The linear dependency of $L-$functions and meromorphic functions sharing finite sets

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Abstract. In this paper we investigate the linear dependency of L-functions and meromorphic functions sharing finite sets. As a consequence, we present some classes of subsets S, T in $\mathbb{C}$ such that for a meromorphic function f and an L-function L, the condition that f and L share S and T, respectively (counting multiplicity) implies $f = hL$ for a non-zero constant h. We discuss some applications of main result. The main result obtained in this paper improves and extends a recent result due to the authors in [32]. We extend previous results of Yuan, Li and Yi [32] by considering distinct finite sets S, T and establishing linear dependency between f and L. Our results are inspired by a work of Yuan, Li, and Yi in [32] and Khoai et al. in [11] and [13].

1. Introduction. Main results

Let f be a non-constant meromorphic function in $\mathbb{C}$, $a \in \mathbb{C} \cup \{\infty\}$. Denote by $E_f(a)$ the set of all $a-$ points of f where an $a-$point is counted with its multiplicity, and by $\overline{E}_f(a)$ where an $a-$point is counted only one time. For a non-empty subset $S \subset \mathbb{C} \cup \{\infty\}$, define $E_f(S) = \cup_{a \in S} E_f(a)$, and similarly for $\overline{E}_f(S)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{M}(\mathbb{C})$. Two non-constant meromorphic functions f, g of $\mathcal{F}$ are said to *share S*, *counting multiplicity*, (share S CM), if

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$E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity*, (share S IM), if $\overline{E}_f(S) = \overline{E}_g(S)$. If the condition $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$) implies $f = g$ for any two non-constant meromorphic (entire) functions f, g of $\mathcal{F}$, then S is called a unique range set for meromorphic (entire) functions of $\mathcal{F}$ counting multiplicity (resp. ignoring multiplicity).

The uniqueness problem for entire or meromorphic functions sharing sets was initiated by a famous question of Gross in [6]. Since, many results have been obtained for this and related topics (see [7, 9, 10, 17, 20, 24, 25, 26, 29]).

In the last few years, the value distribution and uniqueness of L-functions has been studied extensively (see [1, 2, 3, 16, 11, 12, 13, 14, 18, 19, 22, 23, 30, 32]).

Let us recall some basic notations and known results on the value distribution of L-functions (see [3, 22, 20, 11, 32, 13]).

In this paper an L-function always means a non-constant L-function in the Selberg class $\mathcal{S}$, which is defined to be a Dirichlet series

$$L(s) = \sum_{i=0}^{\infty} \frac{a(n)}{n^s},$$

with the normalized condition $a(1) = 1$, satisfying the following axioms:

(i) Ramanujan hypothesis: for all positive ϵ, $a(n) \ll n^{\epsilon}$;

(ii) Analytic continuation: there exists a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;

(iii) Functional equation: there are positive real numbers Q, λ_i, and there exists a positive integer K, and there are complex numbers μ_i,ω with $Re\mu_i \geq 0$ and $|\omega| = 1$ such that $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) := L(s)Q^s \prod_{i=1}^{K} \Gamma(\lambda_i s + \mu_i)$;

(iv) Euler product hypothesis: $L(s)$ satisfies $L(s) = \prod_p L_p(s)$, where $L_p(s) = \exp\big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\big)$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

Note that the Riemann Zeta function is an L-function in the Selberg class.

In 2017 Yuan, Li, and Yi ([32]) posed the following question:

Question A. *What can be said about the relationship between a meromorphic function f and an L-function L if $E_f(S) = E_L(S)$.*

In this direction, they obtained the following result:

Theorem A. ([32]) *Let f be a non-constant meromorphic function having finitely many poles, and let L be an L-function. Let $P(z) = z^n + az^m + b$, where m, n are positive integers, satisfying $n > 2m + 4$, and $(m, n) = 1$, $a, b \in \mathbb{C}$ are nonzero constants. Denote by S the zero set of P. If f and L share S CM, then $f = L$.*

From Theorem A it follows the existence of a class of subsets S with 7 elements,

which are zero sets of Yi's polynomials, such that if $E_f(S) = E_L(S)$, then $f = L$, where f is a non-constant meromorphic function having finitely many poles, L is an L-function.
In 2023, Khoai et al. [13] improved Theorem A. By using a class of polynomials, which are not Yi's polynomials, they showed the existence of a new class of subset S of 5 elements, such that if $E_f(S) = E_L(S)$, then $f = L$, where f is a meromorphic functions having finitely many poles.
Concerning Question A, Pakovitch posed the following question:
Question B. *Under what conditions on compact subsets S, T and polynomials f, g, the following relation holds:*

$$f^{-1}(S) = g^{-1}(T). \tag{1.1}$$

In the case $S = T = \{1, -1\}$ Question B posed by Yang ([26]), and in [24, 25] it is proved that for any compact set $K \in \mathbb{C}$ containing at least two points and polynomials f, g of the same degree, the equality $f^{-1}(K) = g^{-1}(K)$ implies that $f = h(g)$ for some $h = az + b, a, b \in \mathbb{C}$, such that $h(K) = K$. Dinh ([4]) obtained some results for polynomials of arbitrary degree.
In response to Question B, the authors ([10]) showed the following theorem for meromorphic functions having finitely many poles and L-functions.
Let S, T be zero sets of polynomials, having no multiple zeros, of the following form

$$P(z) = az^n + bz^{n-m} + c. \tag{1.2}$$

Note that these polynomials, introduced by Yi ([29]), play an important role in the uniqueness theory for meromorphic functions. A polynomial of the form (1.2) is called a $Y_{(n,m)}$-polynomial.

In response to Question B, the authors ([11]) showed following theorem.

Theorem B. ([11, Theorem 1.1]) *Let n, m be positive integers, $n \geq 2m+8$, and let P, Q be $Y_{(n,m)}$-polynomials, S, T be the zero sets of P, Q, respectively. Suppose $L^{-1}(S) = f^{-1}(T)$ for a non-constant meromorphic function f with finitely many poles in the complex plane, and a non-constant L-function L, then we have:*

1. *There exists a non-zero constant h such that $f = hL$.*
2. *If $(n, m) = 1$, then $f = L$.*

Regarding Theorem A and Theorem B it is natural to ask the following question which motivates us to write this paper.

Question 1. *What can be said about the relationship between a meromorphic function f and an L-function L if $E_L(S) = E_f(T)$, where S, T are the zero sets of P, Q in Theorem B, respectively.*

In this paper, we apply the arguments used in [2], [11] and [13] to answer to Question 1.

Now let us describe main results of the paper. Consider polynomials $P(z), Q(z) \in \mathbb{C}[z]$ of degree n of the form:

$$P(z) = az^n + bz^{n-m} + c, \text{ where } a, b, c \neq 0; \tag{1.3}$$

$$Q(z) = uz^n + vz^{n-m} + t, \text{ where } u, v, t \neq 0. \tag{1.4}$$

Assume that:

$$\frac{a^{n-m}c^m}{b^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}, \tag{1.5}$$

$$\frac{u^{n-m}t^m}{v^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}. \tag{1.6}$$

Note that polynomial $P(z)$ (*resp.* $Q(z)$) has n distinct simple zeros if and only if the condition (1.5) (*resp.* (1.6)) is satisfied (see ([17], Lemma 2.7)).

We shall prove the following main theorems.

Theorem 1. *Let m, n be positive integers, $n \geq 2m+3$, let $P(z), Q(z)$ be polynomials of the form* (1.3) *and* (1.4) *with conditions* (1.5) *and* (1.6), *and let S, T be the zero sets of P, Q, respectively. Let f be a non-constant meromorphic function having finitely many poles in the complex plane and L be a non-constant $L-$function. Then we have:*

1. $E_L(S) = E_f(T)$ if and only if $f = hL$ and $hS = T$, where h is a non-zero constant satisfying $h^n = \dfrac{at}{cu}$ and $h^m = \dfrac{av}{ub}$.

2. In particular, $E_{L_1}(S) = E_{L_2}(T)$ if and only if $L_1 = L_2$ and $\dfrac{a}{u} = \dfrac{b}{v} = \dfrac{c}{t}$ and $S = T$, where L_1, L_2 are non-constant L-functions.

Noting that proof of Theorem 1 is different from Khoai et al.'s ([11]).

Applications. We discuss some applications of Theorem 1. Noting that the identity relationship between an L-function and a meromorphic function is a specific instance of a linear dependency between the same functions.

1/ **Theorem 2.** *Let m, n be positive integers such that $(n, m) = 1$ and $n \geq 2m+3$, let $P(z)$ be polynomial of the form* (1.3) *with condition* (1.5), *and let S be the zero set of P. Suppose that $E_L(S) = E_f(S)$ for an non-constant L-function L and a non-constant meromorphic function having finitely many poles f. Then we have: $f = L$.*

Indeed, applying Theorem 1 with $P(z) = Q(z), S = T$, we get: $f = hL$ and $hS = S$, where h is a non-zero constant satisfying $h^n = 1$ and $h^m = 1$. By $(n, m) = 1$ we obtain $h = 1$. So $f = L$.

2/ **Examples.** Let L be an non-constant $L-$function and let f be a non-constant meromorphic function having finitely many poles, S, T are the zero sets of the polynomials $P(z)$ and $Q(z)$, respectively.

Example 2.1. *Let*

$$P(z) = z^5 - \frac{5}{4}z^4 + 1,\ S = \{a_1, ..., a_5\},\ Q(z) = z^5 - \frac{5}{2}z^4 + 2^5,\ T = \{b_1, ..., b_5\}.$$

Then

$$E_L(S) = E_f(T) \textit{ if and only if } f = 2L, 2S = T.$$

Now we show the necessary condition. We investigate conditions (1.5), (1.6). We have

$$a = u = 1,\ b = -\frac{5}{4},\ c = 1,\ v = -\frac{5}{2},\ t = 2^5, -\frac{4^5}{5^5} \neq -\frac{4^4}{5^5},\ \ 5 = 2.1 + 3.$$

Then, applying Theorem 1 with $n = 5,\ m = 1,\ a = u = 1,\ b = -\frac{5}{4},\ c = 1,$ $v = -\frac{5}{2},\ t = 2^5$, we obtain:

$$f = 2L,\ 2S = T,\ \ \text{where}\ \ 2^5 = \frac{at}{cu} = 2^5, \frac{av}{ub} = 2.$$

Now we show the sufficient condition. Assume that

$$f = 2L,\ \text{and}\ \ 2S = T,\ \text{where}\ \ 2^5 = \frac{at}{cu} = 2^5, \frac{av}{ub} = 2.$$

By

$$Q(f) = f^5 - \frac{5}{2}f^4 + 2^5 = 2^5(L^5 - \frac{5}{4}L^4 + 1) = 2^5 P(L),$$

we get

$$2^5(L - a_1) \cdots (L - a_5) = (f - b_1) \cdots (f - b_5).$$

Example 2.2. *Let* $P(z) = z^5 - \frac{5}{4}z^4 + 1$. *Then*

$$E_L(S) = E_f(S) \ \ \textit{if and only if} \ \ f = L.$$

Now we show the necessary condition. We have $P(z)$ has 5 distinct simple zeros. Then, applying Theorem 2 with $n = 5,\ m = 1$, and noticing that $(5, 1) = 1$, we obtain: $f = L$.

Now we show the sufficient condition. Assume that $f = L$. By

$$P(f) = f^5 - \frac{5}{4}f^4 + 1 = L^5 - \frac{5}{4}L^4 + 1 = P(L), \quad \text{we get}$$

$$(L - a_1) \cdots (L - a_5) = (f - b_1) \cdots (f - b_5).$$

From this it follows that $E_L(S) = E_f(S)$.

2. Preliminaries

We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example, [8], [5], [29]).

Let $f(z)$ be a meromorphic function. The number of poles of $f(z)$ in the disc $\{|z| \leq r\}$ will be denoted by $n(r, f)$, and we assume that a pole of order m contributes m to the value of $n(r, f)$. Then the *counting function* is defined as

$$N(r, f) = \int_o^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and $\overline{N}(r, f)$ is defined in the same way with $n(t, f)$ being replaced by the number of poles of f (ignoring multiplicity) in $\{|z| < t\}$. The *approximating function* is defined as

$$m(r, f) = \frac{1}{2\pi} \int_o^{2\pi} \log^+ |f(re^{i\theta})d\theta, \quad \log^+ |x| = \max(0, \log |x|).$$

The *characteristic function* is defined as

$$T(r, f) = N(r, f) + m(r, f).$$

We have other forms of two Fundamental Theorems of the Nevanlinna theory:

Another form of the First Fundamental Theorem (see [29], Theorem 1.2, p. 8). *Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$ and let $a \in \mathbb{C}$. Then*

$$T(r, \frac{1}{f - a}) = T(r, f) + O(1),$$

where $O(1)$ is a bounded quantity when $r \to +\infty$.

Another form of the Second Fundamental Theorem (see [29], Theorem 1.6', p. 22). Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C}$. Then

$$(q - 1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) - N_0(r, \frac{1}{f'}) + S(r, f),$$

where $N_0(r,\frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of the function $(f-a_1)...(f-a_q)$, and $S(r,f)=o(T(r,f))$ for all r except for a set of finite Lebesgue measure.
A meromorphic function a is said to be a *small function* with respect to a meromorphic function f if $T(r,a)=o(T(r,f))$ when $r\to+\infty$.

Lemma 2.1. ([28] and [5]) *Let $f(z)$ be a non-constant meromorphic function and $P(z)$ be a non-constant polynomial and let $a_0,a_1,a_2,\cdots,a_n$ be distinct meromorphic functions on $\mathbb{C}$ and let*

$$P(f)=a_nf^n+a_{n-1}f^{n-1}+a_{n-2}f^{n-2}+\cdots+a_1f+a_0,\quad \textit{where}\ \ a_n\not\equiv 0.$$

Assume that a_i are small functions with respect to f for all $i=0,1,\cdots,n$. Then

$$T(r,P(f))=nT(r,f)+S(r,f).$$

For the convenience of the reader, we recall Second Fundamental Theorem of the Nevanlinna theory for moving targets (see, for example, [31], [21]).

Lemma 2.2. (Second Fundamental Theorem for moving targets) *Let f be a non-constant meromorphic function and let $a_1,a_2,\cdots,a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i are small functions with respect to f for all $i=1,\cdots,q$. Then, the inequality*

$$(q-2)T(r,f)\le\sum_{i=1}^{q}\overline{N}(r,\frac{1}{f-a_i})+S(r,f)$$

holds for all r except for a set of finite Lebesgue measure.

Lemma 2.3. ([5]) *Let f be an entire function of finite order ρ. If f has no zeros, then $f(z)=e^{h(z)}$, where $h(z)$ is a polynomial of degree less than ρ.*

We shall use the following lemmas on $L-$functions. We denote the order of a meromorphic function f by $\rho(f)$.

Lemma 2.4. ([22]) *Let L be an non-constant $L-$function. Then*

i) $T(r,L)=\frac{d_L}{\pi}r\log r+O(r)$, where $d_L=2\sum_{i=1}^{K}\lambda_i$ is the degree of L, and K,λ_i are respectively the positive integer and positive real number in the functional equation of the definition of L-functions.

ii) $N(r,\frac{1}{L})=\frac{d_L}{\pi}r\log r+O(r)$, $N(r,L)=S(r,L)$.

iii) $\rho(L)=1$.

Lemma 2.5. ([19]) *Suppose L is a non-constant L-function, there is no generalized Picard exceptional value of L in the complex plane.*

Lemma 2.6. ([14]) *Let $L_1,...,L_N$ be distinct non-constant L-functions. Then $L_1,...,L_N$ are linearly independent over $\mathbb{C}$.*

3. Proof of Theorem 1

Recall that

$$P(z) = az^n + bz^{n-m} + c,\ Q(z) = uz^n + vz^{n-m} + t,$$
$$a, b, d, u.v, t \neq 0.$$

Then, we get:

$$P(z) = a(z - a_1)...(z - a_n),\ Q(z) = u(z - b_1)...(z - b_n), \tag{3.1}$$

$$P(L) = a(L - a_1)...(L - a_n),\ Q(f) = u(f - b_1)...(f - b_n). \tag{3.2}$$

1/ **The necessary condition.**
Proof. Let $n \geq 2m + 3$ and $E_L(S) = E_f(T)$.

Lemma 3.1. *We have*

$$(n-1)T(r, L) + S(r, L) \leq nT(r, f) + S(r, f),$$
$$(n-1)T(r, f) + S(r, f) \leq nT(r, L) + S(r, L),\ \textit{in particular, } S(r, f) = S(r, L).$$

Proof. Noting that L has only one possible pole at $s = 1$ and f has finitely many poles and by Lemma 2.4, we have

$$N(r, L) = O(\log r),\quad N(r, f) = O(\log r),\quad N(r, L) = o(T(r, L)),$$
$$= S(r, L), N(r, f) = o(T(r, L)) = S(r, L).$$

Applying another form of the two Fundamental Theorems and noting that $\overline{N}(r, L) = S(r, L) = \overline{N}(r, f),\ E_L(S) = E_f(T)$, we obtain

$$(n-1)T(r, L) \leq \overline{N}(r, L) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L - a_i}) + S(r, L),$$
$$(n-1)T(r, L) + S(r, L) \leq \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f - b_i})$$
$$\leq nT(r, f) + S(r, f).$$

Similarly,

$$(n-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f - b_i}) + S(r, f),$$
$$(n-1)T(r, f) \leq S(r, L) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L - a_i}) + S(r, f),$$
$$(n-1)T(r, f) + S(r, f) \leq nT(r, L) + S(r, L).$$

Combining the above inequalities, we get

$$\frac{n-1}{n}T(r,L)+S(r,L)\le T(r,f)+S(r,f)\le\frac{n}{n-1}T(r,L)+S(r,L),$$

$$\frac{n-1}{n}T(r,f)+S(r,f)\le T(r,L)+S(r,L)\le\frac{n}{n-1}T(r,f)+S(r,f).$$

Therefore $S(r,f)=S(r,L)$.
Lemma 3.1 is proved. ∎

Set $S(r)=S(r,f)=S(r,L)$. From Lemma 3.1 and because f has finitely many poles and by Lemma 2.4 we obtain

$$N(r,L)=S(r)=N(r,f),\ \rho(f)=\rho(L)=1. \tag{3.3}$$

Write

$$P(z)=z^{n-m}R(z)+c,\ Q(z)=z^{n-m}R_1(z)+t$$

where $R(z)=az^m+b$, $R_1(z)=uz^m+v$ are polynomials of degree m. Recall that $n\ge 2m+3$.
Since f and L share S CM, we have

$$\frac{Q(f)}{P(L)}=\frac{f^{n-m}R_1(f)+t}{L^{n-m}R(L)+c}=R_2e^{\varphi(z)}, \tag{3.4}$$

where $R_2\not\equiv 0$ is a rational function and $\varphi(z)$ is an entire function. Then $\rho(R_2)=0$, by ([5], Theorem 1.4) and (3.3) we get

$$\rho(Q(f))=\rho(f)=1,\ \rho(P(L))=\rho(L)=1. \tag{3.5}$$

From (3.4), (3.5), and Lemma 2.3 we have

$$\rho(e^{\varphi(z)})=\rho(\frac{Q(f)}{R_2P(L)})\le\max\{\rho(R_2),\rho(Q(f)),\rho(P(L))\}=\rho(L)=1,$$

$$\varphi(z)=Az+B,$$

where A,B are constants. Set

$$l(z)=R_2e^{\varphi(z)}. \tag{3.6}$$

Claim 1. *We have:*

$$l(z)=\frac{t}{c}.$$

From (3.6) it leads

$$T(r,l)\le T(r,R_2)+T(r,e^{\varphi})=\frac{|A|}{\pi}r+O(\log r).$$

Therefore, from Lemma 2.4 it follows that $T(r,l) = o(T(r,L))$, and by Lemma 3.1, $T(r,l) = o(T(r,f)$, too. This means that l is a small function for the functions f and L. Therefore

$$Q(f) = uf^n + vf^{n-m} + t = lP(L) = l(aL^n + bL^{n-m} + c),$$

$$T(r,l) = S(r),\ T(r,Q(f)) = nT(r,f) + O(1) = nT(r,L) + S(r).$$

So

$$T(r,f) + S(r) = T(r,L) + S(r), \tag{3.7}$$

$$f^{n-m}(uf^m + v) - (lc - t) = lL^{n-m}(aL^m + b). \tag{3.8}$$

Suppose $l \not\equiv \frac{t}{c}$. Then $lc - t \not\equiv 0$. Applying another form of the First Fundamental Theorem and Second Fundamental Theorem for moving targets to the function $f^{n-m}(uf^m + v)$ and the values $0, lc - t$, and by using (3.3), (3.7), (3.8) and Lemma 3.1, we get

$$\begin{aligned}
&\overline{N}(r,\frac{1}{l}) \le T(r,l) + O(1) = S(r),\\
&nT(r,f) + O(1) = T(r,f^{n-m}R_1(f))\\
&\le \overline{N}(r,f^{n-m}R_1(f)) + \overline{N}(r,\frac{1}{f^{n-m}R_1(f)}) + \overline{N}(r,\frac{1}{f^{n-m}R_1(f) - (lc-t)}) + S(r)\\
&\le \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{R_1(f)}) + \overline{N}(r,\frac{1}{l}) + \overline{N}(r,\frac{1}{L^{n-m}R(L)}) + S(r)\\
&\le (1+m)T(r,f) + \overline{N}(r,L) + \overline{N}(r,\frac{1}{L}) + N(r,\frac{1}{R(L)}) + S(r)\\
&\le (1+m)(T(r,f) + T(r,L)) + S(r)\\
&\le (2m+2)T(r,f) + S(r).
\end{aligned}$$

Therefore

$$(n - 2m - 2)T(r,f) \le S(r).$$

This is a contradiction to the assumption that $n \ge 2m + 3$. Thus $l \equiv \frac{t}{c}$, and

$$f^{n-m}R_1(f) = \frac{t}{c}L^{n-m}R(L),\ uf^n + vf^{n-m} = \frac{t}{c}(aL^n + bL^{n-m}). \tag{3.9}$$

For simplicity, set $h = \dfrac{f}{L}$, and $\alpha = \frac{at}{cu} \ne 0; \beta = \frac{bt}{vc} \ne 0$. Then we obtain

$$uL^m(h^n - \frac{at}{cu}) = -v(h^{n-m} - \frac{bt}{vc}),$$

$$L^m(h^n-\alpha)=-\frac{v}{u}(h^{n-m}-\beta),\ L^m=-\frac{v}{u}\frac{h^{n-m}-\beta}{h^n-\alpha}. \tag{3.10}$$

Claim 2. *h is a constant.*
Suppose by contradiction that h is non constant. We consider the following possible cases:
Case 1. Polynomials $z^n-\alpha$ and $z^{n-m}-\beta$ have no common zeros.
Then equation $z^n-\alpha=0$ has n simple zeros α_1,..., α_n, and equation $z^{n-m}-\beta=0$ has $n-m$ simple zeros β_1, ..., β_{n-m} such that $\beta_i\neq\alpha_j$.
Write

$$\begin{aligned} z^n-\alpha&=(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n),\\ z^{n-m}-\beta&=(z-\beta_1)(z-\beta_2)\cdots(z-\beta_{n-m}),\\ h^n-\alpha&=(h-\alpha_1)(h-\alpha_2)\cdots(h-\alpha_n),\\ h^{n-m}-\beta&=(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{n-m}). \end{aligned}$$

Applying another forms of the two Fundamental Theorems to the function h and the values α_1,..., α_n, β_1, ..., β_{n-m}, and noting that

$$mT(r,L)+O(1)=T(r,L^m)=T(r,h^n)=nT(r,h)+O(1),$$

$$S(r)=S(h,r)=N(r,L),\sum_{i=1}^{n}\overline{N}(r,\frac{1}{h-\alpha_i})\leq N(r,L)=S(r),$$

we get

$$(2n-m-2)T(r,h)\leq\sum_{i=1}^{n}\overline{N}(r,\frac{1}{h-\alpha_i})+\sum_{i=1}^{n-m}\overline{N}(r,\frac{1}{h-\beta_i})+S(h,r),$$

$$(2n-m-2)T(r,h)\leq(n-m)T(r,h)+S(h,r),(n-2)T(r,h)\leq S(h,r).$$

This is a contradiction to the assumption that $n\geq 2m+3$.
Case 2. Polynomials $z^n-\alpha$ and $z^{n-m}-\beta$ have common zeros.
Let z_0 be a common zero of $z^n-\alpha$ and $z^{n-m}-\beta$, then we have $z_0^n=\alpha$ and $z_0^{n-m}=\beta$. From this and (3.10) we get

$$\frac{z^{n-m}-\beta}{z^n-\alpha}=\frac{\beta}{\alpha}\frac{(\frac{z}{z_0})^{n-m}-1}{(\frac{z}{z_0})^n-1},\ \frac{h^{n-m}-\beta}{h^n-\alpha}=\frac{\beta}{\alpha}\frac{(\frac{h}{z_0})^{n-m}-1}{(\frac{h}{z_0})^n-1},$$

$$L^m=-\frac{\beta v}{\alpha u}\frac{(\frac{h}{z_0})^{n-m}-1}{(\frac{h}{z_0})^n-1},\ L^m=-\frac{\beta v}{\alpha u}\frac{H^{n-m}-1}{H^n-1},\ \text{ where } H=\frac{h}{z_0}. \tag{3.11}$$

We see that H is non constant from suppose that h is non constant. Set $(n,m)=t$, we have $t\leq m$ and polynomials z^n-1 and $z^{n-m}-1$ have t common

zeros. Then, the roots of $z^{n-m}-1=0$ are different the roots of $z^n-1=0$, except t common zeros. So equation $z^{n-m}-1=0$ has $n-m-t$ simple zeros β_1,..., β_{n-t}, and equation $z^n-1=0$ has $n-t$ simple zeros α_1, ..., α_{n-t} such that $\beta_i \neq \alpha_j$.
From (3.11) and by similar arguments as in the proof of Case 1 and noting that $S(H,r)=S(h,r)=S(r)$, we get

$$(2n-m-2t-2)T(r,H) \leq \sum_{i=1}^{n-t} \overline{N}(r,\frac{1}{H-\alpha_i}) + \sum_{i=1}^{n-m-t} \overline{N}(r,\frac{1}{H-\beta_i}) + S(H,r),$$

$$(2n-m-2t-2)T(r,H) \leq (n-m-t)T(r,H)+S(H,r), (n-t-2)T(r,H) \leq S(H,r),$$

which is a contradiction since $n \geq 2m+3$.
Hence from Case 1 and Case 2 we have that h is a constant and Claim 2 is proved.
That leads

$$l(aL^n+bL^{n-m}+c) = uf^n+vf^{n-m}+t,\ l=\frac{t}{c},$$

$$alL^n+blL^{n-m}+cl = uh^nL^n+vh^{n-m}L^{n-m}+t. \tag{3.12}$$

Now we return proof the necessary condition of Theorem 1.
By equation (3.12) we get

$$l=\frac{t}{c}, h^n=\frac{al}{u},\ h^m=\frac{lv}{b}\frac{at}{uc}.$$

Therefore $h^n=\dfrac{at}{cu}$, $h^m=\dfrac{av}{ub}$. Now we prove $hS=T$. Take $a_i \in S, i=1,...,n$. By Lemma 2.5, there exists $z_0 \in \mathbb{C}$ such that $L(z_0)-a_i=0$. Moreover, from $lP(L)=Q(f)$ and (3.2) we obtain

$$P(L)=a(L-a_1)...(L-a_n),\ u(f-b_1)...(f-b_n)=Q(f);$$

$$la(L-a_1)...(L-a_n) = u(f-b_1)...(f-b_n). \tag{3.13}$$

By (3.13) we see that: $L(z_0)-a_i=0$ if and only if z_0 is a zero of $P(L)$ and $L(z_0)-a_j \neq 0$ with $i \neq j$, and therefore, there exists a unique $b_k \in \{b_1,...,b_n\}$ such that $f(z_0)-b_k=0$. Because $f=hL$, we have $hL(z_0)-b_k=0$, and therefore $ha_i=b_k$. From this and cardinalities of S,T are n it follows that $hS=T$.
The sufficient condition. We have:

$$P(L)=aL^n+bL^{n-m}+c,\ Q(f)=uf^n+vf^{n-m}+t,$$

$$f = hL, h^n = \frac{at}{cu}, h^m = \frac{av}{ub}, Q(f) = uh^nL^n + vh^{n-m}L^{n-m} + t.$$

Therefore,

$$tP(L) = cQ(f) \text{ and } at(L-a_1)...(L-a_n) = cu(f-b_1)...(f-b_n).$$

From this it leads $E_L(S) = E_f(T)$.

2/ **The necessary condition.** We have: $E_{L_1}(S) = E_{L_2}(T)$. Then, by 1/ we get: $L_2 = hL_1$ and $hS = T$, where h is a non-zero constant satisfying $h^n = \frac{at}{cu}$, $h^m = \frac{av}{ub}$. By Lemma 2.6, we get $L_1 = L_2$, and then $h = 1$, $\frac{a}{u} = \frac{b}{v} = \frac{c}{t}$, and $S = T$.

The sufficient condition. By using the arguments similar to the ones in the sufficient condition of 1/ we get the sufficient condition of 2/.
Theorem 1 is proved. ■

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