A new approach to the index of differential-algebraic equations based on the index of matrix pencil

Ngo Thi Thanh Nga (Hanoi, Vietnam)

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Abstract. Differential-algebraic equations (DAEs) play an important role in many applied models, yet their analysis is complicated by algebraic constraints. A central difficulty is the definition of the index. We propose a new index definition for DAEs based on the classical index of a matrix pencil. This approach ensures projection independence and provides explicit solution formulas for DAEs of index at most two, while also clarifying the relation to März's index.

1. Introduction

Since the 1980s, differential-algebraic equations (DAEs) have been the subject of extensive research due to their broad applications in circuit theory, electrical networks, constrained mechanical systems, and control theory (see [1, 2, 3, 7, 12]). Compared with ordinary differential equations (ODEs), DAEs present additional challenges because they involve algebraic constraints that must be treated consistently in both theoretical and numerical analysis. A central concept in this context is the index of a DAE, which plays a decisive role in transforming the system into a semi-explicit form that separates its differential dynamics from its algebraic constraints (see [6, 7]). Important results on stability radii, Lyapunov exponents, the Bohl spectrum, and the Sacker–Sell spectrum for DAEs were obtained by Du, Linh, and Mehrmann (see [4, 5, 8, 9]). Fundamental contributions to index theory, stability analysis,

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and numerical methods for DAEs have been made by Campbell, März, Petzold, Kunkel, Mehrmann, and many others. In particular, R. März introduced an influential definition of the index and derived corresponding solution formulas based on projections onto kernel spaces of matrices (see [10, 11, 6]). Nevertheless, an important issue remains open: although März established projection independence for index-one DAEs, this property has not yet been explicitly demonstrated for index-two systems. As a result, the validity and general applicability of März's definition for index-two DAEs remain uncertain. This raises several natural research questions: (1) Can the traditional concept of the index of a matrix pair be extended to provide a consistent definition of the index for DAEs of order higher than one? (2) Can such an approach ensure projection independence while yielding explicit solution formulas? (3) What is the precise relationship between this new definition and that of März?

The equivalence between index one in the sense of März's definition and the Kronecker index of a matrix pair was established in [13] (Theorem 5.1.4, p. 243). However, no such connection has been identified in the case of index two. Therefore, in this paper we propose a new approach to index two based on the concept of the Kronecker index of a matrix pencil. We demonstrate that this notion of index is independent of the choice of projection and provide an explicit formula-based solution method for this case.

Organization of the paper. Section 2 recalls the necessary preliminaries, including the Kronecker index of a matrix and the index of a matrix pencil, and concludes with the introduction of a new approach for solving autonomous systems of index 2. Section 3 reviews the method of R. März for differential-algebraic equations of index 1 and 2. In Section 4, we further develop our approach to DAEs of index 2 in the general non-autonomous case and provide an illustrative example. The paper ends with concluding remarks and recommendations.

2. Preliminaries

In this section, we recall some basic notions and properties of the index of a matrix, which play an important role in the study of differential-algebraic equations (DAEs). We start with the definition of the Kronecker index (see [6]).

2.1. The Kronecker index of a matrix

Let $A \in \mathbb{R}^{m\times m}$ with $A \neq 0$. Since

$$A^{k+1} = A \cdot A^k = A^k \cdot A,$$

we immediately obtain

$$\mathrm{Ker}\,(A^k) \subset \mathrm{Ker}\,(A^{k+1}), \quad \mathrm{Im}\,(A^{k+1}) \subset \mathrm{Im}\,(A^k).$$

This observation suggests that both the kernel and the image of the iterates of A stabilize after finitely many steps. Motivated by this, we introduce the following quantities:

$$p = \inf\{k \in \mathbb{N} : \mathrm{Im}\,(A^{k+1}) = \mathrm{Im}\,(A^k)\}, \quad q = \inf\{k \in \mathbb{N} : \mathrm{Ker}\,(A^{k+1}) = \mathrm{Ker}\,(A^k)\}.$$

It is not difficult to show that $p = q$. This common value is called the *index* of the matrix A.

Definition 2.1. *The* index *of a matrix A, denoted by $ind\,(A)$, is the smallest nonnegative integer k such that*

$$Ker\,(A^{k+1}) = Ker\,(A^k).$$

It is clear that $\mathrm{ind}\,(A) = 0$ if and only if A is nonsingular.

The next proposition provides some basic algebraic properties of the index, which will be useful for later analysis (see [6]).

Proposition 2.1. *Let $A \in \mathbb{R}^{m\times m}$ with $A \neq 0$. The following statements hold*

(i) If $k = ind\,(A)$, then $\mathbb{R}^m = Im\,(A^k) \oplus Ker\,(A^k)$.

(ii) If W is invertible, then $ind\,(W^{-1}AW) = ind\,(A)$.

(iii) If W is invertible and $WA = AW$, then

$$ind\,(WA) = ind\,(AW) = ind\,(A).$$

The results above provide the basic framework for understanding the algebraic structure of matrices with nonzero index. This foundation will be essential in the subsequent sections, where we extend the notion of index to matrix pairs (A, B) and apply it to the analysis of differential-algebraic equations.

2.2. The index of a matrix pencil

In the previous subsection, we introduced the notion of the index of a single matrix, which captures the stabilization of the image and kernel of its powers. However, in the study of differential-algebraic equations (DAEs), one typically encounters pairs of matrices (A, B) rather than a single matrix. This naturally leads to the concept of the index of a matrix pencil, which generalizes the Kronecker index to pairs and provides a fundamental tool for analyzing the solvability and structure of DAEs (see [6]).

Definition 2.2. *A matrix pencil $\{A, B\}$ is called* regular *if there exists $c \in \mathbb{R}$ such that $cA + B$ is invertible. In this case, the* index *of the matrix pencil $\{A, B\}$, denoted by $ind\{A, B\}$, is defined as*

$$ind\{A, B\} = ind\big((cA + B)^{-1}A\big).$$

Since the index of a matrix pencil is meaningful only for regular pencils, in what follows it is implicitly assumed that all pencils under consideration are regular.

The following result collects some useful properties of the index of a matrix pencil.

Proposition 2.2. *Properties of the index of a matrix pencil*

(i) If W is nonsingular, then

$$ind\{WA, WB\} = ind\{AW, BW\} = ind\{A, B\}.$$

(ii) If A and B commute, then

$$ind\{A, B\} = ind(A).$$

(iii) We have

$$A(cA + B)^{-1}B = B(cA + B)^{-1}A.$$

(iv) If $\{A, B\}$ is regular, then so is $\{A, B + sA\}$, and

$$ind\{A, B\} = ind\{A, B + sA\}.$$

The next two propositions give practical characterizations of pencils of index at most one and two, respectively.

Proposition 2.3. *Equivalent conditions for index less than or equal to* 1*:*

(i) $ind\{A, B\} \leq 1$.

(ii) $A + BQ$ *is nonsingular for any projection* Q *onto* $Ker(A)$.

(iii) $N \cap S = \{\theta\}$, *where* $N = Ker(A)$ *and* $S = \{z \in \mathbb{R}^m : Bz \in Im(A)\}$.

Proposition 2.4. *Equivalent statements for index not exceeding* 2*:*

(i) $ind\{A, B\} \leq 2$.

(ii) $AP_2 + BQ_2$ *is invertible, where* Q_2 *is any projection onto* $N_2 = Ker\{((cA+B)^{-1}A)^2\}$ *and* $P_2 = I - Q_2$.

(iii) $A + BQ_2$ *is invertible.*

(iv) $N_2 \cap S_2 = \{\theta\}$, *where* $N_2 = Ker\{((cA + B)^{-1}A)^2\}$ *and* $S_2 = \{z \in \mathbb{R}^m : Bz \in Im(AP_2)\}$.

The above characterization can be extended to higher indices as follows.

Proposition 2.5. *Equivalent statements for index* n $(n \geq 2)$*:*

(i) $ind\{A, B\} \leq n$.

(ii) $AP_n + BQ_n$ *is invertible, where* Q_n *is any projection onto* $Ker\{((cA + B)^{-1}A)^n\}$ *and* $P_n = I - Q_n$.

(iii) $A + BQ_n$ *is invertible.*

(iv) $N_n \cap S_n = \{\theta\}$, *where* $N_n = Ker\{((cA + B)^{-1}A)^n\}$ *and* $S_n = \{z \in \mathbb{R}^m : Bz \in Im(AP_n)\}$.

An important technical aspect is that the spaces N_n defined above do not depend on the choice of the parameter c.

Proposition 2.6. *The space* $N_2 = Ker\{A(cA + B)^{-1}A\}$ *is independent of the choice of* c.

Proposition 2.7. *The space*

$$N_n = Ker\left(A\{(cA + B)^{-1}A\}^{n-1}\right) = Ker\left(((cA + B)^{-1}A)^n\right)$$

is independent of the choice of c.

These results lay the foundation for studying differential-algebraic equations of index two, which will be the focus of the following sections. Using the concepts of the index of a matrix pair, we examine the solvability of the equation

$$(2.1) \qquad Ax'(t) + Bx(t) = q(t), \quad t \in J \subset \mathbb{R}, \qquad \text{where } A, B \in \mathbb{R}^{m\times m}.$$

Several authors have employed the Kronecker decomposition in order to separate a differential-algebraic equation into a differential component and an algebraic component. This procedure, however, involves a rather large amount of matrix computations. In the sequel, we propose a new approach. For clarity of exposition, we restrict ourselves to the case of index two, i.e. $\operatorname{ind}\{A, B\} = 2$.

Let $\tilde{Q}_2$ be an arbitrary projection onto the subspace

$$N_2 = \operatorname{Ker}\{A(cA + B)^{-1}A\},$$

and define

$$\tilde{G}_2 = A\tilde{P}_2 + B\tilde{Q}_2, \quad Q_2 = \tilde{Q}_2\tilde{G}_2^{-1}B.$$

From Proposition 2.6, we know that N_2 is independent of c; hence the projections Q_2 and $\tilde{Q}_2$ are also independent of c.

Proposition 2.8. *The following statements hold:*

(i) Q_2 *is a projection onto* N_2,

(ii) $P_2G_2^{-1}AQ_2 = 0$,

(iii) $PQ_2G_2^{-1}AQ_2 = 0$,

(iv) $PQ_2 = PQ_2G_2^{-1}B$.

Proof. (i) As established in Proposition 2.4, if $\operatorname{ind}\{A, B\} \leq 2$, then for any projection $\tilde{Q}_2$ onto N_2, the operator $\tilde{G}_2 = A\tilde{P}_2 + B\tilde{Q}_2$ is invertible.

From the relations $\tilde{G}_2\tilde{P}_2 = A\tilde{P}_2 \Rightarrow \tilde{P}_2 = \tilde{G}_2^{-1}A\tilde{P}_2$ and $\tilde{G}_2\tilde{Q}_2 = B\tilde{Q}_2 \Rightarrow \tilde{Q}_2 = \tilde{G}_2^{-1}B\tilde{Q}_2$, it follows that $\tilde{Q}_2 = \tilde{Q}_2\tilde{G}_2^{-1}B\tilde{Q}_2$ and $Q_2^2 = Q_2$.

Moreover, $\operatorname{Im} Q_2 = \operatorname{Im}\tilde{Q}_2 = N_2$ (since $Q_2 = \tilde{Q}_2\tilde{G}_2^{-1}B$ and $Q_2\tilde{Q}_2 = \tilde{Q}_2\tilde{G}_2^{-1}B\tilde{Q}_2 = \tilde{Q}_2$), so Q_2 is indeed a projection onto N_2. Thus, the proof of (i) is completed.

(ii) For all $x \in \mathbb{R}^m$ we prove that $P_2G_2^{-1}AQ_2x = \theta$. Indeed, set $y = G_2^{-1}AQ_2x$. Then $AQ_2x = (AP_2 + BQ_2)y$, using (iii) of Proposition 2.2 we obtain

$$\begin{aligned} A\{(cA + B)^{-1}A\}^2Q_2x =& A(cA + B)^{-1}A(cA + B)^{-1}AP_2y \\ &+ B\{(cA + B)^{-1}A\}^2Q_2y. \end{aligned}$$

Hence,

$$\theta = A\{(cA + B)^{-1}A\}^2y + \theta \quad \Rightarrow \quad A\{(cA + B)^{-1}A\}^2y = \theta.$$

Since $\operatorname{ind}\{A,B\}=2$, it follows that

$$A(cA+B)^{-1}Ay=\theta \text{ i.e. } Q_2y=y.$$

Therefore, $P_2G_2^{-1}AQ_2x=P_2Q_2y=\theta$. This completes the proof of (ii).

(iii) For all $x\in\mathbb{R}^m$ we prove that $PQ_2G_2^{-1}AQ_2x=\theta$. We have

$$(AP_2+BQ_2)y=AQ_2x$$

which yields

$$(AP_2+BQ_2)Q_2y=AQ_2x \quad\Rightarrow\quad BQ_2y=By=AQ_2x.$$

Therefore,

$$A(cA+B)^{-1}By=A(cA+B)^{-1}AQ_2x=\theta.$$

On the other hand, $A(cA+B)^{-1}Ay=A(cA+B)^{-1}AQ_2y=\theta$. Thus $A(cA+B)^{-1}(cA+B)y=\theta$ i.e. $Ay=\theta$. Hence $y=Qy\Rightarrow Py=PQy=\theta$, which implies $PG_2^{-1}AQ_2x=\theta$.

Since $PP_2G_2^{-1}AQ_2x=\theta$, we conclude that $PQ_2G_2^{-1}AQ_2x=\theta$. Thus, (iii) is proved.

(iv) We have

$$\begin{aligned}G_2^{-1}\widetilde{G}_2&=G_2^{-1}A\widetilde{P}_2+G_2^{-1}B\widetilde{Q}_2\\&=G_2^{-1}A(P_2+Q_2)\widetilde{P}_2+G_2^{-1}B(P_2+Q_2)\widetilde{Q}_2\\&=P_2\widetilde{P}_2+G_2^{-1}AQ_2\widetilde{P}_2+Q_2\widetilde{Q}_2+G_2^{-1}BP_2\widetilde{Q}_2.\end{aligned}$$

Therefore,

$$\begin{aligned}PQ_2G_2^{-1}B&=PQ_2\big(P_2\widetilde{P}_2+G_2^{-1}AQ_2\widetilde{P}_2+Q_2\widetilde{Q}_2+G_2^{-1}BP_2\widetilde{Q}_2\big)\widetilde{G}_2^{-1}B\\&=(PQ_2P_2\widetilde{P}_2+PQ_2G_2^{-1}AQ_2\widetilde{P}_2)\widetilde{G}_2^{-1}B\\&\quad+PQ_2\widetilde{Q}_2\widetilde{G}_2^{-1}B+PQ_2G_2^{-1}BP_2\widetilde{Q}_2\widetilde{G}_2^{-1}B.\end{aligned}$$

Thus,

$$PQ_2G_2^{-1}B=PQ_2Q_2+PQ_2G_2^{-1}BP_2Q_2=PQ_2.$$

Hence, the assertion (iv) follows. ■

Back to the DAE. Applying G_2^{-1} to both sides of (2.1) gives

$$G_2^{-1}Ax'(t)+G_2^{-1}Bx(t)=G_2^{-1}q(t). \tag{2.2}$$

Applying successively the operators P_2, PQ_2, and Q_2 to both sides of (2.2), we obtain:

1) With P_2:

$$\begin{aligned}&P_2G_2^{-1}Ax' + P_2G_2^{-1}Bx = P_2G_2^{-1}q\\ \Leftrightarrow\ &P_2G_2^{-1}A(P_2+Q_2)x' + P_2G_2^{-1}B(P_2+Q_2)x = P_2G_2^{-1}q\\ \Leftrightarrow\ &P_2x' + P_2G_2^{-1}BP_2x = P_2G_2^{-1}q.\end{aligned}$$

Setting $P_2x = u$, we obtain the equation:

$$u' + P_2G_2^{-1}Bu = P_2G_2^{-1}q. \tag{2.3}$$

2) With PQ_2:

$$\begin{aligned}&PQ_2G_2^{-1}Ax' + PQ_2G_2^{-1}Bx = PQ_2G_2^{-1}q\\ \Leftrightarrow\ &PQ_2G_2^{-1}A(P_2+Q_2)x' + PQ_2G_2^{-1}B(P_2+Q_2)x = PQ_2G_2^{-1}q\\ \Leftrightarrow\ &PQ_2x = PQ_2G_2^{-1}q.\end{aligned}$$

Setting $PQ_2x = w$, we obtain the equation:

$$w = PQ_2G_2^{-1}q. \tag{2.4}$$

3) With Q_2:

$$\begin{aligned}&Q_2G_2^{-1}Ax' + Q_2G_2^{-1}Bx = Q_2G_2^{-1}q\\ \Leftrightarrow\ &Q_2G_2^{-1}A(P_2+Q_2)x' + Q_2G_2^{-1}B(P_2+Q_2)x = Q_2G_2^{-1}q\\ \Leftrightarrow\ &Q_2G_2^{-1}APQ_2x' + Q_2G_2^{-1}BP_2x = Q_2G_2^{-1}q.\end{aligned}$$

Setting $Q_2x = v$, we obtain the equation:

$$Q_2G_2^{-1}Aw' + v + Q_2G_2^{-1}Bu = Q_2G_2^{-1}q. \tag{2.5}$$

Thus the triple system (2.3)–(2.5) determines u and v, and hence

$$x = u + v.$$

Equivalently,

$$\begin{cases} u' + P_2G_2^{-1}Bu = P_2G_2^{-1}q,\\ w = PQ_2G_2^{-1}q,\\ v = -Q_2G_2^{-1}Aw' - Q_2G_2^{-1}Bu + Q_2G_2^{-1}q,\\ x = u + v.\end{cases}$$

Remark. In the autonomous case, the index of the DAE is defined exactly as the index of the matrix pair $\{A, B\}$. This shows that the above decomposition is canonical and does not depend on the particular choice of projections. It will play a central role in the analysis of stability and numerical methods for index–2 DAEs.

3. März's Method

In the case where A and B are time-varying matrices, we obtain the equation

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J \subset \mathbb{R} \tag{3.1}$$

where $A(t), B(t) \in C(J, L(\mathbb{R}^m))$. For this equation, the use of the Kronecker decomposition to separate the solution is no longer applicable, since in general U and V are time-varying matrices. To overcome this difficulty, März introduced a projection-based method to decouple the solution of this equation for indices 1 and 2, respectively, as follows (see [10]).

3.1. Differential-algebraic equations of index 1

Consider the DAE under the assumption that rank $A(t)$ is constant. Set $N(t) = \mathrm{Ker}\,(A(t))$ for $t \in J$, and let $Q(t)$ be a projection onto $N(t)$, where we assume $Q(t) \in C^1(J, L(\mathbb{R}^m))$. Define

$$\begin{aligned} P(t) &= I - Q(t), \\ S(t) &= \{z \in \mathbb{R}^m : B(t)z \in \mathrm{Im}\, A(t)\}, \\ G_1(t) &= A(t) + B(t)Q(t), \quad t \in J. \end{aligned}$$

Definition 3.1. *Equation* (3.1) *is said to have* index 1 *if*

$$N(t) \cap S(t) = \{\theta\}, \quad \forall\, t \in J.$$

Equivalently, this holds if and only if $G_1(t)$ is invertible for all $t \in J$.

Solution decomposition method. If (3.1) has index 1, then

$$G_1^{-1}A = P, \quad G_1^{-1}B = Q + G_1^{-1}BP.$$

For any projection $\widetilde{Q}(t)$ onto $N(t)$, one can choose

$$Q(t) = \widetilde{Q}(t)\widetilde{G}_1^{-1}(t)B(t),$$

which ensures that $Q(t)$ is a projection onto $N(t)$ along $S(t)$, with

$$Q(t) = Q(t)G_1^{-1}(t)B(t).$$

Applying $G_1^{-1}(t)$ to (3.1) gives

$$Px' + Qx + G_1^{-1}BPx = G_1^{-1}q. \tag{3.2}$$

Splitting (3.2) with P and Q, and noting that $QG_1^{-1}BPx = QPx = \theta$, we obtain

$$\begin{cases} Px' + PG_1^{-1}BPx &= PG_1^{-1}q, \\ Qx &= QG_1^{-1}q. \end{cases}$$

Let $Px = u$, $Qx = v$, so $x = u + v$. The solution decomposition is

$$\begin{cases} u' + (PG_1^{-1}B - P')u &= PG_1^{-1}q + P'QG_1^{-1}q, \\ v &= QG_1^{-1}q, \\ x &= u + v. \end{cases}$$

Remark 3.1. *By rewriting the condition $N(t) \cap S(t) = \{\theta\}$ equivalently as the invertibility of $G_1(t)$, März showed that the definition of index 1 is independent of the choice of the projection $Q(t)$. The equivalence between index one in the sense of März's definition and the Kronecker index of a matrix pair also has been established in [13] (Theorem 5.1.4, p. 243).*

3.2. Differential-algebraic equations of index 2

When $G_1 = A + BQ$ is singular, i.e. $N \cap S \neq \{\theta\}$ for some $t \in J$, equation (3.1) can be reformulated as

$$A(t)(P(t)x(t))' + \overline{B}(t)x(t) = q(t),$$

where $\overline{B}(t) = B(t) - A(t)P'(t)$.

Define

$$\begin{aligned} &A_1(t) = A(t) + \overline{B}(t)Q(t), \text{ assume that rank } A_1(t) \text{ is constant.} \\ &N_1(t) = \operatorname{Ker}(A_1(t)), \\ &S_1(t) = \{z \in \mathbb{R}^m : \overline{B}(t)P(t)z \in \operatorname{Im} A_1(t)\}, \quad t \in J. \end{aligned}$$

Definition 3.2. *Equation* (3.1) *is said to have* index 2 *if*

$$N_1(t) \cap S_1(t) = \{\theta\}, \quad \forall t \in J.$$

Proposition 3.1. *Let* (3.1) *be of index* 2*. If $Q_1(t)$ is a projection onto $N_1(t)$ along $S_1(t)$, then*

$$G_2(t) = A(t) + \overline{B}(t)(Q(t) + P(t)Q_1(t))$$

is nonsingular, and we have

$$\begin{aligned} Q_1 = Q_1G_2^{-1}\overline{B}P = G_2^{-1}\overline{B}PQ_1 = Q_1G_2^{-1}\overline{B}PQ_1, \quad Q_1Q = 0, \\ G_2^{-1}A = P_1P, \quad G_2^{-1}\overline{B} = G_2^{-1}\overline{B}PP_1 + Q_1 + Q. \end{aligned}$$

Solution decomposition method. Applying G_2^{-1} to (3.1) yields

$$P_1P(Px)' + G_2^{-1}\overline{B}PP_1x + Q_1x + Qx = G_2^{-1}q,$$

with $x = PP_1x + PQ_1x + Qx$. Introducing $u = PP_1x$, $v = PQ_1x$, $w = Qx$, one obtains the system

$$\begin{cases} u' - (PP_1)'(u+v) + PP_1G_2^{-1}Bu = PP_1G_2^{-1}q, \\ -QQ_1v' + QQ_1(PQ_1)'(u+v) + w + QP_1G_2^{-1}Bu = QP_1G_2^{-1}q, \\ v = PQ_1G_2^{-1}q. \end{cases}$$

Remark 3.2. *Unlike the index* 1 *case, März did not clarify whether the definition of index* 2 *depends on the particular choice of the projection* $Q(t)$*. The connection between index two and the Kronecker index of a matrix pair has not been established either.*

4. A New Approach to Differential-Algebraic Equations of Index 2

At first sight, März's approach does not reveal any apparent connection with the Kronecker index of the matrix pair $\{A(t), \overline{B}(t)\}$. A natural question arises: if system 3.1 has index 1 or 2 in the sense of the above definition, does the pair $\{A(t), \overline{B}(t)\}$ possess Kronecker index 1 or 2, respectively? In this paper, we aim to address this question and, based on the original definition of the Kronecker index, we propose a new approach to decouple equation (3.1) into its differential and algebraic components.

We now return to the DAE system (3.1).

Definition 4.1. *The differential-algebraic equation* (3.1) *is said to be of index* 2 *if*

$$ind\{A(t), B(t) - A(t)P'(t)\} = 2, \quad \forall t \in J,$$

where $P(t)$ *is a projection onto* $Ker(A(t))$.

Proposition 4.1. *The definition of index* 2 *for the DAE* (3.1) *is well posed. Namely, if* Q, $\overline{Q}$ *are two projections onto* $Ker(A)$, *with* $P = I - Q$, $\overline{P} = I - \overline{Q}$, *then*

$$ind\{A, B - AP'\} = 2 \quad \text{if and only if} \quad ind\{A, B - A\overline{P}'\} = 2.$$

Proof. First, we prove that

$$\{A(cA + B - AP')^{-1}A - A(cA + B - A\overline{P}')^{-1}A\}x = \theta$$

if $A(cA+B-AP')^{-1}Ax=\theta$. Indeed, consider the difference

$$\{A(cA+B-AP')^{-1}A - A(cA+B-A\overline{P}')^{-1}A\}x$$

and use the properties $A\overline{P}=AP=A$, $\overline{P}P=\overline{P}$, and $P(cA+B-AP')^{-1}Ax=\theta$, we obtain

$$\begin{aligned}
&\{A(cA+B-AP')^{-1}A - A(cA+B-A\overline{P}')^{-1}A\}x\\
&= A(cA+B-A\overline{P}')^{-1}(A\overline{P}'-AP')(cA+B-AP')^{-1}Ax\\
&= A(cA+B-A\overline{P}')^{-1}\{A(\overline{P}P)'-AP'\}(cA+B-AP')^{-1}Ax\\
&= A(cA+B-A\overline{P}')^{-1}\{A(\overline{P}'P+\overline{P}P')-AP'\}(cA+B-AP')^{-1}Ax\\
&= A(cA+B-A\overline{P}')^{-1}A\overline{P}'P(cA+B-AP')^{-1}Ax=\theta.
\end{aligned}$$

It follows that

$$A(cA+B-A\overline{P}')^{-1}Ax=\theta \quad \text{if } A(cA+B-AP')^{-1}Ax=\theta,$$

or equivalently,

$$\operatorname{Ker} A(cA+B-AP')^{-1}A \subset \operatorname{Ker} A(cA+B-A\overline{P}')^{-1}A.$$

Similarly, one shows that

$$\operatorname{Ker} A(cA+B-A\overline{P}')^{-1}A \subset \operatorname{Ker} A(cA+B-AP')^{-1}A.$$

Therefore,

$$\operatorname{Ker} A(cA+B-AP')^{-1}A = \operatorname{Ker} A(cA+B-A\overline{P}')^{-1}A = N_2. \tag{4.1}$$

Set $B_0=B-AP'$ and $\overline{B}_0=B-A\overline{P}'$. Assume $\operatorname{ind}\{A,B_0\}\le 2$. Then, by Proposition 2.4, we have $\tilde{G}_2=A\tilde{P}_2+B_0\tilde{Q}_2$ invertible. By Proposition 2.8, we can choose a new projection

$$Q_2=\tilde{Q}_2\tilde{G}_2^{-1}B_0,$$

and thus

$$\begin{aligned}
PQ_2G_2^{-1}A &= PQ_2G_2^{-1}A(P_2+Q_2) \quad (\text{note that } G_2^{-1}AP_2=P_2)\\
&= PQ_2P_2+PQ_2G_2^{-1}AQ_2=0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\overline{G}_2 &= AP_2+\overline{B}_0Q_2\\
&= (AP_2+B_0Q_2)+A(P'-\overline{P}')Q_2\\
&= G_2+A[(P\overline{P})'-\overline{P}']Q_2 \quad (\text{since } P\overline{P}=P)\\
&= G_2+(AP\overline{P}'+AP'\overline{P}-A\overline{P}')Q_2\\
&= G_2(I+G_2^{-1}AP'\overline{P}Q_2).
\end{aligned}$$

Setting $X = I + G_2^{-1}AP'\overline{P}Q_2$ and $Y = I - G_2^{-1}AP'\overline{P}Q_2$, we get

$$\begin{aligned} XY = YX &= I - G_2^{-1}AP'\overline{P}PQ_2G_2^{-1}AP'\overline{P}Q_2 \\ &= I - G_2^{-1}AP'\overline{P}PQ_2G_2^{-1}AP'\overline{P}Q_2 = I \end{aligned}$$

(since $PQ_2G_2^{-1}A = 0$), which implies that X is invertible. Therefore, $\overline{G}_2 = G_2X$ is invertible, and by Proposition 2.8, we obtain

$$\operatorname{ind}\{A, \overline{B}_0\} \leq 2.$$

Similarly, one can prove that if $\operatorname{ind}\{A, \overline{B}_0\} \leq 2$ then $\operatorname{ind}\{A, B_0\} \leq 2$. On the other hand, from (4.1) we have $\operatorname{ind}\{A, B_0\} \leq 1$ if and only if $\operatorname{ind}\{A, \overline{B}_0\} \leq 1$. Hence,

$$\operatorname{ind}\{A, B_0\} = 2 \quad \text{if and only if} \quad \operatorname{ind}\{A, \overline{B}_0\} = 2.$$

■

Reduction of the system. We rewrite (3.1) as

$$A(t)(P(t)x(t))' + B_0(t)x(t) = q(t), \qquad B_0(t) = B(t) - A(t)P'(t).$$

Proceeding as in the autonomous case, with respect to the pair (A, B_0), we obtain

$$\begin{cases} P_2(Px)' + P_2G_2^{-1}B_0P_2x & = P_2G_2^{-1}q, \\ PQ_2x & = PQ_2G_2^{-1}q, \\ Q_2G_2^{-1}AQ_2(Px)' + Q_2x + Q_2G_2^{-1}B_0P_2x & = Q_2G_2^{-1}q. \end{cases}$$

By substituting $B_0 = B - AP'$ into the above system, we obtain

$$\begin{cases} P_2(Px)' + P_2G_2^{-1}BP_2x - P_2G_2^{-1}AP'P_2x = P_2G_2^{-1}q, \\ PQ_2x = PQ_2G_2^{-1}q, \\ Q_2G_2^{-1}AQ_2(Px)' + Q_2x + Q_2G_2^{-1}B_0P_2x = Q_2G_2^{-1}q. \end{cases}$$

We now proceed to analyze each equation of the system.

1) We have

$$\begin{aligned} & P_2(Px)' + P_2G_2^{-1}BP_2x - P_2G_2^{-1}AP'P_2x = P_2G_2^{-1}q, \\ \Leftrightarrow\ & (P_2Px)' - P_2'P(P_2 + Q_2)x - P_2G_2^{-1}A(P_2 + Q_2)P'P_2x = P_2G_2^{-1}q, \\ \Leftrightarrow\ & (P_2x)' - P_2'P(P_2x) - P_2'(PQ_2x) - P_2P'(P_2x) + P_2G_2^{-1}B(P_2x) = P_2G_2^{-1}q. \end{aligned}$$

Note that $Q = Q_2Q$, hence

$$P_2P = P_2 - P_2Q = P_2 - P_2Q_2Q = P_2.$$

Setting $P_2x = u$, $Q_2x = v$, and $PQ_2x = w$, we obtain

$$u' - (P_2'P + P_2P' - P_2G_2^{-1}B)u = P_2G_2^{-1}q + P_2'w,$$

which yields the equation

$$u' - (P_2' - P_2G_2^{-1}B)u = P_2G_2^{-1}q + P_2'w.$$

2) We have

$$w = PQ_2G_2^{-1}q.$$

3) We have

$$\begin{aligned}
&Q_2G_2^{-1}AQ_2(Px)' + Q_2x + Q_2G_2^{-1}BP_2x - Q_2G_2^{-1}AP'P_2x = Q_2G_2^{-1}q,\\
\Leftrightarrow\ &Q_2G_2^{-1}AQ_2(PP_2x + PQ_2x)' + Q_2x + Q_2G_2^{-1}BP_2x - Q_2G_2^{-1}BP_2x\\
&- Q_2G_2^{-1}A(P_2 + Q_2)P'P_2x = Q_2G_2^{-1}q,\\
\Leftrightarrow\ &Q_2G_2^{-1}AQ_2(PQ_2x)' + Q_2x + Q_2G_2^{-1}B(P_2x)\\
&+Q_2G_2^{-1}AQ_2\big[(PP_2x)' - P'P_2x\big] = Q_2G_2^{-1}q,\\
\Leftrightarrow\ &Q_2G_2^{-1}AQ_2(PQ_2x)' + Q_2x + Q_2G_2^{-1}B(P_2x)\\
&+ Q_2G_2^{-1}AQ_2P(P_2x)' = Q_2G_2^{-1}q,\\
\Leftrightarrow\ &G_2^{-1}AQ_2w' + v + Q_2G_2^{-1}Bu + Q_2G_2^{-1}AQ_2Pu' = Q_2G_2^{-1}q.
\end{aligned}$$

Hence, we obtain the equation

$$v = Q_2G_2^{-1}q - G_2^{-1}AQ_2w' - Q_2G_2^{-1}Bu - Q_2G_2^{-1}AQ_2Pu'.$$

Thus, we obtain the following system:

(4.2)
$$u' - (P_2' - P_2G_2^{-1}B)u = P_2G_2^{-1}q + P_2'w,$$

(4.3)
$$w = PQ_2G_2^{-1}q,$$

(4.4)
$$v = Q_2G_2^{-1}q - G_2^{-1}AQ_2w' - Q_2G_2^{-1}Bu - Q_2G_2^{-1}AQ_2Pu'.$$

Conclusion. The equations (4.2)–(4.4) form a closed system for (u, v, w).

Once u and v are computed, the solution to the original DAE is reconstructed as

$$x = u + v.$$

This shows that, even in the non-autonomous setting, the decomposition into differential and algebraic components remains valid and provides a constructive way to solve index–2 DAEs.

4.1. Example

Consider the differential-algebraic equation

$$A(t)X'(t) + B(t)X(t) = q(t),$$

where

$$A(t) = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & -t & t+t^2 & t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 2t & 0 & -1 & -t \\ 1-2t^2 & 1-t^2 & t^2+t+t^3-1 & 2t^2+t^4-1 \\ 0 & 0 & -1 & -t \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

$$q(t) = \begin{pmatrix} t \\ 1 \\ 2t \\ -t \end{pmatrix}, \quad t \in J = (0, +\infty).$$

Choose a projection $Q(t) \in C^1(J, L(\mathbb{R}^4))$ onto $\operatorname{Ker} A(t)$ as follows:

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2t & t & 1-t^2 & -t^3+t^2 \\ 2t & t & 1-t-t^2 & -t^3 \\ -2 & -1 & 1+t & 1+t^2 \end{pmatrix},$$

$$P(t) = I - Q(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2t & 1-t & t^2-1 & t^3-t^2 \\ -2t & -t & t^3+t & t^3 \\ 2 & 1 & -1-t & -t^2 \end{pmatrix}.$$

We have

$$A + BQ = \begin{pmatrix} t & 0 & -1 & -t \\ 2-2t^2 & 1-t-t^2 & 2t^2+2t+t^3-1 & t^3+2t^2+t^4-1 \\ 0 & 0 & -1 & -t \\ 0 & 0 & t & t^2 \end{pmatrix},$$

which is singular for all $t \in (0, +\infty)$.

Define

$$\overline{B}(t) = B - AP' = \begin{pmatrix} 2t & 0 & -1 & -t \\ 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -t \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Since $\overline{B}(t)$ is invertible, we can choose $c = 0$, and hence

$$N_2 = \operatorname{Ker}\{A(cA + B)^{-1}A\} = \operatorname{Ker}\{AB^{-1}A\}.$$

Indeed,

$$AB^{-1}A = \begin{pmatrix} \frac{t}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t \in (0, +\infty).$$

Clearly, we can choose a projection $\tilde{Q}_2(t) \in C^1(J, L(\mathbb{R}^4))$ onto N_2 in the form:

$$\tilde{Q}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{P}_2 = I - \tilde{Q}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\tilde{G}_2(t) = A(t)\tilde{P}_2(t) + \overline{B}(t)\tilde{Q}_2(t) = \begin{pmatrix} t & 0 & -1 & -t \\ 0 & t & -1 & 1-t \\ 0 & t & -1 & -t \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

For $t \in (0, +\infty)$, $\tilde{G}_2(t)$ is invertible and

$$\tilde{G}_2^{-1}(t) = \begin{pmatrix} t^{-1} & 0 & t^{-1} & 0 \\ 0 & t & -1 & 1-t \\ 0 & t & -1 & -t \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

According to Proposition 2.4, we obtain $\operatorname{ind}\{A, B\} = 2$. Hence, the given equation has index 2. We now construct the canonical projection as follows:

$$Q_2(t) = \tilde{Q}_2(t)\tilde{G}_2(t)\overline{B}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad P_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Next,

$$G_2(t) = A(t)P_2(t) + \overline{B}(t)Q_2(t) = \begin{pmatrix} t & 0 & -1 & -t \\ 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -t \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

$$G_2^{-1}(t) = \begin{pmatrix} t^{-1} & 0 & -t^{-1} & 0 \\ -1 & t & t^{-1}-1 & 1-t \\ -1 & t & -1 & -t \\ t^{-1} & -1 & 0 & 1 \end{pmatrix}.$$

Let $P_2x = u,\ Q_2x = v$. We obtain the following system:

$$\begin{cases} u' - (P_2' - P_2G_2^{-1}B)u = P_2G_2^{-1}q + P_2'PQ_2G_2^{-1}q, \\ v = -G_2^{-1}AQ_2(PQ_2G_2^{-1}q)' - Q_2G_2^{-1}Bu - Q_2G_2^{-1}AQ_2Pu' + Q_2G_2^{-1}q. \end{cases}$$

Solving this system, we obtain u, v and hence $x = u + v$. The detailed results are:

$$\begin{cases} u' - \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1-2t & 0 & 0 & 0 \\ 1-2t & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} u = \begin{pmatrix} -1 \\ t \\ t \\ -1 \end{pmatrix}, \\ v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ t^3-t & t^3-1 & -t^2-t^3-t^4 & -2t^3-t^5 \\ t^3-t & t^3 & -1-t^2-t^3-t^4 & -2t^3-t^5 \\ 1-t^2 & -t^2 & t+t^2+t^3 & 2t^2+t^4-1 \end{pmatrix} u + \\ \qquad + \begin{pmatrix} 0 \\ -2t+4t^3+t^4-2t^5 \\ -t+4t^3+t^4-2t^5 \\ -1-4t^2-t^3+2t^4 \end{pmatrix}. \end{cases}$$

From the first equation we obtain

$$u(t) = \begin{pmatrix} -\frac{1}{2} + C_1e^{-2t} \\ -C_1te^{-2t} + \frac{t}{2} + C_2 \\ -C_1te^{-2t} + \frac{t}{2} + C_3 \\ C_1e^{-2t} + C_4 \end{pmatrix}, \quad t \in (0, +\infty).$$

Since $u(t) = P_2(t)x(t)$, we have

$$\begin{cases} -C_1te^{-2t} + \frac{t}{2} + C_2 = -t(-\frac{1}{2} + C_1e^{-2t}), \\ -C_1te^{-2t} + \frac{t}{2} + C_3 = -t(-\frac{1}{2} + C_1e^{-2t}), \\ -C_1e^{-2t} + C_4 = -\frac{1}{2} + C_1e^{-2t}. \end{cases}$$

This system yields $C_2 = 0,\ C_3 = 0,\ C_4 = -\frac{1}{2}$. Hence

$$u(t) = \begin{pmatrix} -\frac{1}{2} + C_1 e^{-2t} \\ -C_1 t e^{-2t} + \frac{t}{2} \\ -C_1 t e^{-2t} + \frac{t}{2} \\ -\frac{1}{2} + C_1 e^{-2t} \end{pmatrix}, \quad t \in (0, +\infty).$$

Substituting into the second equation, we obtain

$$v(t) = \begin{pmatrix} 0 \\ \frac{1}{2} t^5 \\ 0 \\ 0 \end{pmatrix}, \quad t \in (0, +\infty).$$

Therefore, the general solution is

$$x(t) = u(t) + v(t) = \begin{pmatrix} -\frac{1}{2} + C_1 e^{-2t} \\ -C_1 t e^{-2t} + \frac{t}{2} + \frac{t^5}{2} \\ -C_1 t e^{-2t} + \frac{t}{2} \\ -\frac{1}{2} + C_1 e^{-2t} \end{pmatrix}.$$

Conclusion and recommendations

In this paper, we have investigated differential-algebraic equations (DAEs) of index up to two. The main contributions of the study can be summarized as follows:

- We provided a definition of the differential-algebraic equation

  $$A(t)x'(t) + B(t)x(t) = q(t),$$

  with index less than or equal to 2, based on the condition that the index of the matrix pair $(A(t), B(t) - A(t)P'(t))$ is less than or equal to 2, where $Q(t)$ is a projection onto $\operatorname{Ker} A(t)$ and $P(t) = I - Q(t)$. It was proved that this definition does not depend on the choice of projection.

- We proposed a solution method, including explicit computational formulas, for solving differential-algebraic equations with index less than or equal to 2.

- We presented a numerical example to illustrate the applicability of the proposed approach.

The obtained results provide a new perspective on the theory of DAEs of index 2 and contribute to the development of practical solution techniques. Future research may focus on extending the analysis to DAEs of higher index, as well as investigating efficient numerical implementations and applications to real-world models.

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Ngo Thi Thanh Nga
Thang Long University
Hanoi
Vietnam
ngantt@thanglong.edu.vn