

## A survey on different variants of shift operators of meromorphic function normally or partially sharing values or small functions

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**Abstract.** Qi [24] first demonstrated that when an entire function and its shift operator share two distinct periodic functions, they must coincide. This result has inspired further investigation into similar situations within a more generalized framework. In this survey, we systematically develop the theory of periodicity and relevant topics for various shift operators associated with meromorphic functions, all within the framework of normal and partial sharing conditions. This work offers a fresh perspective on the theory of shift operators, streamlining existing results to present them in a compact form. It also includes our own contributions to this specific area of literature with a number illustrative examples.

### 1. Introduction

In the mathematical field of complex analysis, Nevanlinna theory is part of the theory of meromorphic functions. It was devised in 1925, by Rolf Nevanlinna. Hermann Weyl called it "one of the few great mathematical events of (the twentieth) century. The theory describes the asymptotic distribution of solutions of the equation  $f(z) = a$ , as  $a$  varies. A fundamental tool is the

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Nevanlinna characteristic  $T(r, f)$  which measures the rate of growth of a meromorphic function.

The basic notations of Nevanlinna Theory such as  $T(r, f)$ ,  $m(r, f)$ ,  $N(r, f)$ ,  $N(r, \frac{1}{f})$ ,  $S(r, f)$  etc. can be found in W.K. Hayman [11].

**Definition 1.1.** Let  $f(z)$  be a meromorphic function in the complex plane  $\mathbb{C}$ . Then the counting function of  $f(z)$  is defined by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r.$$

**Definition 1.2.** Let  $f(z)$  be a meromorphic function in  $\mathbb{C}$ , proximity function of  $f(z)$  is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

**Definition 1.3.** The characteristic function of a meromorphic function is defined by

$$T(r, f) = N(r, f) + m(r, f),$$

where  $m(r, f)$  is the proximity function and  $N(r, f)$  is the counting function of poles of  $f(z)$ .

**Definition 1.4.** A meromorphic function  $a$  is said to be small function of a meromorphic function  $f$  if  $T(r, a) = S(r, f)$ , where  $S(r, f)$  is any quantity satisfying  $o(T(r, f))$ ,  $r \rightarrow \infty$ , outside possibly an exceptional set of finite logarithmic measure. That means growth of  $a$  is comparatively lesser than growth of  $f$  and we denote  $\mathcal{S}(f)$  by the collection of all small functions of  $f$  and  $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$ .

**Definition 1.5.** For a meromorphic function  $f$ ,  $\delta(a, f)$  at a value  $a \in \mathbb{C}$  is defined by:

$$\delta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a)}{T(r, f)}.$$

**Definition 1.6.** For a meromorphic function  $f(z)$  in  $\mathbb{C}$ , the order is defined as

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

**Definition 1.7.** For a meromorphic function  $f(z)$  in  $\mathbb{C}$ , the hyper-order of  $f$  as

$$\rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

**Definition 1.8.** Let  $S$  be a set of distinct elements of  $\mathbb{C} \cup \{\infty\}$  and  $f(z)$  be a meromorphic function. We denote by  $E_f(S) = E(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$ , where each zero is counted according to its multiplicity. If we do not count the multiplicities then the set  $\bigcup_{a \in S} \{z : f(z) - a = 0\}$  is denoted by  $\overline{E}_f(S)$ .

Using these definitions, we describe various types of set sharing between meromorphic functions  $f$  and  $g$ :

**Definition 1.9.** If  $E(S, f) = E(S, g)$ , we say that  $f$  and  $g$  share the set  $S$  CM.

**Definition 1.10.** If  $\overline{E}(S, f) = \overline{E}(S, g)$ , we say that  $f$  and  $g$  share the set  $S$  IM.

**Definition 1.11.** If  $E(S, f) \subseteq E(S, g)$ , we say that  $f$  and  $g$  partially share  $S$  CM.

**Definition 1.12.** If  $\overline{E}(S, f) \subseteq \overline{E}(S, g)$ , we say that  $f$  and  $g$  partially share  $S$  IM.

When  $S$  is a singleton, the above definition corresponds to value sharing.

Let  $c \in \mathbb{C} \setminus \{0\}$ , then the shift operator is denoted by  $f(z + c)$ .

## 2. Results related to traditional IM-CM sharing:

We will begin our discussion by the results of Heittokangas et al. [12], which are as follows:

**Theorem 2.1.** [12] Let  $f$  be a meromorphic function of finite order and  $c \in \mathbb{C}$ . If  $f(z)$  and  $f(z + c)$  share three distinct periodic functions  $a_i$ ,  $i = 1, 2, 3$  with period  $c$  CM, then  $f(z) = f(z + c)$ .

**Theorem 2.2.** [12] Under the same condition of the above theorem, if  $f(z)$  and  $f(z + c)$  share  $a_1, a_2$  CM and  $a_3$  IM, then  $f(z) = f(z + c)$ .

### Shift operator sharing ‘2 values IM’

Next, Qi [24] provided the following result:

**Theorem 2.3.** Let  $f$  be an entire function of finite order. If  $f(z)$  and  $f(z + c)$  share two distinct periodic functions  $a_1, a_2$  IM with period  $c \neq 0$ , then  $f(z) = f(z + c)$ .

**Definition 2.1. Definition of difference and  $k$ -th difference operators:**

We define difference and  $k$ -th ( $k \in \mathbb{N}, k \geq 2$ ) difference operators of  $f(z)$  by

$$\begin{aligned}\Delta_c f(z) &= f(z+c) - f(z), \\ \Delta_c^k f(z) &= \Delta_c(\Delta_c^{k-1} f(z)) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(z+ic).\end{aligned}$$

**Difference operator sharing ‘2 values CM +  $\infty$  CM’**

In 2013, Chen-Yi [8] established a result on the uniqueness of meromorphic functions and their first-order difference operators when they share three distinct values. Chen-Yi’s result is as follows:

**Theorem 2.4.** [8] *Let  $f$  be a transcendental meromorphic function such that its order of growth  $\rho(f)$  is not an integer or infinity. If  $f(z)$  and  $\Delta_c f(z)$  share three distinct values  $a_1, a_2, \infty$  CM; then  $f(z) \equiv \Delta_c f(z)$ .*

**Difference operator sharing ‘3 values CM’**

Moreover, in the same paper Chen-Yi [8] conjectured that the condition “order of growth  $\rho(f)$  is not an integer or infinite” in the above theorem can be omitted.

In 2015, Li et al. [17] proved that the above conjecture is true and they presented the following result:

**Theorem 2.5.** [17] *Let  $f(z)$  be a non-constant meromorphic function of finite order. Suppose that  $f(z)$  and  $\Delta_c f(z)$  share three distinct values  $a_i, i = 1, 2, 3$  CM in the extended complex plane. Then  $f(z) \equiv \Delta_c f(z)$ .*

**Difference operator sharing ‘2 values CM’**

We would like to consider the result of Zhang-Liao [28], who studied the uniqueness of the entire function of finite order under two CM shared values.

**Theorem 2.6.** [28] *Let  $f(z)$  be a transcendental entire function of finite order. If  $f(z)$  and  $\Delta_c f(z)$  share two distinct complex constants  $a_1, a_2$  CM, then  $f(z) \equiv \Delta_c f(z)$ . Furthermore,  $f(z)$  is of the form  $f(z) = 2^z h(z)$ , where  $h(z)$  is an entire periodic function with period 1.*

For meromorphic functions, Jiang-Chen [13] obtained the following result:

**Theorem 2.7.** [13] *Let  $f(z)$  be a non-constant meromorphic function of finite order such that  $N(r, \infty; f) = S(r, f)$ . If  $f$  and  $\Delta_c f(z)$  share two nonzero distinct finite complex constants  $a_i, i = 1, 2$  CM, then*

$$f(z) \equiv \Delta_c f(z).$$

In 2017, Lu-Lu [20] removed the condition  $N(r, \infty; f) = S(r, f)$  for transcendental meromorphic function sharing  $a_1, a_2$  and  $\infty$  CM.

### Difference operator sharing ‘2 small functions CM + $\infty$ CM’

Deng et al. [10] proved a theorem for meromorphic function sharing “2 small functions CM +  $\infty$  CM” with its difference operator as follows:

**Theorem 2.8.** [10] *Let  $f(z)$  be a transcendental meromorphic function of finite order and let  $a_1(z), a_2(z) \in \mathcal{S}(f)$  be two distinct small functions of  $f(z)$ . If  $f(z)$  and  $\Delta_c f(z)$  share  $a_1(z), a_2(z), \infty$  CM, then  $f(z) \equiv \Delta_c f(z)$ .*

### Higher order difference operator sharing ‘2 values CM’

In the direction of higher order difference operator, numerous researchers conducted a lot of investigations. Li-Gao [16] proved the following result:

**Theorem 2.9.** [16] *Let  $f$  be a non-constant entire function of finite order. Suppose that  $f(z)$  and  $\Delta_c^k f(z)$  share two distinct finite values  $a_1, a_2$  CM and one of the following cases is satisfied:*

1.  $a_1 a_2 = 0$ ,
2.  $a_1 a_2 \neq 0$  and  $\rho(f) \notin \mathbb{N}$ , then  $f(z) \equiv \Delta_c^k f(z)$ .

### Higher order difference operator sharing ‘1 value CM+ 1 value IM’

In 2017, Li-Mei-Chen [18] proved the following results:

**Theorem 2.10.** [18] *Let  $a_1 \in \mathbb{C}$ ,  $k \in \mathbb{N}$  and let  $f(z)$  be a non-constant entire function of finite order. If  $f(z)$  and  $\Delta_c^k f(z) (\neq 0)$  share  $a_1$  CM,  $0$  CM then  $f(z) \equiv \Delta_c^k f(z)$ .*

**Theorem 2.11.** [18] *Under the same condition of above theorem, if  $f(z)$  and  $\Delta_c^k f(z) (\neq 0)$  share  $a_1 (\neq 0)$  CM and  $a_2$  IM, and if  $N(r, a_1; f(z)) = T(r, f) + S(r, f)$ , then  $f(z) \equiv \Delta_c^k f(z)$ .*

Recently in 2025, Pramanik-Dey [23] proved the following theorem in this direction:

**Theorem 2.12.** [23] *Let  $f$  be a transcendental meromorphic function of hyper-order less than one and let  $c \in \mathbb{C} \setminus \{0\}$ . Also let  $f'(z)$  and  $f(z+c)$  share  $0$  CM and  $1$  IM. If  $N(r, f) < \gamma T(r, f)$  holds for  $\gamma \in [0, \frac{1}{17})$ , then*

$$f'(z) = f(z+c).$$

### Higher order difference operator sharing ‘3 values CM’

In 2016, with respect to meromorphic function, Cui-Chen [9] proved the following theorem for the  $k$ -th difference operator.

**Theorem 2.13.** [9] *Let  $f(z)$  be a non-constant meromorphic function of finite order. If  $f(z)$  and  $\Delta_c^k f(z)$  share two distinct complex constants  $a_1, a_2$  and  $\infty$  CM then  $f(z) \equiv \Delta_c^k f(z)$ .*

### Higher order difference operator sharing ‘2 values IM’

In 2020, related to two values IM sharing problem, Qi-Yang [25] proved the following result:

**Theorem 2.14.** [25] *Let  $f(z)$  be a transcendental meromorphic function  $\rho_2(f) < 1$ . If  $f(z)$  and  $\Delta_c^k f(z)$  share two distinct values  $a_1, a_2$  IM, and*

$$\overline{N}(r, a_1; f) + \overline{N}(r, \infty; f) = S(r, f),$$

then  $f(z) \equiv \Delta_c^k f(z)$ .

### Different operators

To proceed further we require the following generalizations of shift operators, which includes  $k$ -th difference operators and shift operators. We define Linear shift operator as follows:

$$L_1 f(z) = \hat{a}_0(z)f(z) + \sum_{i=1}^k \hat{a}_i(z)f(z + c_i),$$

where  $\hat{a}_i(z) (i = 0, 1, \dots, k) \in \mathcal{S}(f)$  and all  $c_i$ 's are non-zero complex constants. In particular, taking  $c_i = ic$  in the definition of  $L_1 f(z)$ , which is called Linear  $c$ -shift operator, denoted by  $L_k f(z)$  and defined as

$$L_k f(z) = \sum_{i=0}^k \hat{a}_i f(z + ic).$$

In Linear  $c$ -shift operator, for some complex constant  $a$ , we define Restricted  $c$ -shift operator as follows:

$$L_c^a f = \sum_{i=0}^k \hat{a}_i f(z + ic), \quad \text{where} \quad \sum_{i=0}^k \hat{a}_i = a.$$

In restricted  $c$ -shift operator, taking  $\hat{a}_i = (-1)^{k-i}\hat{b}_i$  and  $a = 0$ , we get a special operator denoted by  $L_c^r f$  and call it reduced  $c$ -shift operator. Putting  $\hat{b}_k = \binom{k}{k}$ ,  $\hat{b}_{k-1} = \binom{k}{k-1}$ ,  $\hat{b}_{k-2} = \binom{k}{k-2}$ ,  $\dots$ ,  $\hat{b}_0 = \binom{k}{0}$  in  $L_c^r f$ , we can easily verify that  $L_c^r f(z) = \Delta_c^k f(z)$ .

We shall assume  $\Delta_c f(z) \neq 0$ , i.e.,  $f$  is non periodic. However, it is to be noted that  $L_c^r f(z)$  may be  $\neq 0$ , even if the function is periodic.

### Shift differential and differential operator

Define shift differential and differential operators as follows:

$$L_2 f(z) = \sum_{i=1}^s \hat{b}_i(z) f^{(i)}(z + c_i),$$

$$L_3 f(z) = \sum_{i=1}^t \hat{d}_i(z) f^{(i)}(z),$$

where  $\hat{b}_i(z) (i = 1, \dots, s)$ ;  $\hat{d}_i(z) (i = 1, \dots, t) \in \mathcal{S}(f)$  and all  $c_i$ 's are non-zero complex constants.

### Delay differential and linear shift delay differential operator

For the sake of convenience, we shall call

$$\tilde{L}(f(z)) = L_2(f(z)) + L_3(f(z)) = \sum_{i=1}^l \hat{b}_i(z) f^{(i)}(z + c_i)$$

as Delay differential operator where  $c_i$ 's are complex constants. We write

$$L(f(z)) = L_1(f(z)) + L_2(f(z)) + L_3(f(z))$$

and call it Linear shift delay differential operator. By virtue of the definition it becomes a larger class to include the class of shift or difference operators.

### Reduced $c$ -shift operator sharing '2 values CM'

In 2019, Banerjee-Bhattacharyya [2] extended the previous results for the operator  $L_c^r f$ .

**Theorem 2.15.** [2] *Let  $f(z)$  be a nonconstant meromorphic function of finite order. Suppose  $f(z)$  and  $L_c^r f(z)$  share two distinct nonzero finite constants  $a_i$ ,  $i = 1, 2$  CM. If*

$$(k+1)N(r, \infty; f) + 2(k+2)\overline{N}(r, \infty; f) < T(r, f),$$

then  $f(z) \equiv L_c^r f(z)$ .

### Reduced $c$ -shift operator sharing ‘3 values CM’

In the same paper, Banerjee-Bhattacharyya proved the following result:

**Theorem 2.16.** [2] *Let  $f(z)$  be a transcendental meromorphic function of finite order that is not of period  $c$ . Suppose  $f(z)$  and  $L_c^r f(z)$  share two distinct finite constants  $a_1, a_2$  and  $\infty$  CM, then  $f(z) \equiv L_c^r f(z)$ .*

### Linear shift operator sharing ‘2 small functions CM’

In 2014, Liu et al. [19] investigated the uniqueness theorem for a finite order entire function sharing two small functions with its linear shift operator as follows:

**Theorem 2.17.** [19] *Let  $f(z)$  be a non-constant entire function of finite order and  $a_1(z), a_2(z) \in S(f)$  be two distinct small functions of  $f(z)$ . If  $f(z)$  and  $L_1(f(z))$  share  $a_1(z), a_2(z)$  CM, then  $f(z) \equiv L_1(f(z))$ .*

Further results were investigated by Roy-Banerjee [27] such as:

**Theorem 2.18.** [27] *Let  $f(z)$  be a transcendental meromorphic function of finite order and let  $a(z) (\neq 0) \in S(f)$  be an entire function. If  $L_1(f(z))$  and  $f(z+c)$  share  $(a(z), \infty)$  and  $(\infty, \infty)$  with  $\delta(0; f) > 0$ , then  $L_1(f(z)) \equiv f(z+c)$ .*

## 3. Results related to partial IM-CM sharing:

### Shift operator sharing ‘4 values partially IM’

In view of partial sharing of values, in 2016, Charak et al. [5] proved the following result under the notion of partial sharing.

**Theorem 3.1.** [5] *Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$ . Let  $a_i, i = 1, 2, 3, 4$  be four distinct periodic functions with period  $c$ . If  $\delta(a, f) > 0$  for some  $a$  periodic function and  $\bar{E}(a_i, f(z)) \subseteq \bar{E}(a_i, f(z+c))$ ,  $i = 1, 2, 3, 4$ , the.  $f(z) = f(z+c)$  for all  $z \in \mathbb{C}$ .*

### Shift operators sharing ‘2 values partially IM + $0, \infty$ CM’

To remove the condition  $\delta(a, f) > 0$ , recently Kaish-Mondal [15] replaced two periodic functions by 0 and  $\infty$  to obtain the following result:

**Theorem 3.2.** [15] Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$  and let  $a$  be a non-zero finite complex number. If  $\overline{E}(a, f(z)) \subseteq \overline{E}(a, f(z+c))$ ,  $\overline{E}(-a, f(z)) \subseteq \overline{E}(-a, f(z+c))$  and  $f(z)$  and  $f(z+c)$  share  $0, \infty$  CM, then  $f(z) = f(z+c)$  for all  $z \in \mathbb{C}$ .

### Difference operator sharing ‘1 value CM + 0, $\infty$ partially CM’

In 2018, Chen [6] proved following result:

**Theorem 3.3.** [6] Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$ . If  $f(z)$  and  $\Delta_c f(z)$  share the value 1 CM and satisfy  $E(0, f(z)) \subseteq E(0; \Delta_c f(z))$  and  $E(\infty; \Delta_c f(z)) \subseteq E(\infty; f(z))$ , then  $f(z) = \Delta_c f(z)$ .

### Linear $c$ -shift operator sharing ‘3 values partially CM’

In 2021, in terms of generalized operator Banerjee-Roy [4], have relaxed the CM sharing conditions in the above theorem.

**Theorem 3.4.** [4] Let  $f(z)$  be a transcendental meromorphic function of finite order and let  $a_1$  be a nonzero complex constant. If

$$E(a_1, f(z)) \subseteq E(a_1, L_c^0 f(z)), \quad E(0, f(z)) \subseteq E(0, L_c^0 f(z)) \text{ and} \\ E(\infty, L_c^0 f(z)) \subseteq E(\infty, f(z)),$$

then  $f(z) = L_c^0 f(z)$ .

This theorem improves the result of Banerjee-Bhattacharyya [2].

### Example

By the following example we can show that the above theorem is not true when the strictly reverse case of  $E(\infty, L_c^0 f(z)) \subseteq E(\infty, f(z))$  i.e.,  $E(\infty, f(z)) \subset E(\infty, L_c^0 f(z))$  holds.

**Example 3.1.** Let  $f(z) = \frac{1}{1 - e^{\frac{\pi iz}{c}}}$ . Then  $L_c^0 f(z) = \Delta_c f(z) = \frac{2e^{\frac{\pi iz}{c}}}{e^{\frac{2\pi iz}{c}} - 1}$ .

Now,

$$f(z) = 1 \implies \frac{1}{1 - e^{\frac{\pi iz}{c}}} = 1 \implies e^{\frac{\pi iz}{c}} = 0.$$

Since  $f$  has no zeros and no 1-points, so clearly

$$E(1, f(z)) \subseteq E(1, L_c^0 f(z)) \quad E(0, f(z)) \subseteq E(0, L_c^0 f(z)), \\ \text{but } E(\infty, L_c^0 f(z)) \not\subseteq E(\infty, f(z)).$$

Here  $f(z) \neq L_c^0 f(z)$ .

**Linear  $c$ -shift operator sharing ‘3 values partially CM’**

The next theorem are on the uniqueness of  $f(z)$  and  $L_c^0 f(z)$  under partial sharing given by Banerjee-Maity [3].

**Theorem 3.5.** [3] *Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$ . For  $c \in \mathbb{C} \setminus \{0\}$ , let  $a_1(z) \in \mathcal{S}(f)$  be a non-zero periodic function of period  $c$ . If*

$$E(a_1(z), f(z)) \subseteq E(a_1(z), L_c^0 f(z)), \quad E(0, f(z)) \subseteq E(0, L_c^0 f(z)) \text{ and} \\ E(\infty, L_c^0 f(z)) \subseteq E(\infty, f(z)),$$

then  $f(z) = L_c^0 f(z)$ .

This theorem improves the result of Noulorvang-Pham [21]:

**Theorem 3.6.** [21] *Let  $f$  be a meromorphic function of hyper-order  $< 1$  and let  $c \in \mathbb{C} \setminus \{0\}$ . Let  $a_1, a_2, a_3 \in \hat{\mathcal{S}}(f)$  be three distinct periodic functions with period  $c$ . Assume that  $f(z)$  and  $f(z+c)$  share partially  $a_1, a_2$  CM and share partially  $a_3$  IM, i.e.,*

$$E(a_1, f(z)) \subseteq E(a_1, f(z+c)), \quad E(a_2, f(z)) \subseteq E(a_2, f(z+c))$$

and

$$\overline{E}(a_3, f(z)) \subseteq \overline{E}(a_3, f(z+c)).$$

If  $\Theta(a, f) > 0$  for some  $a \in \hat{\mathcal{S}}(f) \setminus \{a_3\}$ , then  $f(z) = f(z+c)$  for all  $z \in \mathbb{C}$ .

**Example**

The following example shows that the condition  $\rho_2(f) < 1$  is sharp in the above theorem.

**Example 3.2.** Consider  $f(z) = e^{e^{\frac{z \log 3}{c}}} - 1$  and  $L_c^0 f(z) = \Delta_c f(z)$ . Notice that

$$\rho_2(f) = 1 \quad \text{and} \quad L_c^0 f(z) = e^{e^{\frac{z \log 3}{c}}} \left( e^{2e^{\frac{z \log 3}{c}}} - 1 \right).$$

So it is clear that  $E(0, f) \subset E(0, L_c^0 f)$ . Letting  $t = e^{e^{\frac{z \log 3}{c}}}$ , we have

$$f(z) = \frac{-3 + \sqrt{5}}{2} \implies t = \frac{-1 + \sqrt{5}}{2}, \quad \text{and} \\ L_c^0 f(z) = \frac{-3 + \sqrt{5}}{2} \implies t^3 - t - \frac{-3 + \sqrt{5}}{2} = 0$$

$$\implies \left( t - \frac{-1 + \sqrt{5}}{2} \right) \left[ t^2 + \left( \frac{-1 + \sqrt{5}}{2} \right) t + \left( \frac{-1 + \sqrt{5}}{2} \right)^2 - 1 \right] = 0.$$

So  $E\left(\frac{-3+\sqrt{5}}{2}, f\right) \subset E\left(\frac{-3+\sqrt{5}}{2}, L_c^0 f\right)$  and clearly  $E(\infty, f) = E(\infty, L_c^0 f)$ .  
But notice that  $f(z) \not\equiv L_c^0 f(z)$ .

In the same paper [3], authors have investigated the case for the partial sharing of a non-zero small function instead of 0.

**Theorem 3.7.** [3] Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$ . For  $c \in \mathbb{C} \setminus \{0\}$ , let  $a_1(z), a_2(z) \in \mathcal{S}(f)$  be any two non-zero distinct periodic functions of period  $c$ . If

$$E(a_1(z), f(z)) \subseteq E(a_1(z), L_c^0 f(z)), \quad E(a_2(z), f(z)) \subseteq E(a_2(z), L_c^0 f(z)),$$

$$E(\infty, L_c^0 f(z)) \subseteq E(\infty, f(z)) \quad \text{and} \quad \delta(0, f(z)) > 0,$$

then  $f(z) = L_c^0 f(z)$ .

### Derivative and shift of a function sharing

#### '1 value CM + 2 values partially CM'

In 2020, for  $f'(z)$  and  $f(z+c)$ , Qi et al. [26] proved the following result:

**Theorem 3.8.** [26] Let  $f(z)$  be a non-constant meromorphic function of finite order. If  $f'(z)$  and  $f(z+c)$  share  $a_1 \in \mathbb{C} \setminus \{0\}$  CM and satisfy

$$E(0, f(z+c)) \subset E(0, f'(z)), \quad E(\infty, f'(z)) \subset E(\infty, f(z+c)),$$

then  $f'(z) \equiv f(z+c)$ . Further,  $f(z)$  is a transcendental entire function.

### Shift and $k$ -th derivative of a meromorphic function sharing

#### '2 values partially CM + 2 values CM'

In 2023, Kaish-Mandal [14] provided the following theorem:

**Theorem 3.9.** [14] Let  $f(z)$  be a non-constant meromorphic function of  $\rho_2(f) < 1$ , and  $c \in \mathbb{C} \setminus \{0\}$ . Let  $a_1, a_2$  be two non-zero distinct elements in  $\mathcal{S}(f)$ . If

$$\overline{E}(a_i, f(z+c)) \subseteq \overline{E}(a_i, f^{(k)}(z)), \quad i = 1, 2,$$

and  $f^{(k)}(z)$  and  $f(z+c)$  share  $0, \infty$  CM, where  $k \geq 1$  is a positive integer, then  $f^{(k)}(z) \equiv f(z+c)$  for all  $z \in \mathbb{C}$ .

### Differentiable periodic Function

**Definition 3.1.** Let  $F(z)$  be a non-constant meromorphic function.  $F(z)$  is said to be differentially periodic if  $F^{(l)}(z) \equiv F^{(s)}(z+c)$ , where  $0 \leq l < s$ .

#### Differentiable periodicity of a meromorphic function sharing

##### ‘3 values partially CM’

Chen-Huang [7] presented the following two results focusing on partial and normal sharing, as outlined below:

**Theorem 3.10.** [7] Let  $f(z)$  be a meromorphic function of finite order,  $a_1 \in \mathbb{C} \setminus \{0\}$  and  $l, s$  be integers with  $0 \leq l < s$ . If  $f^{(l)}(z)$  and  $f^{(s)}(z+c)$  share  $a_1$  CM and satisfy  $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$  and  $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$ , then  $f^{(l)}(z) \equiv f^{(s)}(z+c)$ .

**Theorem 3.11.** [7] Let  $f(z)$  be a transcendental meromorphic function of finite order,  $0 \leq l < s$ . Suppose that,  $f^{(l)}(z)$  and  $f^{(s)}(z+c)$  share a finite value  $a_1 (\neq 0)$  IM and satisfy  $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$  and  $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$ . If  $N(r, 0; f^{(l)}(z)) + \overline{N}(r, 0; f(z)) = S(r, f)$ , then  $f^{(l)}(z) \equiv f^{(s)}(z+c)$ .

#### Differentiable periodicity of a meromorphic function sharing

##### ‘1 value IM + 2 values partially CM’

Very recently, Pramanik-Sarkar [22] have been able to successfully remove the conditions over counting function in previous theorem. The result is as follows:

**Theorem 3.12.** [22] Let  $f(z)$  be a non-constant meromorphic function of finite order,  $0 \leq l < s$ . If  $f^{(l)}(z)$  and  $f^{(s)}(z+c)$  share a finite value  $a_1 (\neq 0)$  IM and satisfy  $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$  and  $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$ , then  $f^{(l)}(z) \equiv f^{(s)}(z+c)$ .

#### Linear $c$ -shift operator and it’s shift sharing ‘2 values partially IM

##### + $\infty$ partially CM’

Recently, Banerjee-Banerjee [1] have investigated the periodicity of the linear  $c$ -shift operator based solely on partial IM sharing, which was not addressed earlier.

**Theorem 3.13.** [1] Let  $f(z)$  be a non-constant meromorphic function with  $\rho_2(f) < 1$  and let  $a_1, a_2 \in \mathbb{C}$  be two distinct numbers. If  $\overline{E}(a_1, L_k f(z)) \subseteq \overline{E}(a_1, L_k f(z+c))$ ,  $\overline{E}(a_2, L_k f(z)) \subseteq \overline{E}(a_2, L_k f(z+c))$ ,  $E(\infty, L_k f(z+c)) \subseteq E(\infty, L_k f(z))$  with  $\overline{N}(r, \infty; f) = S(r, f)$ , then  $L_k f(z) = L_k f(z+c)$ .

### Examples

The following example shows that, by no means, the linear  $c$ -shift operator  $L_k f(z+c)$  in the theorem, can be replaced by an arbitrary linear shift operator.

**Example 3.3.** Consider the function  $L_0 f(z) = f(z) = \frac{e^{\frac{2\pi iz}{c}} - 1}{e^{\frac{2\pi iz}{c}} + 1}$ , where  $c \in \mathbb{C} \setminus \{0\}$ . Choose a linear shift operator  $\widehat{L}_3 f(z)$  by

$$\widehat{L}_3 f(z) = 3f(z+3c) - f(z+2c) - 2f(z+c) - f(z) = -\frac{e^{\frac{2\pi iz}{c}} - 1}{e^{2\frac{\pi iz}{c}} + 1}.$$

1 is an e.v.P. of both  $f(z)$  and  $\widehat{L}_3 f(z)$ , hence  $E(1, L_0 f(z)) = E(1, \widehat{L}_3 f(z))$ . Clearly,  $E(0, L_0 f(z)) = E(0, \widehat{L}_3 f(z))$  and  $E(\infty, L_0 f(z)) = E(\infty, \widehat{L}_3 f(z))$ , but  $f(z) \neq \widehat{L}_3 f(z)$ . Here also,  $\overline{N}(r, \infty; f) \neq S(r, f)$ .

The next example shows that for entire functions, in the Theorem 1,  $f(z+c)$  can not be replaced by  $\Delta_c f(z)$ .

**Example 3.4.** Take  $L_0 f(z) = f(z) = \left(\frac{e^{\frac{\pi iz}{c}} - 1}{2}\right)^2 + 1$ . Then

$$\Delta_c f(z) = \left(\frac{e^{\frac{\pi iz}{c}} + 1}{2}\right)^2 - \left(\frac{e^{\frac{\pi iz}{c}} - 1}{2}\right)^2 = e^{\frac{\pi iz}{c}}.$$

Here,

$$f(z) = 1 \implies e^{\frac{\pi iz}{c}} = 1, 1$$

and

$$\Delta_c f(z) = e^{\frac{\pi iz}{c}} = 1,$$

hence  $f$  and  $\Delta_c f$  share 1 IM. Similarly, it can be shown that,  $f(z)$  and  $\Delta_c f(z)$  share 0 IM, but  $f \neq \Delta_c f$ .

### Differentiable periodicity of Linear $c$ -shift operator sharing

#### ‘3 values partially CM’

The following theorem formalizes the extensions of the previous theorems:

**Theorem 3.14.** [1] Let  $f(z)$  be a non-constant meromorphic function with  $\rho_2(f) < 1$ ,  $a_1 \in \mathbb{C} \setminus \{0\}$  and  $l, s$  be two integers such that  $0 \leq l < s$ . If  $E(a_1, L_k f^{(l)}(z)) \subseteq E(a_1, L_k f^{(s)}(z+c))$ ,  $E(0, L_k f^{(l)}(z)) \subseteq E(0, L_k f^{(s)}(z+c))$ ,  $E(\infty, L_k f^{(s)}(z+c)) \subseteq E(\infty, L_k f^{(l)}(z))$ , then  $L_k f^{(l)}(z) \equiv L_k f^{(s)}(z+c)$ .

### Example

The following example shows that, sharing of the non-zero element in Theorem 2 can not be dropped.

**Example 3.5.** Set  $L_0 f(z) = \sin z$ , then  $L_0 f'(z + \frac{\pi}{2}) = -\sin z$ .

$$\sin z = 0 \iff -\sin z = 0.$$

Clearly,  $E(0, L_0 f(z)) = E(0, L_0 f'(z + \frac{\pi}{2}))$  and  $E(\infty, L_0 f(z)) = E(\infty, L_0 f'(z + \frac{\pi}{2}))$ , but  $E(1, L_0 f(z)) \not\subseteq E(1, L_0 f'(z + \frac{\pi}{2}))$  and  $L_0 f(z) \neq L_0 f'(z + \frac{\pi}{2})$ .

### Differentiable periodicity Linear $c$ -shift operator sharing

#### '1 value partially IM + 1 value partially CM + $\infty$ partially CM'

We have investigated the previous theorems of  $L_k f(z)$  under the framework of partial IM value sharing over  $\mathbb{C}$ .

**Theorem 3.15.** [1] Let  $f(z)$  be a non-constant meromorphic function with  $\rho_2(f) < 1$ ,  $a_1 \in \mathbb{C} \setminus \{0\}$  and  $l, s$  be two integers such that  $0 \leq l < s$ . If  $\bar{E}(a_1, L_k f^{(l)}(z)) \subseteq \bar{E}(a_1, L_k f^{(s)}(z+c))$ ,  $E(0, L_k f^{(l)}(z)) \subseteq E(0, L_k f^{(s)}(z+c))$  and  $E(\infty, L_k f^{(s)}(z+c)) \subseteq E(\infty, L_k f^{(l)}(z))$  and  $\bar{N}(r, 0; L_k f^{(l)}(z)) = S(r, f)$ , then  $L_k f^{(l)}(z) \equiv L_k f^{(s)}(z+c)$ .

### Example

The following example shows that sharing structure of  $\infty$  in the above theorem can not be reversed.

**Example 3.6.** Take  $L_0 f(z) = \frac{2e^{\frac{2\pi iz}{c}}}{e^{\frac{2\pi iz}{c}} - 1}$ , where  $c \in \mathbb{C} \setminus \{0\}$ . Then,

$$L_0 f'(z+c) = \frac{-4e^{\frac{2\pi iz}{c}}}{(e^{\frac{2\pi iz}{c}} - 1)^2},$$

$$L_0 f(z) = 1 \implies e^{\frac{2\pi iz}{c}} = -1,$$

and

$$L_0 f'(z+c) = 1 \implies (e^{\frac{2\pi iz}{c}} + 1)^2 = 0.$$

Clearly,  $\bar{E}(1, L_0 f(z)) \subset \bar{E}(1, L_0 f'(z+c))$ . Similarly,  $E(0, L_0 f(z)) = E(0, L_0 f'(z+c))$  and  $\bar{N}(r, \frac{1}{L_0 f(z)}) = S(r, f)$ , but  $E(\infty, L_0 f'(z+c)) \not\subseteq E(\infty, L_0 f(z))$  and  $L_0 f(z) \neq L_0 f'(z+c)$ .

### Differentiable periodicity of Linear $c$ -shift operator sharing

#### ‘1 value partially IM + 1 value partially CM + $\infty$ partially CM’

We have proved the following theorem for derivative and shift derivative of linear shift operator, which extends and improves the result of [22]:

**Theorem 3.16.** [1] *Let  $f(z)$  be a transcendental meromorphic function with  $\rho_2(f) < 1$ ,  $a_1 \in \mathbb{C} \setminus \{0\}$  and  $l, s$  be two integers such that  $0 \leq l < s$ . If  $L_k f^{(l)}(z)$  and  $L_k f^{(s)}(z+c)$  share  $a_1$  IM,  $E(0, L_k f^{(l)}(z)) \subseteq E(0, L_k f^{(s)}(z+c))$  and  $E(\infty, L_k f^{(s)}(z+c)) \subseteq E(\infty, L_k f^{(l)}(z))$ , then  $L_k f^{(l)}(z) \equiv L_k f^{(s)}(z+c)$ .*

#### Examples

The following two examples show that Theorem 1 cannot be directly extended to set sharing structure.

**Example 3.7.** Let  $L_0 f(z) = \frac{2e^{\frac{\pi iz}{c}}}{1+e^{\frac{\pi iz}{c}}}$ ,  $c \in \mathbb{C} \setminus \{0\}$ . Then we see that for a non-zero constant  $\alpha$ ,  $E(\{\alpha, -\alpha\}, L_0 f(z)) = E(\{\alpha, -\alpha\}, L_0 f(z+c))$ ,  $E(\infty, L_0 f(z)) = E(\infty, L_0 f(z+c))$  but  $L_0 f(z) \neq L_0 f(z+c)$ .

**Example 3.8.** Let  $L_0 f(z) = \frac{1+e^{\frac{\pi iz}{c}}}{1-e^{\frac{\pi iz}{c}}}$ ,  $c \in \mathbb{C} \setminus \{0\}$ . Then we see that for a non-zero constant  $\alpha$ ,  $E(\{\alpha, \frac{1}{\alpha}\}, L_0 f(z)) = E(\{\alpha, \frac{1}{\alpha}\}, L_0 f(z+c))$ ,  $E(\infty, L_0 f(z)) = E(\infty, L_0 f(z+c))$  but  $L_0 f(z) \neq L_0 f(z+c)$ .

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