix}\omega_2\\ \zeta_2\\ g_2\\ h_2\end{bmatrix}(t)\right|\right|\right|dt=0.$$

Inserting (3.15) into (3.2), we obtain

$$\begin{aligned}
u(t) &= -\int_{-\infty}^{t}e^{(t-s)\Delta}\nabla\cdot[(\omega_1+\omega_2)\nabla(\zeta_1+\zeta_2)](s)ds\\
&\quad+\int_{-\infty}^{t}e^{(t-s)\Delta}(g_1+g_2)(s)ds\\
&= -\int_{-\infty}^{t}e^{(t-s)\Delta}\nabla\cdot(\omega_1\nabla\zeta_1)(s)ds+\int_{-\infty}^{t}e^{(t-s)\Delta}g_1(s)ds\\
&\quad-\int_{-\infty}^{t}e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)ds+\int_{-\infty}^{t}e^{(t-s)\Delta}g_2(s)ds\\
&= u_1(t)+u_2(t), \qquad (3.17)
\end{aligned}$$

where

$$u_1(t) = -\int_{-\infty}^{t}e^{(t-s)\Delta}\nabla\cdot(\omega_1\nabla\zeta_1)(s)ds+\int_{-\infty}^{t}e^{(t-s)\Delta}g_1(s)ds \tag{3.18}$$

and

$$u_2(t) = -\int_{-\infty}^{t} e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)ds + \int_{-\infty}^{t} e^{(t-s)\Delta}g_2(s)ds.$$

Inserting (3.15) and (3.17) into (3.3), we get

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}(u_1+u_2)(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}(h_1+h_2)(s)ds \\ &= \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}u_1(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}h_1(s)ds \\ &\quad + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}u_2(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}h_2(s)ds \\ &= v_1(t)+v_2(t), \end{aligned} \tag{3.20}$$

where

$$v_1(t) = \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}u_1(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}h_1(s)ds \tag{3.21}$$

and

$$v_2(t) = \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}u_2(s)ds + \int_{-\infty}^{t} e^{(t-s)(\Delta-\mathrm{Id})}h_2(s)ds. \tag{3.22}$$

Now, we prove that the function $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ is almost periodic in $\mathcal{X}$ with repsect to the norm

$$\left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}(t) \right\| = \|u_1(t)\|_{L^{\frac{p}{2}}} + \|v_1(t)\|_{L^{\frac{p}{2}}} + \|\nabla v_1(t)\|_{L^p},\ t\in\mathbb{R} \tag{3.23}$$

and the function $\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$ satisfying that

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} \left\| \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}(t) \right\| dt = 0. \tag{3.24}$$

Combining these with equation (3.14), we complete the proof. In particular, we establish the above properties by the following two steps:
Step 1: Since the function $(\omega_1, \zeta_1, g_1, h_1)$ is almost periodic in $C_b(\mathbb{R}, L^{\frac{p}{2}}(\mathbb{H}^n))\times L^{\frac{p}{2}}(\mathbb{H}^n))\times L^{\frac{p}{2}}(\mathbb{H}^n))\times L^{\frac{p}{2}}(\mathbb{H}^n)))$ with respect to the norm (3.16), we have that: for each $\epsilon>0$, there exists $l_\epsilon>0$ such that every interval of length l_ϵ contains at least a number T satisfying

$$\|\omega_1(t+T)-\omega_1(t)\|_{L^{\frac{p}{2}}} + \|\zeta_1(t+T)-\zeta_1(t)\|_{L^{\frac{p}{2}}} + \|\nabla\zeta_1(t+T)-\nabla\zeta_1(t)\|_{L^p}$$

$$+\|g_1(t+T)-g_1(t)\|_{L^{\frac{p}{2}}}+\|h_1(t+T)-h_1(t)\|_{L^{\frac{p}{2}}}<\epsilon,\ t\in\mathbb{R}. \tag{3.25}$$

Using (3.25) and the similar estimates as (3.10) in the proof of Lemma 3.1 we can obtain that

$$\begin{aligned}
&\|u_1(t+T)-u_1(t)\|_{L^{\frac{p}{2}}}\\
\leqslant\ & \int_{-\infty}^{t}\left\|e^{(t-s)\Delta}\nabla\cdot[(\omega_1\nabla\zeta_1)(s+T)-(\omega_1\nabla\zeta_1)(s)]\right\|_{L^{\frac{p}{2}}}ds\\
&+\int_{-\infty}^{t}\left\|e^{(t-s)\Delta}(g_1(s+T)-g_1(s))\right\|_{L^{\frac{p}{2}}}ds\\
\leqslant\ & k_2\int_{-\infty}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(t-s)(\gamma_{p/3,p/2})}\\
&\times\Big(\|(\omega_1(s+T)-\omega_1(s))\nabla\zeta(s+T)\|_{L^{\frac{p}{3}}}\\
&+\|\omega_1(s)\nabla(\zeta(s+T)-\zeta_1(s))\|_{L^{\frac{p}{3}}}\Big)ds\\
&+k_1\int_{-\infty}^{t}e^{-(t-s)(\gamma_{p/2,p/2})}\|g_1(s+T)-g_1(s)\|_{L^{\frac{p}{2}}}ds\\
\leqslant\ & k_2\|\omega_1(\cdot+T)-\omega_1(\cdot)\|_{\infty,L^{\frac{p}{2}}}\|\nabla\zeta_1\|_{\infty,L^p}\\
&\times\int_0^{+\infty}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(t-s)(\gamma_{p/3,p/2})}dz\\
&+k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\|\nabla\zeta_1(\cdot+T)-\nabla\zeta_1(\cdot)\|_{\infty,L^p}\\
&\times\int_0^{+\infty}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(t-s)(\gamma_{p/3,p/2})}dz\\
&+k_1\|g_1(\cdot+T)-g_1(\cdot)\|_{\infty,L^{\frac{p}{2}}}\int_0^{+\infty}e^{-(t-s)(\gamma_{p/2,p/2})}dz\\
\leqslant\ & k_2C\left(\|\omega_1\|_{\infty,L^{\frac{p}{2}}}+\|\nabla\zeta_1\|_{\infty,L^p}+\frac{k_1}{\lambda_1}\right)\epsilon,
\end{aligned}\tag{3.26}$$

where the constant C is given as in the proof of Lemma 3.1. By the same way as (3.11) in the proof of Lemma 3.1 we have

$$\begin{aligned}
&\|v_1(t+T)-v_1(t)\|_{L^{\frac{p}{2}}}\\
\leqslant\ & \left\|\int_{-\infty}^{t+T}e^{(t+T-s)(\Delta-\mathrm{Id})}u_1(s)-\int_{-\infty}^{t}e^{(t-s)(\Delta-\mathrm{Id})}u_1(s)\right\|_{L^{\frac{p}{2}}}ds\\
&+\left\|\int_{-\infty}^{t+T}e^{(t+T-s)(\Delta-\mathrm{Id})}h_1(s)ds-\int_{-\infty}^{t}e^{(t-s)(\Delta-\mathrm{Id})}h_1(s)ds\right\|_{L^{\frac{p}{2}}}\\
\leqslant\ & \left\|\int_{-\infty}^{t}e^{(t-s)(\Delta-\mathrm{Id})}(u_1(s+T)-u_1(s))\right\|_{L^{\frac{p}{2}}}ds\\
&+\left\|\int_{-\infty}^{t}e^{(t-s)(\Delta-\mathrm{Id})}(h_1(s+T)-h_1(s))ds\right\|_{L^{\frac{p}{2}}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant k_1 \int_{-\infty}^t e^{-(\lambda_1+1)(t-s)} \|u_1(s+T) - u_1(s)\|_{L^{\frac{p}{2}}} ds \\
&\quad + k_1 \int_{-\infty}^t e^{-(\lambda_1+1)(t-s)} \|h_1(s+T) - h_1(s)\|_{L^{\frac{p}{2}}} ds \\
&\leqslant \frac{k_1}{\lambda_1+1} \left(\|u_1(\cdot+T) - u_1(\cdot)\|_{\infty, L^{\frac{p}{2}}} + \|h_1(\cdot+T) - h_1(\cdot)\|_{\infty, L^{\frac{p}{2}}} \right) \\
&\leqslant \left\{ \frac{k_1 k_2 C}{\lambda_1+1} \left(\|\omega_1\|_{\infty, L^{\frac{p}{2}}} + \|\nabla\zeta_1\|_{\infty, L^p} \right) + \frac{k_1}{\lambda_1+1} \right\} \epsilon, \qquad (3.27)
\end{aligned}$$

where the last holds by using estimation (3.26) and inequality (3.25). Moreover, we also have that

$$\begin{aligned}
&\|\nabla v_1(t+T) - \nabla v_1(t)\|_{L^p} \\
&\leqslant \int_{-\infty}^t \left\| \nabla e^{(t-s)(\Delta - \mathrm{Id})} (u_1(s+T) - u_1(s)) \right\|_{L^p} ds \\
&+ \int_{-\infty}^t \left\| \nabla e^{(t-s)(\Delta - \mathrm{Id})} (h_1(s+T) - h_1(s)) \right\|_{L^p} ds \\
&\leqslant k_2 \int_{-\infty}^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p}} \right) e^{-(\lambda_1+1)(t-s)} \|u_1(s+T) - u_1(s)\|_{L^{\frac{p}{2}}} ds \\
&+ k_2 \int_{-\infty}^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p}} \right) e^{-(\lambda_1+1)(t-s)} \|h_1(s+T) - h_1(s)\|_{L^{\frac{p}{2}}} ds \\
&\leqslant k_2 C'' \left(\|u_1(\cdot+T) - u_1(\cdot)\|_{\infty, L^{\frac{p}{2}}} + \|h_1(\cdot+T) - h_1(\cdot)\|_{\infty, L^{\frac{p}{2}}} \right) \\
&\leqslant \left\{ k_2^2 C C'' \left(\|\omega_1\|_{\infty, L^{\frac{p}{2}}} + \|\nabla\zeta_1\|_{\infty, L^p} + \frac{k_1}{\lambda_1} \right) + k_2 C'' \right\} \epsilon. \qquad (3.28)
\end{aligned}$$

Combining estimations (3.26), (3.27) and (3.28) we get

$$\left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} (t+T) - \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} (t) \right\| \leqslant \widetilde{C} \epsilon, \; t \in \mathbb{R}, \qquad (3.29)$$

where

$$\begin{aligned}
\widetilde{C} &= k_2 C \left(\|\omega_1\|_{\infty, L^{\frac{p}{2}}} + \|\nabla\zeta_1\|_{\infty, L^p} + \frac{k_1}{\lambda_1} \right) \\
&\quad + \frac{k_1 k_2 C}{\lambda_1+1} \left(\|\omega_1\|_{\infty, L^{\frac{p}{2}}} + \|\nabla\zeta_1\|_{\infty, L^p} \right) + \frac{k_1}{\lambda_1+1} \\
&\quad + k_2^2 C C'' \left(\|\omega_1\|_{\infty, L^{\frac{p}{2}}} + \|\nabla\zeta_1\|_{\infty, L^p} + \frac{k_1}{\lambda_1} \right) + k_2 C''. \qquad (3.30)
\end{aligned}$$

Therefore, the function $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ is almost periodic in $\mathcal{X}$.

Step 2: We remain to prove the limit (3.24) which is equivalent to

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^L \left(\|u_2(t)\|_{L^{\frac{p}{2}}} + \|v_2(t)\|_{L^{\frac{p}{2}}} + \|\nabla v_2(t)\|_{L^p} \right) dt = 0. \qquad (3.31)$$

Below, we prove in detail that

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\|u_2(t)\|_{L^{\frac{p}{2}}}dt=0. \tag{3.32}$$

The same limits of $v_2(t)$ and $\nabla v_2(t)$ hold by the same way amd we obtain the limit (3.24). Now, by using formula (3.32) we have

$$\begin{aligned}
&\|u_2(t)\|_{L^{\frac{p}{2}}}\\
&\leqslant \int_{-\infty}^{t}\left\|e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)\right\|_{L^{\frac{p}{2}}}ds\\
&+\int_{-\infty}^{t}\left\|e^{(t-s)\Delta}g_2(s)\right\|_{L^{\frac{p}{2}}}ds\\
&=\int_{-\infty}^{-L}\left(\left\|e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)\right\|_{L^{\frac{p}{2}}}+\left\|e^{(t-s)\Delta}g_2(s)\right\|_{L^{\frac{p}{2}}}\right)ds\\
&+\int_{-L}^{t}\left(\left\|e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)\right\|_{L^{\frac{p}{2}}}+\left\|e^{(t-s)\Delta}g_2(s)\right\|_{L^{\frac{p}{2}}}\right)ds\\
&=\psi(t)+\varphi(t), \qquad (3.33)
\end{aligned}$$

where

$$\begin{aligned}
\varphi(t) &= \int_{-L}^{t}\left(\left\|e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)\right\|_{L^{\frac{p}{2}}}+\left\|e^{(t-s)\Delta}g_2(s)\right\|_{L^{\frac{p}{2}}}\right)ds\\
\psi(t) &= \int_{-\infty}^{-L}\left(\left\|e^{(t-s)\Delta}\nabla\cdot[\omega_1\nabla\zeta_2+\omega_2\nabla\zeta](s)\right\|_{L^{\frac{p}{2}}}+\left\|e^{(t-s)\Delta}g_2(s)\right\|_{L^{\frac{p}{2}}}\right)ds
\end{aligned}$$

under the assumption that $\omega_2,\zeta_2,g_2\in PAP_0(\mathbb{R},L^{\frac{p}{2}}(\mathbb{H}^n))$. Indeed, using arguments analogous to those employed in Lemma 3.1, we obtain the following estimate

$$\begin{aligned}
&\varphi(t)\\
\leqslant\ & k_2\int_{-L}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)\\
&\times e^{-(t-s)(\gamma_{p/3,p/2})}\left(\|(\omega_1(s))\nabla\zeta_2(s)\|_{L^{\frac{p}{3}}}+\|\omega_2(s)\nabla(\zeta(s))\|_{L^{\frac{p}{3}}}\right)ds\\
&+k_1\int_{-L}^{t}e^{-(t-s)(\gamma_{p/2,p/2})}\|g_2(s)\|_{L^{\frac{p}{2}}}ds\\
\leqslant\ & k_2\int_{-L}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)\\
&\times e^{-(t-s)(\gamma_{p/3,p/2})}\left(\|(\omega_1(s))\|_{L^{\frac{p}{2}}}\|(\nabla\zeta_2(s)\|_{L^p}+\|\omega_2(s)\|_{L^{\frac{p}{2}}}\|\nabla(\zeta(s))\|_{L^p}\right)ds\\
&+k_1\int_{-L}^{t}e^{-(t-s)(\gamma_{p/2,p/2})}\|g_2(s)\|_{L^{\frac{p}{2}}}ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\int_{-L}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(t-s)(\gamma_{p/3,p/2})}\left\|(\nabla\zeta_2(s)\right\|_{L^p}ds\\
&\quad+k_2\left\|\nabla\zeta\right\|_{\infty,L^p}\int_{-L}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(t-s)(\gamma_{p/3,p/2})}\|\omega_2(s)\|_{L^{\frac{p}{2}}}ds\\
&\quad+k_1\int_{-L}^{t}e^{-(t-s)(\gamma_{p/2,p/2})}\|g_2(s)\|_{L^{\frac{p}{2}}}ds\\
&\leqslant k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\int_{0}^{t+L}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\left\|(\nabla\zeta_2(t-z)\right\|_{L^p}dz\\
&\quad+k_2\left\|\nabla\zeta\right\|_{\infty,L^p}\int_{0}^{t+L}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\|\omega_2(t-z)\|_{L^{\frac{p}{2}}}dz\\
&\quad+k_1\int_{0}^{t+L}e^{-z(\gamma_{p/2,p/2})}\|g_2(t-z)\|_{L^{\frac{p}{2}}}dz.
\end{aligned}
$$

Consequently, we estimate

$$
\begin{aligned}
0 &\leqslant \frac{1}{2L}\int_{-L}^{L}\varphi(t)dt\\
&\leqslant k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\frac{1}{2L}\int_{-L}^{L}\left(\int_{0}^{t+L}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\left\|(\nabla\zeta_2(t-z)\right\|_{L^p}dz\right)dt\\
&\quad+k_2\left\|\nabla\zeta\right\|_{\infty,L^p}\frac{1}{2L}\int_{-L}^{L}\left(\int_{0}^{t+L}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\|\omega_2(t-z)\|_{L^{\frac{p}{2}}}dz\right)dt\\
&\quad+k_1\frac{1}{2L}\int_{-L}^{L}\left(\int_{0}^{t+L}e^{-z(\gamma_{p/2,p/2})}\|g_2(t-z)\|_{L^{\frac{p}{2}}}dz\right)dt\\
&= k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\frac{1}{2L}\int_{0}^{2L}\left(\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\int_{z-L}^{L}\left\|(\nabla\zeta_2(t-z)\right\|_{L^p}dt\right)dz\\
&\quad+k_2\left\|\nabla\zeta\right\|_{\infty,L^p}\frac{1}{2L}\int_{0}^{2L}\left(\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\int_{z-L}^{L}\|\omega_2(t-z)\|_{L^{\frac{p}{2}}}dt\right)dz\\
&\quad+k_1\frac{1}{2L}\int_{0}^{2L}\left(e^{-z(\gamma_{p/2,p/2})}\int_{z-L}^{L}\|g_2(t-z)\|_{L^{\frac{p}{2}}}dt\right)dz\\
&= k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\int_{0}^{2L}\left(\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\frac{1}{2L}\int_{-L}^{L-z}\left\|(\nabla\zeta_2(\tau))\right\|_{L^p}d\tau\right)dz\\
&\quad+k_2\left\|\nabla\zeta\right\|_{\infty,L^p}\int_{0}^{2L}\left(\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}\frac{1}{2L}\int_{-L}^{L-z}\|\omega_2(\tau))\|_{L^{\frac{p}{2}}}d\tau\right)dz\\
&\quad+k_1\int_{0}^{2L}\left(e^{-z(\gamma_{p/2,p/2})}\frac{1}{2L}\int_{-L}^{L-z}\|g_2(\tau))\|_{L^{\frac{p}{2}}}d\tau\right)dz\\
&\leqslant k_2\|\omega_1\|_{\infty,L^{\frac{p}{2}}}\int_{0}^{\infty}\left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-z(\gamma_{p/3,p/2})}dz\frac{1}{2L}\int_{-L}^{L}\left\|(\nabla\zeta_2(\tau))\right\|_{L^p}d\tau
\end{aligned}
$$

$$
\begin{aligned}
& +k_2 \left\|\nabla \zeta\right\|_{\infty, L^p} \int_0^{\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z(\gamma_{p/3,p/2})} dz \frac{1}{2L} \int_{-L}^{L} \left\|\omega_2(\tau))\right\|_{L^{\frac{p}{2}}} d\tau \\
& +k_1 \int_0^{\infty} e^{-z(\gamma_{p/2,p/2})} dz \frac{1}{2L} \int_{-L}^{L} \left\|g_2(\tau))\right\|_{L^{\frac{p}{2}}} d\tau \\
= \; & k_2 \|\omega_1\|_{\infty, L^{\frac{p}{2}}} C \frac{1}{2L} \int_{-L}^{L} \left\|(\nabla \zeta_2(\tau))\right\|_{L^p} d\tau \\
& +k_2 \left\|\nabla \zeta\right\|_{\infty, L^p} C \frac{1}{2L} \int_{-L}^{L} \left\|\omega_2(\tau))\right\|_{L^{\frac{p}{2}}} d\tau + \frac{k_1}{\gamma_{p/2,p/2}} \frac{1}{2L} \int_{-L}^{L} \left\|g_2(\tau))\right\|_{L^{\frac{p}{2}}} d\tau.
\end{aligned}
$$

Since $\omega_2, \zeta_2, g_2 \in PAP_0(\mathbb{R}, L^{\frac{p}{2}}(\mathbb{H}^n))$ it implies that

$$
\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \varphi(t) dt = 0 \;\; \forall s > 0. \tag{3.34}
$$

Using arguments analogous to those employed in Lemma 3.1 again, we have

$$
\begin{aligned}
\psi(t) \;\leqslant\; & k_2 \int_{-\infty}^{-L} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)(\gamma_{p/3,p/2})} \\
& \times \left(\left\|(\omega_1(s))\nabla \zeta_2(s)\right\|_{L^{\frac{p}{3}}} + \left\|\omega_2(s) \nabla(\zeta(s))\right\|_{L^{\frac{p}{3}}} \right) ds \\
& +k_1 \int_{-\infty}^{-L} e^{-(t-s)(\gamma_{p/2,p/2})} \|g_2(s)\|_{L^{\frac{p}{2}}} ds \\
\leqslant\; & k_2 \|\omega_1\|_{\infty, L^{\frac{p}{2}}} \int_{t+L}^{\infty} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)(\gamma_{p/3,p/2})} \left\|(\nabla \zeta_2(s)\right\|_{L^p} ds \\
& +k_2 \left\|\nabla \zeta\right\|_{\infty, L^p} \int_{t+L}^{\infty} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)(\gamma_{p/3,p/2})} \|\omega_2(s)\|_{L^{\frac{p}{2}}} ds \\
& +k_1 \int_{t+L}^{\infty} e^{-(t-s)(\gamma_{p/2,p/2})} \|g_2(s)\|_{L^{\frac{p}{2}}} ds \\
\leqslant\; & k_2 \|\omega_1\|_{\infty, L^{\frac{p}{2}}} \left\|\nabla \zeta_2\right\|_{\infty, L^p} \int_{2L}^{\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z(\gamma_{p/3,p/2})} dz \\
& +k_2 \left\|\nabla \zeta\right\|_{\infty, L^p} \|\omega_2\|_{\infty, L^{\frac{p}{2}}} \int_{2L}^{\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z(\gamma_{p/3,p/2})} dz \\
& +k_1 \|g_2\|_{\infty, L^{\frac{p}{2}}} \int_{2L}^{\infty} e^{-z(\gamma_{p/2,p/2})} dz.
\end{aligned}
$$

Hence, we clearly have

$$
\begin{aligned}
0 \;\leqslant\; & \frac{1}{2L} \int_{-L}^{L} \psi(t) dt \\
\leqslant\; & k_2 \|\omega_1\|_{L^{\infty, \frac{p}{2}}} \left\|\nabla \zeta_2\right\|_{\infty, L^p} \frac{1}{2L} \int_{-L}^{L} \left(\int_{2L}^{\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z(\gamma_{p/3,p/2})} dz \right) dt \\
& +k_2 \left\|\nabla \zeta\right\|_{\infty, L^p} \|\omega_2\|_{\infty, L^{\frac{p}{2}}} \frac{1}{2L} \int_{-L}^{L} \left(\int_{2L}^{\infty} \left(1+z^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-z(\gamma_{p/3,p/2})} dz \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&+k_1\|g_2\|_{\infty,L^{\frac{p}{2}}}\frac{1}{2L}\int_{-L}^{L}\left(\int_{2L}^{\infty}e^{-z(\gamma_{p/2,p/2})}dz\right)dt\\
\leqslant\ & k_2\|\omega_1\|_{L^{\infty},\frac{p}{2}}\|\nabla\zeta_2\|_{\infty,L^p}\left(1+(2L)^{-\frac{1}{2}-\frac{n}{2p}}\right)\frac{e^{-2L(\gamma_{p/3,p/2})}}{\gamma_{p/3,p/2}}\\
&+k_2\|\nabla\zeta\|_{\infty,L^p}\|\omega_2\|_{\infty,L^{\frac{p}{2}}}\left(1+(2L)^{-\frac{1}{2}-\frac{n}{2p}}\right)\frac{e^{-2L(\gamma_{p/3,p/2})}}{\gamma_{p/3,p/2}}\\
&+k_1\|g_2\|_{\infty,L^{\frac{p}{2}}}\frac{e^{-2L(\gamma_{p/2,p/2})}}{\gamma_{p/2,p/2}}.
\end{aligned}
$$

This leads to the limit

$$
\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\psi(t)dt=0. \tag{3.35}
$$

Combining equations (3.34) and (3.35), we conclude the desired limit (3.32). A similar argument applies to the remaining terms on the right-hand side of (3.31). Our proof is complete. ■

Similar to the linear system, we define the mild solutions of semi-linear system (2.1) by the solutions of the following integral equations

$$
u(t)=-\int_{-\infty}^{t}e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds+\int_{0}^{t}e^{(t-s)\Delta}g(s)ds \tag{3.36}
$$

and

$$
v(t)=\int_{-\infty}^{t}e^{(t-s)(\Delta-\mathrm{Id})}u(s)ds+\int_{0}^{t}e^{(t-s)(\Delta-\mathrm{Id})}h(s)ds \tag{3.37}
$$

for u satisfies (3.2). In the matrix form, these equations are equivalent to

$$
\begin{bmatrix}u\\v\end{bmatrix}(t)=\mathcal{B}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)(s)+\mathcal{F}\left(\begin{bmatrix}u\\0\end{bmatrix}\right)(s), \tag{3.38}
$$

where

$$
\mathcal{B}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)(s)=-\int_{-\infty}^{t}e^{-(t-s)\mathcal{A}}\begin{bmatrix}\nabla\cdot(u\nabla v)\\0\end{bmatrix}(s)ds \tag{3.39}
$$

and

$$
\mathcal{F}\left(\begin{bmatrix}u\\0\end{bmatrix}\right)(s)=\int_{-\infty}^{t}e^{-(t-s)\mathcal{A}}F\left(s,\begin{bmatrix}u\\0\end{bmatrix}\right)ds=\int_{-\infty}^{t}e^{-(t-s)\mathcal{A}}\begin{bmatrix}g\\u+h\end{bmatrix}(s)ds. \tag{3.40}
$$

We state and prove the main results of this section in the following theorem.

Theorem 3.2. *Let $2 \leqslant n$ and $\max\{3,n\} < p$. For a given PAP-function $\begin{bmatrix} g \\ h \end{bmatrix} \in C_b(\mathbb{R}, L^{\frac{p}{2}}(\Omega) \times L^{\frac{p}{2}}(\Omega))$. If the norm $\left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}}$ is small enough, then there exists a unique PAP-mild solution $(\hat{u}, \hat{v})$ of Keller-Segel (P-P) system* (2.1) *satisfying integral equations* (3.36) *and* (3.37)*.*

Proof. We denote B_ρ^{PAP} which is a ball centered at 0 and radius $\rho > 0$ and consists of all PAP-function in $\mathcal{X}$. For each given function $\begin{bmatrix} \omega \\ \zeta \end{bmatrix} \in B_\rho^{AAP}$ we consider the following linear integral equation

$$\begin{bmatrix} u \\ v \end{bmatrix}(t) = \mathcal{B}\left(\begin{bmatrix} \omega \\ \zeta \end{bmatrix}\right)(s) + \mathcal{F}\left(\begin{bmatrix} u \\ 0 \end{bmatrix}\right)(s). \tag{3.41}$$

By Theorem 3.1, this integral has a unique solution $\begin{bmatrix} u \\ v \end{bmatrix}$ and we can set a mapping $\Phi : B_\rho^{PAP} \to B_\rho^{PAP}$ by

$$\Phi\left(\begin{bmatrix} \omega \\ \zeta \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \tag{3.42}$$

which is a unique solution of (3.41). Now, we prove that Φ maps B_ρ^{PAP} into itself and is contraction. Indeed, by using Lemma 3.1 we have

$$\begin{aligned} \left\| \Phi\left(\begin{bmatrix} \omega \\ \zeta \end{bmatrix}\right)(t) \right\| &\leqslant C_1 \left\| \begin{bmatrix} \omega \\ \zeta \end{bmatrix} \right\|_{\mathcal{X}}^2 + C_2 \left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}} \\ &\leqslant C_1 \rho^2 + C_2 \left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}} \\ &\leqslant \rho \end{aligned} \tag{3.43}$$

provided that ρ and $\left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}}$ are small enough. Hence, the mapping Φ map B_ρ^{PAP} into itself.

Now, for given functions $\begin{bmatrix} \omega_1 \\ \zeta_1 \end{bmatrix}, \begin{bmatrix} \omega_2 \\ \zeta_2 \end{bmatrix} \in B_\rho^{PAP}$ we have

$$\begin{aligned} & \Phi\left(\begin{bmatrix} \omega_1 \\ \zeta_1 \end{bmatrix}\right) - \Phi\left(\begin{bmatrix} \omega_2 \\ \zeta_2 \end{bmatrix}\right) \\ = \; & \mathcal{B}\left(\begin{bmatrix} \omega_1 \\ \zeta_1 \end{bmatrix}\right)(s) - \mathcal{B}\left(\begin{bmatrix} \omega_2 \\ \zeta_2 \end{bmatrix}\right)(s) \\ & + \mathcal{F}\left(\begin{bmatrix} u_1 \\ 0 \end{bmatrix}\right)(s) - \mathcal{F}\left(\begin{bmatrix} u_2 \\ 0 \end{bmatrix}\right)(s) \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} \nabla\cdot(-\omega_1\nabla\zeta_1+\omega_2\nabla\zeta_2) \\ 0 \end{bmatrix}(s)ds \\
&\quad + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} 0 \\ u_1-u_2 \end{bmatrix}(s)ds \\
&= \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} \nabla\cdot[-(\omega_1-\omega_2)\nabla\zeta_1+\omega_2\nabla(-\zeta_1+\zeta_2)] \\ 0 \end{bmatrix}(s)ds \\
&\quad + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} 0 \\ u_1-u_2 \end{bmatrix}(s)ds \\
&= \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} \nabla\cdot[-(\omega_1-\omega_2)\nabla\zeta_1] \\ 0 \end{bmatrix}(s)ds \\
&\quad + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} \nabla\cdot[\omega_2\nabla(-\zeta_1+\zeta_2)] \\ 0 \end{bmatrix}(s)ds \\
&\quad + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} \begin{bmatrix} 0 \\ u_1-u_2 \end{bmatrix}(s)ds, \qquad (3.44)
\end{aligned}
$$

where we set

$$
\Phi\left(\begin{bmatrix}\omega_1\\ \zeta_1\end{bmatrix}\right)=\begin{bmatrix}u_1\\ v_1\end{bmatrix} \text{ and } \Phi\left(\begin{bmatrix}\omega_2\\ \zeta_2\end{bmatrix}\right)=\begin{bmatrix}u_2\\ v_2\end{bmatrix}.
$$

By the same way as in the proof of Lemma 3.1 we can estimate from (3.44) that

$$
\begin{aligned}
\left\|\Phi\left(\begin{bmatrix}\omega_1\\ \zeta_1\end{bmatrix}\right)-\Phi\left(\begin{bmatrix}\omega_2\\ \zeta_2\end{bmatrix}\right)\right\|_{\mathcal{X}} &\leqslant 2C_1\left(\left\|\begin{bmatrix}\omega_1\\ \zeta_1\end{bmatrix}+\begin{bmatrix}\omega_2\\ \zeta_2\end{bmatrix}\right\|_{\mathcal{X}}\right)\left(\left\|\begin{bmatrix}\omega_1\\ \zeta_1\end{bmatrix}-\begin{bmatrix}\omega_2\\ \zeta_2\end{bmatrix}\right\|_{\mathcal{X}}\right) \\
&\leqslant 4C_1\rho\left(\left\|\begin{bmatrix}\omega_1\\ \zeta_1\end{bmatrix}-\begin{bmatrix}\omega_2\\ \zeta_2\end{bmatrix}\right\|_{\mathcal{X}}\right). \qquad (3.45)
\end{aligned}
$$

Hence, the mapping Φ is contraction provided that $\rho < \dfrac{1}{4C_1}$.

By using fixed point arguments, there is a fixed point $\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}$ of Φ which is clearly solution of integral equation (3.38). Therefore, the Keller-Segel system (2.1) has a mild solution. The uniqueness holds by using core estimate (3.45).

■

Remark 3.2. *In this paper, we prove the existence and uniqueness of pseudo almost periodic mild solutions for the parabolic–parabolic Keller–Segel system* (1.1). *Their exponential stability follows from earlier results (see [34, Theorem 3.2*(*ii*)*, Theorem 4.3*(*iii*)*]), and the remaining details are left to the reader.*

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References

[1] **M. Ahmedou**, **T. Bartsch** and **Z. Hu**, *Blow-up solutions for the steady state of the Keller-Segel system on Riemann surfaces*, preprint (2025) arXiv:2409.00519

[2] **A. Blanchet**, **J. Dolbeault** and **B. Perthame**, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differ. Eq., 2006 (2006), 1–33.

[3] **P. Biler**, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), 715–43.

[4] **V. Calvez** and **L. Corrias**, *The parabolic–parabolic Keller–Segel model in R2*, Commun. Math. Sci. **6** (2008), 417–47.

[5] **X. Cao**, *Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces*, Discrete and Continuous Dynamical Systems, 2015, 35(5): 1891–1904.

[6] **X. Chen**, *Well-posedness of the Keller-Segel system in Fourier-Besov-Morrey spaces*, Z. Anal. Anwend., 37 (2018), 417–433.

[7] **L. Corrias** and **B. Perthame**, *Critical space for the parabolic–parabolic Keller–Segel model in Rd*, C. R. Math. Acad. Sci. Paris **342** (2006), 745–50.

[8] **L. Corrias**, **M. Escobedo** and **J. Matos**, *Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller–Segel system in the plane*, J. Differential Equations, **257** (2014), 1840–1878.

[9] **T. Diagana**, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer International Publishing Switzerland (2013).

[10] **J. Dolbeault**, **B. Perthame**, *Optimal critical mass in the two dimensional Keller-Segel model in R^2*, C.R.Acad.Sci. (2004).

[11] **Lucas C.F. Ferreira** and **Juliana C. Precioso**, *Existence and asymptotic behaviour for the parabolic–parabolic Keller–Segel system with singular data*, Nonlinearity **24** (2011), 1433.

[12] **Lucas C.F. Ferreira**, *On the uniqueness of mild solutions for the parabolic-elliptic Keller-Segel system in the critical L^p-space*, Mathematics in Engineering, Vol. 4 (6) (2021), 1–14.

[13] **M. Hieber** and **C. Stinner**, *Strong time periodic solutions to Keller-Segel systems: An approach by the quasilinear Arendt-Bu theorem*, J. Differential Equations, Vol. 269, Iss. 2 (2020), 1636–1655.

[14] **Hao Yu**, **W. Wang** and **S. Zheng**, *Global classical solutions to the Keller–Segel–Navier–Stokes system with matrix-valued sensitivity*, Journal of Mathematical Analysis and Applications Vol. 461, Iss. 2 (2018), 1748-1770.

[15] **T. Iwabuchi**, *Global well-posedness for Keller-Segel system in Besov type spaces*, J. Math. Anal. Appl., 379 (2011), 930–948.

[16] **J. Jiang**, *Global stability of Keller–Segel systems in critical Lebesgue spaces*, Discrete and Continuous Dynamical Systems, (2020), 40(1): 609-634.

[17] **J. Jiang**, *Global stability of homogeneous steady states in scaling-invariant spaces for a Keller–Segel–Navier–Stokes system*, Journal of Differential Equations, Vol. 267, Iss. 2 (2019), Pages 659-692.

[18] **E.F. Keller** and **L.A. Segel**, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. 26 (3) (1970), pages 399–415.

[19] **H. Kozono** and **M. Nakao**, *Periodic solution of the Navier–Stokes equations in unbounded domains*, Tôhoku Math. J. 48, 33–50 (1996).

[20] **K. Klaus**, *Time-Periodic Solutions to Bidomain, Chemotaxis-Fluid, and Q-Tensor Models*, Ph.D. Thesis, Technische Universität Darmstadt (2020). doi: 10.25534/tuprints-00013505

[21] **H. Kozono** and **Y. Sugiyama**, *The Keller–Segel system of parabolic–parabolic type with initial data in weak-$L^{\frac{n}{2}}(\mathbb{R}^n)$ and its application to self-similar solutions*, Indiana Univ. Math. J. **57** (2008), pages1467–500.

[22] **H. Kozono**, **Y. Sugiyama** and **Y. Yahagi**, *Existence and uniqueness theorem on weak solutions to the parabolic-elliptic Keller-Segel system*, J. Differ. Equations, **253** (2012), 2295–2313.

[23] **H. Kozono** and **Y. Sugiyama**, *Strong solutions to the Keller-Segel system with the weak-$L^{n/2}$ initial data and its application to the blow-up rate*, Math. Nachr., **283** (2010), 732–751.

[24] **Y. Li** and **W. Wang**, *Finite-time blow-up and boundedness in a 2D Keller-Segel system with rotation*, Nonlinearity, Vol. 36, No. 1 (2023), pages 287–318.

[25] **P. Maheux** and **V. Pierfelice**, *The Keller-Segel System on the Two-Dimensional-Hyperbolic Space*, SIAM Journal on Mathematical Analysis, Vol. 52, Iss. 5 (2020), 5036-5065.

[26] **A. Montaru**, *Exponential speed of uniform convergence of the cell density toward equilibrium for subcritical mass in a Patlak-Keller-Segel model*, J. Differential Equations Vol. 258, Iss. 3 (2015), 621-668.

[27] **D.S. Mitrinović**, **J.E. Pečarić** and **A.M. Fink**, *Inequalities Involving Functions and Their Integrals and Derivatives*, Part of the book series: Mathematics and its Applications (MAEE, volume 53) (1991). doi: 10.1007/978-94-011-3562-7

[28] **V. Pierfelice**, *The incompressible Navier-Stokes equations on non-compact manifolds*, Journal of Geometric Analysis, 27(1) (2017), 577-617.

[29] **T. Nagai**, **R. Syukuinn** and **M. Umesako**, *Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in* $\mathbb{R}^n$, Funkcial. Ekvac. **46** (2003), 383–407.

[30] **Q. Wang** and **J. Zhang** and **L. Zhang**, *Time-periodic and stable patterns of a two-competing-species Keller-Segel chemotaxis model: effect of cellular growth*, Discrete Contin. Dyn. Syst. Ser. B, Vol. 22, No. 9 (2017), 3547–3574.

[31] **K. Watanabe**, *Strong time-periodic solutions to chemotaxis–Navier–Stokes equations on bounded domains*, Discrete and Continuous Dynamical Systems, Vol. 42, Iss. 11 (2022), 5577-5590.

[32] Y. Sugiyama and H. Kunii, *Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term*, J. Diff. Eqns **227** (2006), 333–64.

[33] **Van, Nguyen Thi**, **Nguyet, Tran Minh**, **Loan, Nguyen Thi** and **Xuan, Pham Truong**. *On pseudo almost periodic solutions of the parabolic-elliptic Keller-Segel systems*, Mathematica Slovaca, vol. 75, no. 3, (2025), 633-648.

[34] **Thi Van, N.**, **The Sac, L.** and **Truong Xuan, P.**, *Well-posedness and exponential stability of periodic and almost periodic solutions for parabolic–parabolic Keller–Segel systems.* Bol. Soc. Mat. Mex. 31, 64 (2025).

[35] **P.T. Xuan**, **N.T. Van** and **B. Quoc**, *On Asymptotically Almost Periodic Solution of Parabolic Equations on real hyperbolic Manifolds*, J. Math. Anal. Appl., Vol. 517, Iss. 1 (2023), 1-19.

[36] **P.T. Xuan** and **N.T. Van**, *On asymptotically almost periodic solutions to the Navier–Stokes equations in hyperbolic manifolds.* J. Fixed Point Theory Appl. 25, 71 (2023), 32 pages.

[37] **P.T. Xuan**, **V.A. Nguyen Thi**, **T.V. Thuy**, and **N.T.Loan**, *Periodic solutions of the parabolic–elliptic Keller–Segel system on whole spaces*, Math. Nachr. (2024), 3003–3023.

[38] **P.T. Xuan**, **N.T. Van** and **T.V. Thuy**, *On asymptotically almost periodic solutions of the parabolic-elliptic Keller-Segel system on real hyperbolic manifolds*, Dynamical Systems, Vol. 39, No. 4 (2024), 665-682.

[39] **P.T. Xuan** and **N.T. Loan**, *Almost periodic solutions of the parabolic-elliptic Keller–Segel system on the whole space*, Arch. Math. **123** (2024), 431–446.

[40] **P.T. Xuan**, *Asymptotically Almost Periodic Solutions of a Chemotaxis Model on Bounded Domains*, Siberian Mathematical Journal, Vol. 66 (2025), 857–873.

[41] **Z. Zhai**, *Global well-posedness for nonlocal fractional Keller–Segel systems in critical Besov spaces*, Nonlinear Anal. **72** (2010), 3173–89.

[42] **M. Winkler**, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations 248, (2010), 2889–2905.

[43] **M. Winkler**, *Global large-data solutions in a chemotaxis–(Navier–)Stokes system modeling cellular swimming in fluid drops*, Comm. Partial Differential Equations, 37 (2012), pp. 319–351.

[44] **M. Winkler**, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller–Segel system*, J. Math. Pures Appl., 100 (2013), 748-767.

[45] **M. Winkler**, *How unstable is spatial homogeneity in Keller-Segel systems? A new critical mass phenomenon in two- and higher-dimensional parabolic-elliptic cases*, Math. Ann. 373 (2019), 1237–1282.

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