J. Math. Math. Sci. (2025) 73–91

A Korenblum Maximum Principle for weighted Hilbert spaces of entire Dirichlet series with real frequencies

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(Received Sept. 24, 2025; accepted Oct. 8, 2025)

Abstract. In this paper, we study a Korenblum Maximum Principle for weighted Hilbert spaces of entire Dirichlet series with real frequencies. We investigate dominating sets for which the Korenblum Maximum Principle must hold. The results obtained imply that a dominating set, if exists, must be a left half-plane. This provides a new perspective for studying Korenblum Maximum Principle on function spaces containing the entire Dirichlet series.

1. Introduction

The Korenblum Maximum Principle is an important open problem in complex analysis as it acts as one of the fundamental properties of complex function spaces that remains unsolved. First conjectured in 1991, the principle was introduced by Boris Korenblum for the classical Bergman space $A^2(\mathbb{D})$ in the following way [7].

Conjecture 1.1. *There exists a numerical constant c, $0 < c < 1$, such that if f and g are holomorphic in the unit disk $\mathbb{D}$ and $|f(z)| \leq |g(z)|$ for all z with $c < |z| < 1$, then $\|f\|_{A^2} \leq \|g\|_{A^2}$.*

Key words and phrases: Hilbert space, Korenblum constant, Dominating set, Entire functions, Dirichlet series, Maximum principle

2020 Mathematics Subject Classification: Primary 30D15, 30B50, 30C80

The first-named author is supported by the grant NVCC43.02/25-25 of Vietnam Academy of Science and Technology

In [7], Korenblum defined c as the *Korenblum constant* and κ as the largest possible value of c. The exact value of κ remains unknown. In the same paper, Korenblum also proved that $\kappa_{A^2} \leq \frac{1}{\sqrt{2}} \approx 0.7071$.

Initial progress on the Korenblum Maximum Principle was made through a series of partial results reported in works such as [8, 9, 10, 12], among others. The existence of the Korenblum constant for the Bergman space $A^2(\mathbb{D})$ was first rigorously established in 1999 by [5], with an estimate of $\kappa_{A^2} = 0.04$. Subsequent research has focused on refining both lower and upper bounds of κ_{A^2}, yielding a rich body of results contributed by various authors. Renewed interest in recent years has led to significant developments not only for classical function spaces but also for their intersections and generalizations. This resurgence highlighted the need for a comprehensive review of key findings related to the Korenblum Maximum Principle. Additionally, several studies have explored modified versions of the principle (see, e.g., [11, 13]).

In our recent survey [14], we investigate the Korenblum Maximum Principle in the setting of weighted function spaces, highlighting recent progress in bounding Korenblum constants and identifying cases of failure—such as in weighted Bergman, Hardy, Bloch, Fock, and mixed norm spaces [15]. We also present a collection of open problems, both classical and newly proposed. Notably, we generalize existing results on weighted Fock spaces to broader families [16]. Special emphasis is placed on the Gamma function, which satisfies Ramanujan's Master Theorem [3] and connects to Mellin transforms of Dirichlet series and generalized hypergeometric functions [1, 3].

There is an interesting question to ask: How about spaces of Dirichlet series? This stems from the fact that Dirichlet series, from classical to generalized, have many important applications in different fields. We refer the reader to [4, 2] for detailed information about these series.

To our knowledge, this question, which seems probably very difficult, has never been addressed before.

2. Basic Definitions and Notations

Consider the Dirichlet series with real frequencies

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C}, \tag{2.1}$$

where $0 \leq (\lambda_n) \uparrow \infty$ is a given sequence of real numbers.

Let

$$L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n}.$$

Note that $L \geq 0$ and it can be ∞.

Another quantity associated with this series is

$$D = \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n}.$$

Note that D can be a finite real number, $-\infty$, or ∞.

We note the following elementary results.

Lemma 2.1. *Let $0 \leq (\lambda_n) \uparrow \infty$ be given. Then the following statements hold true.*

1. *$L < \infty$ if and only if there exists a real number r for which $\sum e^{-\lambda_n r} < \infty$.*
2. *In the case $L < \infty$, we have $\sum e^{-\lambda_n r} < \infty$ if $r > L$, and $\sum e^{-\lambda_n r} = \infty$ if $r < L$.*
3. *If ρ is a real number such that $\sum e^{-\lambda_n \rho} < \infty$, then $L \leq \rho$.*
4. *$L = 0$ if and only if $\sum e^{-\lambda_n r}$ converges for all $r > 0$, if and only if $\sum \alpha^{\lambda_n} < \infty$ for all $0 < \alpha < 1$.*

As is well known, in case $L < \infty$, the series (2.1) represents an entire function in $\mathbb{C}$ if and only if $D = -\infty$.

Throughout this paper, the condition $L < \infty$ is supposed to hold unless otherwise stated.

2.1. Weighted spaces of entire Dirichlet series

Let $0 \leq (\lambda_n) \uparrow \infty$ satisfying condition $L < \infty$, be given. Consider the normed space of entire Dirichlet series

$$\mathcal{H}(E) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n) \in E \right\},$$

where

$$E = \left\{ (a_n) : \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \text{ or } \lim_{n\to\infty} |a_n|^{1/\lambda_n} = 0 \right\}.$$

The norm in this space is defined by the inner product

$$\langle f, g\rangle := \sum_{n=1}^{\infty} a_n \overline{b_n}, \quad f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}.$$

It should be noted that the space $\mathcal{H}(E)$ is never complete with respect to the norm above. Then a natural question to ask is: how to define subspaces of $\mathcal{H}(E)$ that can be Hilbert spaces? To study this question, one introduces and considers some weighted spaces.

2.2. Weighted spaces

Let $\beta = (\beta_n)$ be a sequence of real positive numbers. To each β, we associate the following *weighted sequence space*

$$\ell^2_\beta = \left\{ a = (a_n) \subset \mathbb{C} \colon \|a\|_{\ell^2_\beta} = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\},$$

which is a Hilbert space with the inner product

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2, \quad (a_n), (b_n) \in \ell^2_\beta, \tag{2.2}$$

Such sequence spaces ℓ^2_β have many important applications in studying operators on function spaces.

Consider the following *weighted function space* $\mathcal{H}^2(\beta)$ of entire Dirichlet series induced by weight β

$$\mathcal{H}^2(\beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \text{ entire} \colon \|f\| := \|(a_n)\|_{\ell^2_\beta} < \infty \right\}. \tag{2.3}$$

This space $\mathcal{H}^2(\beta)$ is an inner product space, where $\langle f, g\rangle_\beta = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2$, for any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$ in $\mathcal{H}^2(\beta)$.

Depending on β, the induced space $\mathcal{H}^2(\beta)$ may not be complete in its norm, and so it is not necessarily a Hilbert space. To characterize a completeness we put

$$\liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \beta_*, \quad \limsup_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \beta^*,$$

and prove the following result.

Proposition 2.2. *There are three alternative possibilities*

1. $\ell^2_\beta \subset E$, *which is equivalent to* $\beta_* = \infty$.
2. $E \subset \ell^2_\beta$, *which is equivalent to* $\beta^* < \infty$.
3. $\ell^2_\beta \setminus E \neq \emptyset$ *and* $E \setminus \ell^2_\beta \neq \emptyset$, *which is equivalent to* $\beta_* < \beta^* = \infty$.

As a consequence, these spaces never coincide.

Proof. As the proof of (1) and (2) are quite similar, we prove (2). Combining (1) and (2) yields (3).

(2): Let $E \subset \ell^2_\beta$. Assume that $\beta^* = \infty$. In this case there exist $(M_p) \uparrow \infty$ and $(n_p) \uparrow \infty$ such that

$$\frac{\log \beta_{n_p}}{\lambda_{n_p}} > M_{n_p}, \ \forall p \geq 1.$$

Define a sequence (a_n) as follows

$$a_n = \begin{cases} e^{-M_p \lambda_{n_p}}, & \text{if } n = n_p, \ p = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $(a_n) \in E$, but (a_n) clearly is not in ℓ^2_β: a contradiction.

Conversely, suppose that $\beta^* < \infty$. In this case there exist $M > 0$ and $N_1 \in \mathbb{N}$ such that

$$\frac{\log \beta_n}{\lambda_n} \leq M, \ \forall n > N_1.$$

Let $\varepsilon > 0$ be given. For an arbitrary $(a_n) \in E$, there is $N_2 \in \mathbb{N}$ such that $|a_n|^{1/\lambda_n} < \varepsilon < e^{-M-L-1}$. Then for all $n > \max\{N_1, N_2\}$ we have

$$\begin{aligned} \sum_{n>N} |a_n|^2 \beta_n^2 &\leq \sum_{n>N} \varepsilon^{2\lambda_n} \beta_n^2 \leq \sum_{n>N} \varepsilon^{2\lambda_n} e^{2M\lambda_n} \\ &= \sum_{n>N} (e^{-2(M+L+1)} e^{2M})^{\lambda_n} = \sum_{n>N} (e^{-2(L+1)})^{\lambda_n} < \infty, \end{aligned}$$

due to Lemma 2.1(2). That is, $E \subset \ell^2_\beta$. ■

The following theorem provides a criterion on the weight β for $\mathcal{H}^2(\beta)$ to be complete.

Theorem 2.3. *The space $\mathcal{H}^2(\beta)$ of entire Dirichlet series induced by a sequence of positive real numbers $\beta = (\beta_n)$, as defined in* (2.3)*, is a Hilbert space if and only if the following condition holds,*

$$\liminf_{n \to \infty} \frac{\log \beta_n}{\lambda_n} = \infty. \tag{2.4}$$

It is clear that if condition (2.4) holds, then the space $\mathcal{H}^2(\beta)$ automatically becomes a Hilbert space of entire functions, so we can drop the word "entire" in (2.3).

In the sequel the condition $\beta_* = \infty$ is always supposed to hold. That is, the sequence of positive real numbers (β_n) satisfies the condition

$$(E) \qquad \liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty, \text{ or the same, } \lim_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty.$$

Proposition 2.2 leads us to the following definition.

Definition 2.4. Let (β_n) be the sequence of positive real numbers such that $\beta_* = \infty$. The Hilbert space of entire Dirichlet series with real frequencies $0 \le (\lambda_n) \uparrow \infty$ induced by (β_n) is defined as

$$\mathcal{H}^2(\beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : \|f\|_{\mathcal{H}^2(\beta)} := \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\}.$$

Note that if $\lambda_1 = 0$, then the space $\mathcal{H}^2(\beta)$ contains all constant functions. On the other hand, it contains no nonzero constant functions if $\lambda_1 > 0$.

We refer the reader to the forthcoming monograph [6] for having more information on Dirichlet series with real frequencies and related topics.

3. Korenblum Maximum Principle for $\mathcal{H}^2(\beta)$

Recall that (λ_n) and (β_n) which satisfy $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty$ and

$$\beta_* = \liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty, \text{ or the same, } \lim_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty.$$

Let $\mathcal{H}^2(\beta)$ be a weighted Hilbert space of entire Dirichlet series with weights (β_n) and real frequencies (λ_n).

Definition 3.1. Let $-\infty \le c_1 < c_2 \le \infty$. A trip

$$S_{c_1,c_2} = \{z \in \mathbb{C} : c_1 \le \operatorname{Re}(z) \le c_2\}$$

is called a *dominating* set for the space $\mathcal{H}^2(\beta)$, if for any pair $f(z), g(z) \in \mathcal{H}^2(\beta)$ the following implication holds

$$|f(z)| \le |g(z)| \text{ for all } z \in S_{c_1,c_2} \implies \|f\|_{\mathcal{H}^2} \le \|g\|_{\mathcal{H}^2}.$$

We wish to investigate the estimates for pairs (c_1, c_2) as well as the two constants κ_1 (the largest possible c_1) and κ_2 (the smallest possible c_2).

Without loss of generality, we may assume that $(\beta_n)_{n\geq 1}$ is strictly increasing.

3.1. Results for pairs (λ_1, λ_2) and (β_1, β_2)

3.1.1. Upper bounds

Theorem 3.2. *Suppose that c is a real number satisfying the condition*

$$c > \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2}.$$

Then there exists a pair of monomials $f(z),\ g(z) \in \mathcal{H}^2(\beta)$ such that

$$|f(z)| \leq |g(z)|, \quad \forall z : \mathrm{Re}\,(z) \geq c,$$

but $\|f\|_{\mathcal{H}^2(\beta)} > \|g\|_{\mathcal{H}^2(\beta)}$.

Proof. Regardless $\lambda_1 = 0$ or $\lambda_1 \neq 0$, the following proof works well.

There is an $\varepsilon > 0$ such that $c = \dfrac{1}{\lambda_2 - \lambda_1} \log \left(\dfrac{\beta_1}{\beta_2} + \varepsilon \right)$. Take an arbitrary pair of positive numbers (a, b) satisfying the following conditions

$$\frac{\beta_1}{\beta_2} < \frac{a}{b} \leq \frac{\beta_1}{\beta_2} + \varepsilon,$$

and consider the following monomials from $\mathcal{H}^2(\beta)$.

$$f(z) = ae^{-\lambda_2 z} \quad \text{and} \quad g(z) = be^{-\lambda_1 z},$$

for which

$$\begin{cases} |f(z)| = ae^{-\lambda_2 \mathrm{Re}\,(z)}, & |g(z)| = be^{-\lambda_1 \mathrm{Re}\,(z)}, \\ \|f\|_{\mathcal{H}^2(\beta)} = a\beta_2, & \|g\|_{\mathcal{H}^2(\beta)} = b\beta_1. \end{cases}$$

On the one hand, for all z with $\mathrm{Re}\,(z) \geq c$,

$$\frac{|f(z)|}{|g(z)|} = \frac{a}{b} e^{-(\lambda_2 - \lambda_1)\mathrm{Re}\,(z)} \leq \frac{a}{b} e^{-(\lambda_2 - \lambda_1)c} = \frac{a}{b} \frac{1}{\left(\frac{\beta_1}{\beta_2} + \varepsilon \right)} \leq 1.$$

On the other hand, $\|f\|_{\mathcal{H}^2(\beta)} = a\beta_2 > b\beta_1 = \|g\|_{\mathcal{H}^2(\beta)}$. ■

As an immediate corollary of Theorem 3.2, we have.

Corollary 3.3. *For any real numbers c_1, c_2 satisfying*

$$\frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2} < c_1 < c_2 \leq \infty,$$

the trip S_{c_1,c_2} cannot be dominating for the space $\mathcal{H}^2(\beta)$.

In other words, for a trip S_{c_1,c_2} to be dominating, it is necessary that

$$c_1 \leq \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2} \implies \kappa_1 \leq \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2}.$$

3.1.2. Lower bounds

For the lower bound, the approach is almost similar, with some technical modifications.

We have the following result.

Theorem 3.4. *Suppose that c is a real number satisfying the condition*

$$c < \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2}.$$

Then there exists a pair of monomials $f(z),\ g(z) \in \mathcal{H}^2(\beta)$ such that

$$|f(z)| \leq |g(z)|, \quad \forall z : \mathrm{Re}\,(z) \leq c,$$

but $\|f\|_{\mathcal{H}^2(\beta)} > \|g\|_{\mathcal{H}^2(\beta)}$.

Proof. Note that

$$\frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2} < 0,$$

then so does c.

For a better exposition, we denote

$$-c = \tau > \frac{1}{\lambda_1 - \lambda_2} \log \frac{\beta_1}{\beta_2} > 0.$$

From this it follows that

$$(\lambda_2 - \lambda_1)\tau > \frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_2} \log \frac{\beta_1}{\beta_2} = \log \frac{\beta_2}{\beta_1},$$

which gives $e^{(\lambda_2 - \lambda_1)\tau} > \frac{\beta_2}{\beta_1}$.

The last estimate allows us to choose a pair of positive numbers (a, b) satisfying the following conditions

$$\frac{\beta_2}{\beta_1} < \frac{a}{b} \leq e^{(\lambda_2 - \lambda_1)\tau},$$

and determine two monomials from $\mathcal{H}^2(\beta)$:

$$f(z) = ae^{-\lambda_1 z} \quad \text{and} \quad g(z) = be^{-\lambda_2 z},$$

for which

$$\begin{cases} |f(z)| = ae^{-\lambda_1 \mathrm{Re}\,(z)}, & |g(z)| = be^{-\lambda_2 \mathrm{Re}\,(z)}, \\ \|f\|_{\mathcal{H}^2(\beta)} = a\beta_1, & \|g\|_{\mathcal{H}^2(\beta)} = b\beta_2. \end{cases}$$

On the one hand, for all z with $\mathrm{Re}\,(z) \leq c$, we have

$$\frac{|f(z)|}{|g(z)|} = \frac{a}{b} e^{(\lambda_2 - \lambda_1)\mathrm{Re}\,(z)} \leq \frac{a}{b} e^{(\lambda_2 - \lambda_1)c} = \frac{a}{b} e^{(\lambda_1 - \lambda_2)\tau} \leq 1.$$

On the other hand, $\|f\|_{\mathcal{H}^2(\beta)} = a\beta_1 > b\beta_2 = \|g\|_{\mathcal{H}^2(\beta)}$. ■

As a consequence of Theorem 3.4, we have the following result.

Corollary 3.5. *For any real numbers c_1, c_2 satisfying*

$$-\infty \leq c_1 < c_2 < \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2},$$

the trip S_{c_1,c_2} cannot be dominating for the space $\mathcal{H}^2(\beta)$.

In other words, for a trip S_{c_1,c_2} to be dominating, it is necessary to have

$$c_2 \geq \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2} \implies \kappa_2 \geq \frac{1}{\lambda_2 - \lambda_1} \log \frac{\beta_1}{\beta_2}.$$

3.2. Results for pairs $(\lambda_N, \lambda_{N+1})$ and (β_N, β_{N+1}), $N \geq 2$

Analyzing discussions in the previous subsection, we see that the results for pairs (λ_1, λ_2) and (β_1, β_2) can be generalized to arbitrary pairs $(\lambda_N, \lambda_{N+1})$ and (β_N, β_{N+1}), $N \geq 2$, replacing indexes β_1 and β_2 by β_N and β_{N+1} respectively.

Now, for every $n \in \mathbb{N}$, let us denote

$$\Omega_n := \frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{\beta_n}{\beta_{n+1}} < 0.$$

We have the following estimates:

$$\kappa_1 \leq \Omega_n \leq \kappa_2, \quad \text{for all } n \in \mathbb{N}.$$

Consequently, if a dominating set S_{c_1,c_2} exists, then the open interval (c_1, c_2) must contain the real parts of all vertical lines

$$\mathrm{Re}\, z = \Omega_n \left(= \frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{\beta_n}{\beta_{n+1}} \right), \quad n \in \mathbb{N}.$$

Moreover, we have the estimates

$$\kappa_1 \leq \inf_{n\in\mathbb{N}}\{\Omega_n\} \leq \sup_{n\in\mathbb{N}}\{\Omega_n\} \leq \kappa_2. \tag{3.1}$$

For completeness, recall the following standard relations between the infimum, supremum, limit inferior, and limit superior of a sequence of real numbers. If (u_n) is a sequence of real numbers, then

$$\inf_{n\in\mathbb{N}} u_n \leq \liminf_{n\to\infty} u_n \leq \limsup_{n\to\infty} u_n \leq \sup_{n\in\mathbb{N}} u_n. \tag{3.2}$$

Remark 3.6. If the sequence is unbounded above or below, then the supremum or infimum may be $\pm\infty$, and the inequalities still hold in the extended real line. Moreover, equality at the two middle terms occurs exactly when the ordinary limit exists and equals that common value.

Combining inequalities (3.1) and (3.2) yields the following result.

Proposition 3.7. *The bounds κ_1 and κ_2 control the asymptotic behavior of (Ω_n) in the following sense:*

$$\kappa_1 \leq \inf_{n\in\mathbb{N}}\{\Omega_n\} \leq \liminf_{n\to\infty} \Omega_n \leq \limsup_{n\to\infty} \Omega_n \leq \sup_{n\in\mathbb{N}}\{\Omega_n\} \leq \kappa_2. \tag{3.3}$$

Proposition 3.7 reduces the problem to determining the possible values of

$$\inf_{n\in\mathbb{N}}\{\Omega_n\}, \quad \sup_{n\in\mathbb{N}}\{\Omega_n\}, \quad \liminf_{n\to\infty} \Omega_n, \quad \text{and} \quad \limsup_{n\to\infty} \Omega_n.$$

4. A study of $\{\Omega_n\}$

Recall, in a general situation, that $(\lambda_n)_{n\geq 1}$ is a strictly increasing sequence of positive reals with $\lambda_n \uparrow \infty$ satisfying the condition $L < \infty$ and $(\beta_n)_{n\geq 1}$ is a strictly increasing sequence of positive real numbers for which $\beta_* = \infty$.

Put $\Delta\lambda_n := \lambda_{n+1} - \lambda_n > 0$ and $a_n := \log \beta_n$.

It is convenient to have the standard equivalence

$$\beta_* = \liminf_{n\to\infty} \beta_n^{1/\lambda_n} = \infty \iff \liminf_{n\to\infty} \frac{a_n}{\lambda_n} = \infty, \tag{4.1}$$

and introduce the "weighted increment"

$$S_n := \frac{a_{n+1} - a_n}{\Delta\lambda_n} > 0.$$

Then we have $\Omega_n = -S_n$.

4.1. The quantity $\inf_{n\in\mathbb{N}}\{\Omega_n\}$ and $\liminf\limits_{n\to\infty}\Omega_n$

We have the following result.

Proposition 4.1. *If $\beta_* = \infty$ (equivalently $\liminf \dfrac{a_n}{\lambda_n} = \infty$), then*

$$\limsup_{n\to\infty} S_n = \infty.$$

Consequently,

$$\liminf_{n\to\infty} \Omega_n = -\infty \quad \textit{and hence} \quad \inf_{n\in\mathbb{N}}\{\Omega_n\} = -\infty.$$

Proof. Suppose, to the contrary, that $\limsup S_n < \infty$. Then there exists $C > 0$ with $S_n \le C$ for all n. Summing telescopically,

$$a_n - a_1 = \sum_{k=1}^{n-1}(a_{k+1} - a_k) = \sum_{k=1}^{n-1} S_k\,\Delta\lambda_k \le C\sum_{k=1}^{n-1}\Delta\lambda_k = C(\lambda_n - \lambda_1).$$

Dividing by λ_n and letting $n \to \infty$ yields

$$\limsup_{n\to\infty}\frac{a_n}{\lambda_n} \le C,$$

contradicting the assumption $\liminf \dfrac{a_n}{\lambda_n} = \infty$. Therefore $\limsup S_n = \infty$.

Since $\Omega_n = -S_n \le 0$, it follows that

$$\liminf_{n\to\infty}\Omega_n = -\infty,$$

which also implies $\inf\limits_{n\in\mathbb{N}} \Omega_n = -\infty$. ∎

By (3.3), we have $\kappa_1 = -\infty$. In view of Proposition 4.1, this yields the following corollary.

Corollary 4.2. *Every dominating set for the space $\mathcal{H}^2(\beta)$ has lower bound $-\infty$. Consequently, if a dominating set exists, it must be of the form $(-\infty, c]$ with $c < \infty$.*

4.2. The quantities $\limsup\limits_{n\to\infty}\Omega_n$ and $\sup\limits_{n\in\mathbb{N}}\Omega_n$

In contrast with Proposition 4.1, the behavior of $\limsup\limits_{n\to\infty}\Omega_n$ and $\sup\limits_{n\in\mathbb{N}}\Omega_n$ is more delicate.

The recursion $a_{n+1} = a_n + S_n \, \Delta\lambda_n$ yields

$$a_n = a_1 + \sum_{k=1}^{n-1} S_k \, \Delta\lambda_k. \tag{4.2}$$

We will also use the standard equivalence (4.1).

4.2.1. General facts about $\sup \Omega_n$ and $\limsup \Omega_n$

Since $\Omega_n < 0$ for all n, we always have

$$\limsup_{n\to\infty} \Omega_n \;\le\; \sup_{n\in\mathbb{N}} \Omega_n \;\le\; 0.$$

Moreover, because $\Omega_n = -S_n$,

$$\sup_{n\in\mathbb{N}} \Omega_n \;=\; -\inf_{n\in\mathbb{N}} S_n. \tag{4.3}$$

In particular:

- $\sup \Omega_n = 0 \iff \inf S_n = 0$;
- for any $c > 0$, $\sup \Omega_n = -c \iff \inf S_n = c$.

4.2.2. Feasibility of prescribed $\inf S_n$

We give a construction which works for all (λ_n) (no additional growth control on λ_n is needed).

Lemma 4.3. *Let $c > 0$. There exists a sequence (S_n) of positive real numbers with $\inf_n S_n = c$ such that*

$$a_{n+1} = a_n + S_n \, \Delta\lambda_n, \qquad \liminf_{n\to\infty} \frac{a_n}{\lambda_n} = \infty.$$

Consequently, with $\beta_n := e^{a_n}$, we have $\liminf_{n\to\infty} \beta_n^{1/\lambda_n} = \infty$ and

$$\sup_n \Omega_n = -\inf_n S_n = -c \in (-\infty, 0).$$

Proof. Define

$$g(\lambda) := (1+\lambda)\log(1+\lambda) - \lambda, \qquad \lambda > 0.$$

Then $g'(\lambda) = \log(1+\lambda)$ is increasing and unbounded, so g is increasing.

Set

$$S_n := c + \frac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta\lambda_n}, \qquad a_{n+1} = a_n + S_n\,\Delta\lambda_n.$$

Since g is increasing, the forward difference quotient is nonnegative; hence $S_n \geq c > 0$. In fact, by the mean value theorem there exists $\xi_n \in (\lambda_n, \lambda_{n+1})$ with

$$\frac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta\lambda_n} = g'(\xi_n) = \log(1+\xi_n) \geq \log(1+\lambda_n),$$

so $S_n \geq c + \log(1+\lambda_n)$.

By (4.2),

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} S_k\,\Delta\lambda_k = a_1 + \sum_{k=1}^{n-1}\Big(c\,\Delta\lambda_k + g(\lambda_{k+1}) - g(\lambda_k)\Big) \\ &= a_1 + c(\lambda_n - \lambda_1) + g(\lambda_n) - g(\lambda_1). \end{aligned}$$

Therefore

$$\frac{a_n}{\lambda_n} = c + \frac{g(\lambda_n)}{\lambda_n} + O\Big(\frac{1}{\lambda_n}\Big).$$

Since

$$\frac{g(\lambda)}{\lambda} = \log(1+\lambda) - 1 + \frac{\log(1+\lambda)}{\lambda} \xrightarrow[\lambda\to\infty]{} \infty,$$

we obtain $\liminf_{n\to\infty} \frac{a_n}{\lambda_n} = \infty$.

As constructed, $S_n \geq c$ for all n, with $S_n > c$ (strictly) for all n. To enforce $\inf_n S_n = c$, modify a single index m by setting $S_m := c$. This changes a_n only by a fixed constant for $n \geq m+1$ (it removes the single increment $g(\lambda_{m+1}) - g(\lambda_m)$), so the asymptotics of a_n/λ_n are unaffected. Thus $\inf_n S_n = c$, and since $\Omega_n = -S_n$, we have $\sup_n \Omega_n = -c$. ■

Remark 4.4.

1) The case $c = 0$ can be handled similarly: take $S_n = \dfrac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta\lambda_n} > 0$ and, if desired, lower S_n along a sparse subsequence to approach 0. The dominant $g(\lambda_n)$ term still forces $\liminf \dfrac{a_n}{\lambda_n} = \infty$.

2) In Lemma 4.3, no hypothesis on $L = \limsup_{n\to\infty} \dfrac{\log n}{\lambda_n}$ is needed. The proof only requires $\lambda_n \uparrow \infty$, since $\dfrac{g(\lambda)}{\lambda} \to \infty$ ensures $\liminf \dfrac{a_n}{\lambda_n} = \infty$ for the constructed a_n.

4.2.3. Range of $\sup \Omega_n$

Theorem 4.5. *Under the standing assumptions* $(\lambda_n) \uparrow \infty$ *and* $(\beta_n) \uparrow$ *with* $\liminf \beta_n^{1/\lambda_n} = \infty$, *one always has* $\Omega_n < 0$, *hence* $\sup \Omega_n \le 0$. *Moreover, for every value* $S \in (-\infty, 0)$ *there exists a strictly increasing sequence* (β_n) *satisfying* $\liminf \beta_n^{1/\lambda_n} = \infty$ *and such that* $\sup_n \Omega_n = S$. *Equivalently, the set of attainable values of* $\sup \Omega_n$ *is exactly* $(-\infty, 0]$, *with the endpoint* 0 *attainable under mild assumptions on the gaps.*

Proof. For $S < 0$, set $c := -S > 0$ and apply Lemma 4.3. For $S = 0$, use the $c = 0$ variant described in the preceding remark.

4.2.4. Relation to $\limsup \Omega_n$

Proposition 4.6. *Let the standing assumptions hold. Then*

1. $\sup \Omega_n = 0 \iff \limsup \Omega_n = 0$.
2. *If* $\limsup \Omega_n = C < 0$, *then* $C \le \sup \Omega_n < 0$, *and every value in* $[C, 0)$ *can occur as* $\sup \Omega_n$ *(by altering finitely many* S_n*).*
3. *If* $\limsup \Omega_n = -\infty$, *then* $\Omega_n \to -\infty$, *so* $\sup \Omega_n$ *is the maximum of finitely many initial terms (a finite negative number). In particular,* $\sup \Omega_n \ne -\infty$.

Proof. (1) If $\limsup \Omega_n = 0$, then $\sup \Omega_n \ge 0$ but $\Omega_n < 0$ forces $\sup \Omega_n = 0$. Conversely, if $\sup \Omega_n = 0$ then Ω_n comes arbitrarily close to 0 infinitely often, so $\limsup \Omega_n = 0$.

(2) By definition of $\limsup$, for every $\varepsilon > 0$ there exists N such that $\Omega_n \le C + \varepsilon$ for all $n \ge N$. Thus

$$\sup_n \Omega_n \le \max\{\Omega_1, \dots, \Omega_{N-1}, C + \varepsilon\}.$$

Letting $\varepsilon \downarrow 0$ shows $\sup \Omega_n \in [C, 0)$. By altering finitely many S_n we can adjust $\inf S_n$ (hence $\sup \Omega_n$) to any target value in $[C, 0)$ without changing $\limsup$.

(3) If $\limsup \Omega_n = -\infty$, then $\Omega_n \to -\infty$. Thus only finitely many terms exceed any fixed negative threshold. The supremum is then the maximum among finitely many values, hence a finite negative number. ∎

Now we consider three interesting issues on $\limsup \Omega_n$ and $\sup \Omega_n$.

A) Achieving $\sup \Omega_n = -c$ *and* $\limsup \Omega_n = -\infty$.

Fix $c > 0$. Start from the baseline with $d = c$:

$$S_n := c + \frac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta \lambda_n} \quad (> c).$$

Then overwrite at one index m by setting $S_m := c$ (this forces $\inf S_n = c$ and hence $\sup \Omega_n = -c$). By the mean value theorem,

$$\frac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta\lambda_n} = g'(\xi_n) = \log(1+\xi_n) \geq \log(1+\lambda_n) \xrightarrow[n\to\infty]{} \infty,$$

hence $S_n \to \infty$ and $\Omega_n = -S_n \to -\infty$. Thus $\limsup \Omega_n = -\infty$, while the baseline term still yields $\liminf \dfrac{a_n}{\lambda_n} = \infty$.

B) Prescribe $\limsup \Omega_n = C < 0$ *and any* $\sup \Omega_n \in [C, 0)$.

Let $C = -d$ with $d > 0$. Begin with the baseline S_n^{base} (with this d). Then:

- To make $\limsup \Omega_n = C$, ensure that S_n assumes the value d *infinitely often*: choose an infinite set I_0 and overwrite $S_n := d$ for $n \in I_0$. Then $\liminf S_n = d$ and hence $\limsup \Omega_n = -d = C$.

- To choose $\sup \Omega_n = S \in [C, 0)$, pick any $c \in (0, d]$ with $S = -c$, and at one (or finitely many) indices overwrite $S_m := c$. Then $\inf_n S_n = c$ while $\liminf S_n = d$, so

$$\limsup \Omega_n = -d = C, \quad \sup \Omega_n = -c = S.$$

Since only finitely or countably many *local* overwrites are made on top of the baseline that already gives

$$a_n \;=\; a_1 + d(\lambda_n - \lambda_1) + g(\lambda_n) - g(\lambda_1),$$

we still have $\liminf \dfrac{a_n}{\lambda_n} = \infty$.

C) Achieving $\sup \Omega_n = 0$ *(i.e.* $\inf S_n = 0$*)*.

We now produce $\inf S_n = 0$ while keeping $\liminf \dfrac{a_n}{\lambda_n} = \infty$.

C1. Bounded gaps. If $\sup_n \Delta\lambda_n < \infty$, then sparse small overwrites contribute only a bounded total loss to a_n. Thus $\liminf a_n/\lambda_n = \infty$ persists and $\sup \Omega_n = 0$.

C2. General gaps. Without any restriction on the gaps, one can still arrange $\inf S_n = 0$ by choosing the overwrite indices (n_j) so sparse that their cumulative loss is controlled by a fixed fraction of the baseline g-growth. This guarantees that

$$\frac{a_n}{\lambda_n} \;\geq\; \tfrac{1}{2}\frac{g(\lambda_n)}{\lambda_n} + o(1) \;\to\; \infty,$$

while $\inf S_n = 0$ ensures $\sup \Omega_n = 0$.

Start from the baseline with $d = 0$:

$$S_n^{\text{base}} = \frac{g(\lambda_{n+1}) - g(\lambda_n)}{\Delta\lambda_n}.$$

Choose inductively an increasing index sequence (n_j) so sparse that for each J,

$$\sum_{j=1}^{J} \big(g(\lambda_{n_j+1}) - g(\lambda_{n_j})\big) \ \le \ \frac{1}{2}\,\big(g(\lambda_{n_J}) - g(\lambda_1)\big).$$

Now overwrite $S_{n_j} := \varepsilon_j$ with $\varepsilon_j \downarrow 0$ (e.g. $\varepsilon_j = 1/j$). Then $\inf_n S_n = 0$ and, for all $n \ge n_J$,

$$\begin{aligned} a_n &= a_1 + \sum_{k<n,\ k\notin\{n_j\}} \big(g(\lambda_{k+1}) - g(\lambda_k)\big) \ + \sum_{j:\ n_j<n} \varepsilon_j\,\Delta\lambda_{n_j} \\ &\ge a_1 + \frac{1}{2}\big(g(\lambda_n) - g(\lambda_1)\big). \end{aligned}$$

Therefore, $\dfrac{a_n}{\lambda_n} \ \ge \ \dfrac{1}{2}\cdot\dfrac{g(\lambda_n)}{\lambda_n} \ + \ o(1) \ \xrightarrow[n\to\infty]{} \ \infty$, and $\sup\Omega_n = \sup(-S_n) = 0$.

4.2.5. Summary of the three issues

- For any $c > 0$: $\sup\Omega_n = -c$ is achieved by setting $S_m = c$ at some index and keeping $S_n \ge c$ elsewhere (A).
- For any $C < 0$ and any $S \in [C, 0)$: set $\liminf S_n = -C$ (i.e. $S_n = d = -C$ infinitely often) and choose $\inf S_n = c = -S \le d$ (B): then $\limsup\Omega_n = C$ and $\sup\Omega_n = S$.
- For $\sup\Omega_n = 0$: make $\inf S_n = 0$ via sparse small overwrites on top of the g-baseline, either under bounded gaps (C1) or by sparse trimming (C2) in full generality; in both cases $\liminf\dfrac{a_n}{\lambda_n} = \infty$ is preserved.

4.2.6. Classification of attainable pairs $(\sup\Omega_n, \limsup\Omega_n)$

Theorem 4.7. *Let $(\lambda_n)\uparrow\infty$ and $(\beta_n)\uparrow$ with $\liminf\beta_n^{1/\lambda_n} = \infty$, and define*

$$S_n = \frac{a_{n+1} - a_n}{\Delta\lambda_n} > 0, \qquad \Omega_n = -S_n < 0.$$

Then the set of attainable pairs $(\sup\Omega_n, \limsup\Omega_n)$ is exactly

$$\big\{(-c, -\infty) : c > 0\big\} \ \cup\ \big\{(S, C) : C < 0,\ S \in [C, 0)\big\} \ \cup\ \{(0, 0)\}.$$

In words:

(A) For every $c > 0$ *there exists a construction with*

$$\sup \Omega_n = -c, \qquad \limsup \Omega_n = -\infty.$$

(B) For every $C < 0$ *and every* $S \in [C, 0)$ *there exists a construction with*

$$\sup \Omega_n = S, \qquad \limsup \Omega_n = C.$$

(C) There exists a construction with

$$\sup \Omega_n = 0, \qquad \limsup \Omega_n = 0,$$

attainable under bounded gaps $\sup_n \Delta\lambda_n < \infty$ *or by sparse trimming in the general case.*

In conclusion, the foregoing results naturally lead to the following question.

Question 4.8. *Does there exist a finite real number* $c < \infty$ *such that the set* $(-\infty, c]$ *is dominating for the space* $\mathcal{H}^2(\beta)$*?*

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