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Robust duality analysis for efficiency via convexificators in nonsmooth nonconvex single - objective optimization problems

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Dedicated to Professor Do Van Luu on the occasion of his 80th birthday

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Abstract.
In this paper, we explore robust duality results between the primal nonsmooth nonconvex single-objective optimization problem with uncertain data (UNMP) and its Mond-Weir-type dual model (DUNMP) in terms of $\epsilon-$upper convexificators: weak $\epsilon-$duality theorem, strong $\epsilon-$duality theorem and converse $\epsilon-$duality theorem, where $\epsilon \geq 0$.

1. Introduction

The first notion of quasi efficient solutions was introduced by Loridan [9] for mathematical programming problems with the Pareto order. The aim of studying $\epsilon-$quasi optimal solutions is to obtain feasible points whose objective value is close to be optimal. In order to treat conditions for quasi efficient solutions, it is necessary to develop new concepts of $\epsilon-$upper/$\epsilon-$lower convexificators. The robust dual theory plays an important role in scalar optimization

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because it provides a powerful tool to derive robust weak, strong and converse optimality conditions for different types of $\epsilon-$quasi efficient solutions in scalar optimization. The robust dual model was introduced by Chen et al. [4] for a class of nonconvex multiobjective optimization problems with data uncertainty in the worst case via the upper semi-regular convexificators in the scheme of Wolf. They constructed a mixed robust Wolfe-type dual model for such a class and discussed the robust weak, strong and converse duality results for a dual-primal pair under suitable hypotheses on the $\hat{\partial}^*-$pseudoconvexity of objective and constraint functions at the vector under consideration. Using the dual approach, Chuong [5] examined robust optimality conditions and robust duality via the limiting subdifferentials/or Clarke subdifferentials (also known as convexificators) for an uncertain nonsmooth multiobjective optimization problem under arbitrary uncertainty nonempty sets. In addition, Su [15] explored robust weak, strong and converse duality results for dual-primal pairs by means of the generalized subdifferentials (without $\epsilon-$upper convexificators). As we have so far, the theory of robust duality is of basic importance in the investigation of nonsmooth nonconvex vector/scalar optimization problems with uncertain data, for instance, see [10, 11, 12]. A robust dual model for such problems is important in practice, for instance, see [15, 16]. However, in the area of robust duality analysis, *there are no papers that concentrate on uncertain nonsmooth nonconvex single-objective optimization problems via $\epsilon-$upper convexificators.*

To the best of our knowledge, minimizing the problem involves nonsmooth nonconvex real-valued functions and data uncertainty in both the objective and constraint functions defined on uncertain sets is one of the most difficult problems in the field of robust optimization. Based on estimations, prediction errors, or lack of information, all researchers do not know precisely when the problem will be solved. Minimax programming problems are special mathematical programming problems, which have many applications: in P. L. Chebyshev's theory of best approximation, in J. von Neumann's game theory, etc. In particular, minimax programming problems can be seen as an effective tool for solving uncertain multiobjective optimization problems, for instance see Chen et al. [6, 7, 17] and the references therein. Especially, Hong et al. [13] showed that minimax programming problems can be seen as an effective tool for solving robust single-objective optimization problems.

In this paper, we study a nonsmooth nonconvex single-objective optimization problem with uncertain data (for brevity, (UNMP)) and construct a its Mond-Weir-type dual model (for brevity, (DUNMP)) through $\epsilon-$upper convexificators. For this aim, some weak, strong and converse $\epsilon-$duality theorems for the same are provided in detail.

2. Preliminaries

Throughout this paper, let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space and its topological dual equipped with the usual Euclidean norm $\| \cdot \|$. The notation $\langle \cdot\, ,\, \cdot \rangle$ signifies the inner product in $\mathbb{R}^n$ and the origin of any $n-$dimensional Euclidean space is expressed as 0_n. The symbols $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ and $\mathbb{R}^n_{++}$ denote the nonnegative orthant, the nonpositive orthant and the topological interior of $\mathbb{R}^n_+$ respectively (resp., for short). Given a nonempty convex set $C \subseteq \mathbb{R}^n$, the interior, the closure, the cone hull, the convex hull and the cardinality of C are denoted resp. by int$C$, cl$C$, cone$C$, co$C$ and $|C|$. Additionally, the distance from the point $x \in \mathbb{R}^n$ to the set C is illustrated by $d_C(x) := \inf_{c \in C} \big\{ \|x - c\| \big\}$. The symbols $\mathcal{S} := \{x \in \mathbb{R}^n | \|x\| = 1\}$, $\mathcal{S}^* := \{x \in \mathbb{R}^n | \|x\| \leq 1\}$, and $B(\overline{x}, r) := \{x \in \mathbb{R}^n | \|x - \overline{x}\| < r\}$, where $\overline{x} \in \mathbb{R}^n$ and $r > 0$. For the sake of brevity, we adopt the robust optimization approach to deal with the original optimization problems in the worst case, $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, the symbol $t_n \downarrow 0$ stands for the sequence of positive real numbers $(t_n)_{n \geq 1}$ with the limit 0.

Definition 2.1. *(Aubin and Frankowska [1]) Let $A \subseteq \mathbb{R}^n$ and $\overline{x} \in cl\, A$.*

(i) The Bouligand tangent cone to the set A at $\overline{x}$ is defined by

$$T(A, \overline{x}) = \big\{ v \in \mathbb{R}^n \,|\, \exists\, v_n \to v,\ \exists t_n \downarrow 0,\ \textit{such that } \overline{x} + t_n v_n \in A \quad \forall\, n \geq 1 \big\}.$$

Especially, if A is convex, then $T(A, \overline{x}) = cl\, cone(A - \overline{x})$.

(ii) The Bouligand normal cone to the set A at $\overline{x}$ is defined by

$$N(A, \overline{x}) = \big\{ \xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \leq 0,\ \ \forall\, v \in T(A, \overline{x}) \big\}.$$

Definition 2.2. *(Clarke [2]) Let an extended-real-valued function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and $\overline{x}$, $v \in \mathbb{R}^n$. Then, the lower and upper Dini directional derivatives of φ defined on $\mathbb{R}^n$ in the direction v at the point $\overline{x}$ are expressed respectively by*

$$d^- \varphi(\overline{x}, v) = \liminf_{t \downarrow 0} \frac{\varphi(\overline{x} + tv) - \varphi(\overline{x})}{t},$$
$$d^+ \varphi(\overline{x}, v) = \limsup_{t \downarrow 0} \frac{\varphi(\overline{x} + tv) - \varphi(\overline{x})}{t}.$$

We observe that the lower and upper Dini directional derivatives may be finite as well as infinite. Especially, if a locally Lipschitz function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, then the lower and upper Dini directional derivatives at $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ of φ are finite and locally Lipschitz in the direction v.

Definition 2.3. *(Jeyakumar and Luc [8]) Let $\overline{x} \in \mathbb{R}^n$. We say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ has*

(i) an upper convexificator $\overline{\partial}^* \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\overline{\partial}^* \varphi(\overline{x})$ is closed and

$$d^- \varphi(\overline{x}, v) \leq \sup_{\xi \in \overline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle \quad \textit{for every } v \in \mathbb{R}^n.$$

(ii) a lower convexificator $\underline{\partial}^* \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\underline{\partial}^* \varphi(\overline{x})$ is closed and

$$d^+ \varphi(\overline{x}, v) \geq \inf_{\xi \in \underline{\partial}^* \varphi(\overline{x})} \langle \xi, v \rangle \quad \textit{for every } v \in \mathbb{R}^n.$$

(iii) a convexificator at $\overline{x}$, denoted by $\partial^* \varphi(\overline{x})$, whenever φ admits an upper convexificator at $\overline{x}$, respectively a lower convexificator at $\overline{x}$.

(iv) an upper semi-regular convexificator $\overline{\partial}^* \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\overline{\partial}^* \varphi(\overline{x})$ is closed and

$$d^+ \varphi(\overline{x}, v) \leq \sup_{\xi \in \overline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle \quad \textit{for every } v \in \mathbb{R}^n.$$

(v) a lower semi-regular convexificator $\underline{\partial}^* \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\underline{\partial}^* \varphi(\overline{x})$ is closed and

$$d^- \varphi(\overline{x}, v) \geq \inf_{\xi \in \underline{\partial}^* \varphi(\overline{x})} \langle \xi, v \rangle \quad \textit{for every } v \in \mathbb{R}^n.$$

(vi) a regular convexificator at $\overline{x}$, denoted by $\partial^* \varphi(\overline{x})$, whenever φ admits an upper semi-regular convexificator at $\overline{x}$, respectively a lower semi-regular convexificator at $\overline{x}$.

Definition 2.4. *(Capâtâ [3]) Let $\epsilon \geq 0$ and $\overline{x} \in \mathbb{R}^n$. We say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ has*

(i) an $\epsilon-$upper convexificator $\overline{\partial}^*_\epsilon \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\overline{\partial}^*_\epsilon \varphi(\overline{x})$ is closed and

$$d^- \varphi(\overline{x}, v) \leq \sup_{\xi \in \overline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle + \epsilon \|v\| \quad \textit{for every } v \in \mathbb{R}^n.$$

(ii) an $\epsilon-$lower convexificator $\underline{\partial}^*_\epsilon \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\underline{\partial}^*_\epsilon \varphi(\overline{x})$ is closed and

$$d^+ \varphi(\overline{x}, v) \geq \inf_{\xi \in \underline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle - \epsilon \|v\| \quad \textit{for every } v \in \mathbb{R}^n.$$

(iii) an $\epsilon-$upper regular convexificator $\overline{\partial}^*_\epsilon \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\overline{\partial}^*_\epsilon \varphi(\overline{x})$ is closed and

$$d^+ \varphi(\overline{x}, v) = \sup_{\xi \in \overline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle + \epsilon \|v\| \quad \textit{for every } v \in \mathbb{R}^n.$$

(iv) *an $\epsilon-$lower regular convexificator $\underline{\partial}^*_\epsilon \varphi(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$ if $\underline{\partial}^*_\epsilon \varphi(\overline{x})$ is closed and*

$$d^- \varphi(\overline{x}, v) = \inf_{\xi \in \underline{\partial}^*_\epsilon \varphi(\overline{x})} \langle \xi, v \rangle - \epsilon \|v\| \text{ for every } v \in \mathbb{R}^n.$$

(v) *an $\epsilon-$convexificator at $\overline{x}$, denoted by $\partial^*_\epsilon \varphi(\overline{x})$, whenever φ admits an $\epsilon-$upper convexificator at $\overline{x}$, respectively an $\epsilon-$lower convexificator at $\overline{x}$.*

Example 2.1. *Let $\epsilon \geq 0$, $\overline{x} = 0 \in \mathbb{R}$ and θ be a real-valued function defined on $\mathbb{R}$ by*

$$\theta(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} - \epsilon|x|, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

An easy computation gives that

$$d^+\theta(\overline{x}; v) = d^+\theta(\overline{x}; v) = \begin{cases} -\epsilon v, & \text{if } v \geq 0, \\ (1+\epsilon)v, & \text{otherwise.} \end{cases}$$

The set $\{0, 1\}$ is an $\epsilon-$lower regular convexificator of θ at $\overline{x} = 0$.

Example 2.2. *Let $\epsilon \geq 0$, $\overline{x} = 1 \in \mathbb{R}$ and Θ be a real-valued function defined on $\mathbb{R}$ by*

$$\Theta(x) = \begin{cases} x^3 - \epsilon(x-1), & \text{if } x \geq 1, \\ (x-2)^2 + \epsilon(x-1), & \text{otherwise.} \end{cases}$$

It is not difficult to verify that Θ is regular in the sense of Clarke at $\overline{x} = 1$ with the Clarke subdifferential of Θ at $\overline{x}$ is

$$\partial_C \Theta(\overline{x}) = [-2-\epsilon, 3-\epsilon].$$

The set $[-2, 3]$ is an $\epsilon-$upper regular convexificator of Θ at $\overline{x} = 1$.

Remark 2.3. *We note that Definition 2.4 is a significant generalization of Definition 2.3. In particular, for the case $\epsilon = 0$, if φ is convex on $\mathbb{R}^n$ and locally Lipschitz at $\overline{x} \in \mathbb{R}^n$, then the Clarke subdifferential coincides with the subdifferential in the sense of Convex Analysis. Such subdifferential is a convexificator of φ at $\overline{x} \in \mathbb{R}^n$. For such function φ, the Clarke subdifferential $\partial_C \varphi(\overline{x})$ is an upper regular convexificator and the convexificator mapping $\partial_C \varphi$ is locally bounded at $\overline{x} \in \mathbb{R}^n$.*

Definition 2.5. *(Capâtâ [3]) Let $\epsilon \geq 0$. A real-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is said to be $\epsilon-$pseudoconvex, if for any $x, y \in \mathbb{R}^n$ with $\Phi(y) + \epsilon\|y - x\| < \Phi(x)$, there exists $\beta := \beta(x, y) > 0$ and $\delta := \delta(x, y) \in (0, 1]$ such that*

$$t\beta + t\epsilon\|y - x\| \leq \Phi(x) - \Phi(x + t(y - x)), \ \ \forall\, t \in (0, \delta).$$

Definition 2.6. *(Rockafellar [14]) Let a real-valued function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *be convex. The set of all subgradients of* Φ *at* $\overline{x}$*, denoted by* $\partial\Phi(\overline{x})$*, is defined as*

$$\partial\Phi(\overline{x}) := \Big\{\xi \in \mathbb{R}^n \mid \langle \xi, x - \overline{x}\rangle \le \Phi(x) - \Phi(\overline{x}),\ \ \forall\, x \in \mathbb{R}^n\Big\}.$$

It is known that $\partial\Phi(\overline{x})$ *is called the subdifferential in the case of convex analysis.*

Remark 2.4. *We remark that any* $\epsilon-$*upper regular convexificator is an* $\epsilon-$*upper convexificator and each convexificator is a subset of an* $\epsilon-$*convexificator. In addition, the definitions in Definition 2.4 coincide with the convexificators notion in the case of Jeyakumar and Luc [8], provided* $\epsilon = 0$*. In this case, the symbol* "$0-$" *is omitted in the whole paper.*

Proposition 2.5. *(Capâtâ [3]) Let* $\epsilon \ge 0$*. A continuous function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *has a locally bounded* $\epsilon-$*convexificator map* $\partial^*_\epsilon\varphi$ *at* $\overline{x} \in \mathbb{R}^n$ *if and only if* φ *is locally Lipschitz at* $\overline{x}$*.*

Theorem 2.6. *([3, 8]) Let* $\epsilon_1, \epsilon_2 \ge 0$*. If the functions* $\varphi_1, \varphi_2 : \mathbb{R}^n \to \mathbb{R}$ *admit* ϵ_1, ϵ_2-*upper convexificators* $\overline{\partial}^*_{\epsilon_1}\varphi_1(\overline{x})$*, respectively,* $\overline{\partial}^*_{\epsilon_2}\varphi_2(\overline{x})$ *at* $\overline{x} \in \mathbb{R}^n$ *and one of them is upper regular at* $\overline{x}$*, then* $\overline{\partial}^*_{\epsilon_1}\varphi_1(\overline{x}) + \overline{\partial}^*_{\epsilon_2}\varphi_2(\overline{x})$ *is an* $\epsilon_1 + \epsilon_2-$*upper convexificator of* $\varphi_1 + \varphi_2$*.*

Theorem 2.7. *([3, 8]) Let* $\epsilon \ge 0$*,* $\varphi_1, \varphi_2 : \mathbb{R}^n \to \mathbb{R}$ *be continuous functions and define* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *as* $\Phi(x) = \max\{\varphi_1(x), \varphi_2(x)\}$ *for every* $x \in \mathbb{R}^n$*. If* φ_1, φ_2 *admit* $\epsilon-$*upper convexificators* $\overline{\partial}^*_\epsilon\varphi_1(\overline{x})$*, respectively,* $\overline{\partial}^*_\epsilon\varphi_2(\overline{x})$ *at* $\overline{x} \in \mathbb{R}^n$*, then* $\overline{\partial}^*_\epsilon\Phi(\overline{x}) = \bigcup\limits_{i\in\tau}\overline{\partial}^*_\epsilon\varphi_i(\overline{x})$ *is an* $\epsilon-$*upper convexificator of* Φ *at* $\overline{x}$*, where* $\tau := \big\{i \in \{1, 2\} \mid \Phi(\overline{x}) = \varphi_i(\overline{x})\big\}$*.*

3. Main results

In this section, we consider the following nonsmooth nonconvex minimization problem with uncertain data (UNMP) is of the following form

$$\text{(UNMP)} \qquad \begin{aligned} &\min_{x\in C\subseteq\mathbb{R}^n}\ f(x)\\ &\text{subject to }\ g_j(x, w_j) \le 0,\ \ w_j \in \Omega_j,\ \ j = \overline{1..m}, \end{aligned}$$

where C is convex, $x \in C \subseteq \mathbb{R}^n$ instead of the decision variable; $w_j \in \Omega_j$ stands for uncertain parameters; the symbol Ω_j signifies nonempty compact convex subsets of $\mathbb{R}^{m_j}$, $j=\overline{1..m}$, the objective function $f:\mathbb{R}^n \to \mathbb{R}$, the constraint functions $g=(g_1,g_2,\dots,g_m):\mathbb{R}^n\times\Omega\to\mathbb{R}^m$, in which $\Omega:=\prod_{j=1}^m \Omega_j$, $w_j\in\Omega_j$ $(j=\overline{1..m})$, $w\in\Omega\mapsto w=(w_j)_{j\in\overline{1..m}}\in\prod_{j=1}^m\Omega_j$. The feasible solutions set K of problem (UNMP) is given by

$$K:=\Big\{x\in C\subseteq\mathbb{R}^n \,|\, g_j(x,w_j)\le 0 \ (\forall\, w_j\in\Omega_j,\ j=\overline{1..m})\Big\}.$$

Suppose that f_i $(i=\overline{1..p})$ and $\underline{g_j(.\,,w_j)}$ $(j=\overline{1..m})$ are locally Lipschitz continuous for every $w_j\in\Omega_j$, $j=\overline{1..m}$. We set

$$\begin{aligned}
&G_j(x)=\max_{w_j\in\Omega_j} g_j(x,w_j),\ \ G:=(G_1,G_2,\dots,G_m):\mathbb{R}^n\to\mathbb{R}^m \quad \text{and}\\
&\Omega_j(x):=\big\{w_j\in\Omega_j \,|\, G_j(x)=g_j(x,w_j)\big\},\ \ j=\overline{1..m},\\
&\Omega_j^N(x):=\big\{w_j\in\Omega_j \,|\, G_j(x^*)=g_j(x^*,w_j)\ \ \forall\, x^* \text{ near } x\big\},\ \ j=\overline{1..m}.
\end{aligned}$$

It is not hard to verify that

$$K=\Big\{x\in C\subseteq\mathbb{R}^n \,|\, G_j(x)\le 0 \ (j=\overline{1..m})\Big\}.$$

Definition 3.1. *Let $\epsilon\ge 0$. A vector $\overline{x}\in K$ is said to be an $\epsilon-$quasi optimal solution of problem* (UNMP)*, if*

$$f(x)-f(\overline{x})+\epsilon\|x-\overline{x}\|\ge 0,\ \forall\, x\in K.$$

Given $\epsilon\ge 0$, and then, we consider the real-valued function Φ is defined on $\mathbb{R}^n$, is given by

$$\Phi(x)=\sup_{\xi\in\overline{\partial}^*_\epsilon\varphi(\overline{x})}\langle\xi,x\rangle+2\epsilon\|x\| \ \text{ for every }\ x\in\mathbb{R}^n.$$

We have Φ is lower semicontinuous, sublinear and positive with $\Phi(0)=0$, and hence, it follows from [3] that

$$0\in\partial\Phi(0)=\mathrm{cl}\Big(2\epsilon\mathcal{S}^*+\mathrm{co}\overline{\partial}^*_\epsilon\varphi(\overline{x})\Big). \tag{3.1}$$

We mention that $\mathcal{S}^*$ is the closed unit ball in $\mathbb{R}^n$, it is a compact subset in $\mathbb{R}^n$ and so is $2\epsilon\mathcal{S}^*$. This together with (3.1) leads to

$$0\in 2\epsilon\mathcal{S}^*+\mathrm{cl}\Big(\mathrm{co}\overline{\partial}^*_\epsilon\varphi(\overline{x})\Big). \tag{3.2}$$

Based on the relation (3.1) (or even the relation (3.2)), and then, in connection with the uncertain nonsmooth nonconvex minimization problem (UNMP), we define an uncertain dual maximization problem (DUNMP) as follows:

$$\text{(DUNMP)}\quad \begin{aligned} &\max_{(y,\lambda,\mu)\in C\times\mathbb{R}^{1+m}_{+}} f(y)\\ \text{s.t. }\ & 0\in \mathrm{cl}\Big(\lambda \mathrm{co}\overline{\partial}^{*}_{\epsilon} f(y)+\sum_{j=1}^{m}\mu_j \mathrm{co}\overline{\partial}^{*}_{y} g_j(y,w_j)+3\epsilon\mathcal{S}^{*}+N(C,y)\Big),\\ & w_j\in\Omega_j^N(y),\ j=\overline{1..m},\\ & \mu_j g_j(y,w_j)\geq 0,\ j=\overline{1..m},\\ & \mu_j=0,\ j\in J\setminus\hat{J}(y),\quad \sum_{j=1}^{m}\mu_j=1-\lambda. \end{aligned}$$

For the dual problem (DUNMP), we denote D instead of the feasible solutions set of problem (DUNMP).

Definition 3.2. *Consider the primal problem (UNMP) and its dual problem* (DUNMP). *A vector* $(\overline{y},\overline{\lambda},\overline{\mu})\in D$ *is said to be a robust* $\epsilon-$*quasi optimal solution for the problem* (DUNMP), *if for every* $(y,\lambda,\mu)\in D$, *we have*

$$f(y)-f(\overline{y})-\epsilon\|y-\overline{y}\|\leq 0.$$

To treat the theorem of robust strong $\epsilon-$duality, we establish the following robust necessary $\epsilon-$optimality condition for $\epsilon-$quasi optimal solution of problem (UNMP).

Theorem 3.1. *Let* $\epsilon\geq 0$. *If* f *admits an* $\epsilon-$*upper convexificator* $\overline{\partial}^{*}_{\epsilon} f(\overline{x})$ *at* $\overline{x}\in\mathbb{R}^n$, $g_j(.,w_j)$, $j\in J$ *admit upper convexificators* $\overline{\partial}^{*}_{\overline{x}} g_j(\overline{x},w_j)$ *at* $\overline{x}$ *for all* $w_j\in\Omega_j$, $j\in J$, *and* $\overline{x}$ *is a robust* $\epsilon-$*quasi optimal solution of problem (UNMP), then there exist* $(\overline{\lambda},\overline{\mu})\in\mathbb{R}^{1+m}_{+}$ *with* $\sum_{j=1}^{m}\overline{\mu}_j=1-\overline{\lambda}$ *and* $\overline{\overline{w}}_j\in\Omega_j(\overline{x})$ *such that*

$$\overline{\mu}_j g_j(\overline{x},\overline{\overline{w}}_j)=0,\ j\in J, \tag{3.3}$$

$$0\in\mathrm{cl}\Big(\overline{\lambda}\mathrm{co}\overline{\partial}^{*}_{\epsilon} f(\overline{x})+\sum_{j=1}^{m}\overline{\mu}_j\mathrm{co}\overline{\partial}^{*}_{\overline{x}} g_j(\overline{x},\overline{\overline{w}}_j)+3\epsilon\mathcal{S}^{*}+N(C,\overline{x})\Big). \tag{3.4}$$

Proof. Since $\overline{x}$ is a robust $\epsilon-$quasi optimal solution for the problem (UNMP), there exists no $\hat{x}\in C$ such that

$$\begin{cases} f(\hat{x})-f(\overline{x})+\epsilon\|\hat{x}-\overline{x}\|<0,\\ G_j(\hat{x})\leq 0,\ j\in J. \end{cases} \tag{3.5}$$

Consider the extended-real-valued mapping $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}\cup\{+\infty\}$ is defined by

$$\varphi(x)=\max\big\{f(x)-f(\overline{x}),\ \max_{j\in\hat{J}(\overline{x})} G_j(x)-\epsilon\|x-\overline{x}\|\big\},\ x\in\mathbb{R}^n.$$

We need to show that G_j $(j \in \hat{J}(\overline{x}))$ admit upper convexificators $\overline{\partial}^* G_j(\overline{x})$ at $\overline{x}$.

In fact, it follows from the initial assumption that $g_j(., w_j),\ j \in \hat{J}(\overline{x})$ admit upper convexificators $\overline{\partial}^*_{\overline{x}} g_j(\overline{x}, w_j)$ at $\overline{x}$ for all $w_j \in \Omega_j,\ j \in \hat{J}(\overline{x})$, which guarantees that

$$\begin{aligned}
d^- G_j(\overline{x}, v) &= \liminf_{t \downarrow 0} \frac{G_j(\overline{x} + tv) - G_j(\overline{x})}{t} \\
&= \liminf_{t \downarrow 0} \frac{\max_{w_j \in \Omega_j} g_j(\overline{x} + tv, w_j) - \max_{w_j \in \Omega_j} g_j(\overline{x}, w_j)}{t} \\
&\leq \sup_{\xi \in \overline{\partial}^* G_j(\overline{x})} \langle \xi, v \rangle, \ \ \forall\, v \in \mathbb{R}^n.
\end{aligned}$$

On the other hand, for $\overline{w}_j \in \Omega_j(\overline{x})$, one can achieve that

$$\begin{aligned}
d^- g_j(\overline{x}, \overline{w}_j; v) &= \liminf_{t \downarrow 0} \frac{g_j(\overline{x} + tv, \overline{w}_j) - g_j(\overline{x}, \overline{w}_j)}{t} \\
&\leq \liminf_{t \downarrow 0} \frac{\max_{w_j \in \Omega_j} g_j(\overline{x} + tv, w_j) - \max_{w_j \in \Omega_j} g_j(\overline{x}, w_j)}{t} \\
&\leq \sup_{\xi \in \overline{\partial}^* G_j(\overline{x})} \langle \xi, v \rangle, \ \ \forall\, v \in \mathbb{R}^n.
\end{aligned}$$

Therefore, $\overline{\partial}^* G_j(\overline{x})$ $(j \in J)$ are upper convexificators of G_j at $\overline{x}$, where

$$\overline{\partial}^* G_j(\overline{x}) = \bigcup_{\overline{w}_j \in \Omega_j(\overline{x})} \overline{\partial}^*_{\overline{x}} g_j(\overline{x}, \overline{w}_j).$$

Applying Theorem 2.7, one can obtain that

$$\bigcup_{j \in \hat{J}(\overline{x})} \overline{\partial}^* G_j(\overline{x}) = \bigcup_{j \in \hat{J}(\overline{x})} \bigcup_{\overline{w}_j \in \Omega_j(\overline{x})} \overline{\partial}^*_{\overline{x}} g_j(\overline{x}, \overline{w}_j)$$

is an upper convexificator of $\max_{j \in \hat{J}(\overline{x})} G_j$ at $\overline{x}$. By virtue of the initial hypotheses, f admits an $\epsilon-$upper convexificator $\overline{\partial}^*_{\epsilon} f(\overline{x})$ at $\overline{x}$, which together with Theorem 2.6 ensures that φ admits an $\epsilon-$upper convexificator $\overline{\partial}^*_{\epsilon} \varphi(\overline{x})$ at $\overline{x}$, where

$$\overline{\partial}^*_{\epsilon} \varphi(\overline{x}) = \overline{\partial}^*_{\epsilon} f(\overline{x}) \bigcup \Big\{ \bigcup_{j \in \hat{J}(\overline{x})} \bigcup_{\overline{w}_j \in \Omega_j(\overline{x})} \overline{\partial}^*_{\overline{x}} g_j(\overline{x}, \overline{w}_j) - \epsilon \mathcal{S}^* \Big\}.$$

Thus, one can find $\overline{\overline{w}}_j \in \Omega_j(\overline{x})$ $(j \in \hat{J}(\overline{x}))$ such that

$$\overline{\partial}^*_{\epsilon} \varphi(\overline{x}) = \overline{\partial}^*_{\epsilon} f(\overline{x}) \bigcup \Big\{ \bigcup_{j \in \hat{J}(\overline{x})} \overline{\partial}^*_{\overline{x}} g_j(\overline{x}, \overline{\overline{w}}_j) - \epsilon \mathcal{S}^* \Big\}. \tag{3.6}$$

We have from the construction of φ and (3.5) that $\overline{x} \in K$ is an $\epsilon-$quasi optimal solution for φ on C, we mean that $\varphi(x)-\varphi(\overline{x})+\epsilon\|x-\overline{x}\| \geq 0,\ \ \forall\, x \in C$. Now, for $\tau > 0$ arbitrarily taken, let us may consider the function $\Delta_\tau : X \to \mathbb{R} \cup \{+\infty\}$, be given by $\Delta_\tau(x) = \varphi(x) + \tau d_C(x), \forall\, x \in \mathbb{R}^n$. Taking $\overline{\tau} > 0$ small enough such that $\overline{x}$ is is a local $\epsilon-$quasi optimal solution of $\Delta_{\overline{\tau}}$ on $\mathbb{R}^n$. In view of Clarke [2], $d_C(.) : X \to \mathbb{R}$ is Lipschitz on X, and $d_C(.)$ is convex because C is convex. Hence we can choose the subdifferential $\partial d_C(\overline{x})$ as an upper regular convexificator of $d_C(.)$ at $\overline{x}$ (see [16]). It is well-known that $\partial d_C(\overline{x})$ is weakly* compact and convex, according to Rule 4.2 in Ref. [8], we deduce that $\Delta_{\overline{\tau}}$ admits an $\epsilon-$upper convexificator at $\overline{x}$ as $\overline{\partial}^*_\epsilon \varphi(\overline{x}) + \overline{\tau}\partial d_C(\overline{x})$, which along with (3.1), (3.6) and $\overline{\tau}\partial d_C(\overline{x}) \subset N(C,\overline{x})$ leads to

$$\begin{aligned}
0 &\in \mathrm{clco}\Big(\overline{\partial}^*_\epsilon \varphi(\overline{x}) + 2\epsilon\mathcal{S}^* + \overline{\tau}\partial d_C(\overline{x})\Big)\\
&\subset \mathrm{cl}\Big(\mathrm{co}\overline{\partial}^*_\epsilon \varphi(\overline{x}) + 2\epsilon\mathcal{S}^* + N(C,\overline{x})\Big)\\
&\subset \mathrm{cl}\Big(\mathrm{co}\Big(\overline{\partial}^*_\epsilon f(\overline{x}) \bigcup \Big\{ \bigcup_{j\in\hat{J}(\overline{x})} \overline{\partial}^*_{\overline{x}} g_j(\overline{x},\overline{\overline{w}}_j) - \epsilon\mathcal{S}^*\Big\}\Big) + 2\epsilon\mathcal{S}^* + N(C,\overline{x})\Big).
\end{aligned}$$

In consequence, one can find $(\overline{\lambda},\overline{\mu}) \in \mathbb{R}^{1+m}_+$ with $\sum_{j=1}^m \overline{\mu}_j = 1-\overline{\lambda}$ satisfying

$$\begin{aligned}
&\overline{\mu}_j = 0,\ \ j \in J \setminus \hat{J}(\overline{x}),\\
&0 \in \mathrm{cl}\Big(\overline{\lambda}\mathrm{co}\overline{\partial}^*_\epsilon f(\overline{x}) + \sum_{j=1}^m \overline{\mu}_j[\mathrm{co}\overline{\partial}^*_{\overline{x}} g_j(\overline{x},\overline{\overline{w}}_j) - \epsilon\mathcal{S}^*] + 2\epsilon\mathcal{S}^* + N(C,\overline{x})\Big),
\end{aligned}$$

whence

$$0 \in \mathrm{cl}\Big(\overline{\lambda}\mathrm{co}\overline{\partial}^*_\epsilon f(\overline{x}) + \sum_{j=1}^m \overline{\mu}_j \mathrm{co}\overline{\partial}^*_{\overline{x}} g_j(\overline{x},\overline{\overline{w}}_j) + 3\epsilon\mathcal{S}^* + N(C,\overline{x})\Big).$$

For $\overline{w}_j \in \Omega_j(\overline{x}),\ j \in \hat{J}(\overline{x})$, it holds that

$$\overline{\mu}_j G_j(\overline{x}) = \overline{\mu}_j \max_{w_j\in\Omega_j} g_j(\overline{x},w_j) = \overline{\mu}_j g_j(\overline{x},\overline{w}_j) = 0,$$

and thus, (3.3) and (3.4) are fulfilled, as we have to prove. ■

Hereafter, we discuss some robust weak, strong and converse $\epsilon-$duality theorems for the $\epsilon-$quasi solutions of the primal problem (UNMP) and its dual model (DUNMP).

Theorem 3.2. *(Robust weak $\epsilon-$duality) Let $\epsilon \geq 0$, $\overline{x} \in K$ and $(y,\lambda,\mu) \in D$. Suppose that the objective functions f admits an $\epsilon-$upper regular convexificator $\overline{\partial}^*_\epsilon f(y)$ at y, respectively the continuous functions $g_j(.\,,w_j)$ $(j \in J)$ admit*

$\epsilon-$upper regular convexificators $\overline{\partial}^* g_j(y, w_j)$ at y for all $w_j \in \Omega_j,\ \ j \in J$. Suppose, furthermore, that the objective function f is $\epsilon-$pseudoconvex and the continuous functions $g_j(.\,, w_j)$ $(w_j \in \Omega_j,\ j \in J)$ are pseudoconvex. Then we have

$$f(y) - f(\overline{x}) - \epsilon\|y - \overline{x}\| \leq 0. \tag{3.7}$$

Proof. Assume to the contrary, that

$$f(y) - f(\overline{x}) - \epsilon\|y - \overline{x}\| > 0. \tag{3.8}$$

Consider the extended-real-valued mapping $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(x) = \max\big\{f(x) - f(y),\ \max_{j\in\hat{J}(y)} G_j(y) - \epsilon\|x - y\|\big\},\ \ x \in \mathbb{R}^n.$$

We need to prove that G_j $(j \in \hat{J}(y))$ admit $\epsilon-$upper regular convexificators $\overline{\partial}^* G_j(y)$ at y. Indeed, it yields from the initial hypotheses that $g_j(., w_j),\ j \in \hat{J}(y)$ admit $\epsilon-$upper regular convexificators $\overline{\partial}^*_{\overline{x}} g_j(y, w_j)$ at y for every $w_j \in \Omega_j,\ j \in \hat{J}(y)$, which ensures that

$$\begin{aligned} d^+ G_j(y, v) &= \limsup_{t\downarrow 0} \frac{G_j(y + tv) - G_j(y)}{t} \\ &= \limsup_{t\downarrow 0} \frac{\max_{w_j\in\Omega_j} g_j(y + tv, w_j) - \max_{w_j\in\Omega_j} g_j(y, w_j)}{t} \\ &= \sup_{\xi\in\overline{\partial}^* G_j(\overline{x})} \langle \xi, v\rangle + \epsilon\|v\|,\ \ \forall\, v \in \mathbb{R}^n. \end{aligned}$$

On one side, for $\overline{w}_j \in \Omega_j^N(y) \subseteq \Omega_j(y)$, one can reach that

$$\begin{aligned} d^+ g_j(y, \overline{w}_j; v) &= \limsup_{t\downarrow 0} \frac{g_j(y + tv, \overline{w}_j) - g_j(y, \overline{w}_j)}{t} \\ &= \limsup_{t\downarrow 0} \frac{\max_{w_j\in\Omega_j} g_j(y + tv, w_j) - \max_{w_j\in\Omega_j} g_j(y, w_j)}{t} \\ &= \sup_{\xi\in\overline{\partial}^* G_j(y)} \langle \xi, v\rangle + \epsilon\|v\|,\ \ \forall\, v \in \mathbb{R}^n. \end{aligned}$$

Consequently, $\overline{\partial}^* G_j(y)$ $(j \in J)$ are $\epsilon-$upper regular convexificators of G_j at y, where

$$\overline{\partial}^* G_j(y) = \bigcup_{\overline{w}_j\in\Omega_j^N(y)} \overline{\partial}^*_y g_j(y, \overline{w}_j).$$

It infers from the initial hypotheses that φ is $\epsilon-$pseudoconvex, and in addition, φ admits an $\epsilon-$upper regular convexificator $\overline{\partial}^*_\epsilon \varphi(y)$ at y, where

$$\overline{\partial}^*_\epsilon \varphi(y) = \overline{\partial}^*_\epsilon f(y) \bigcup \Big\{ \bigcup_{j\in \hat{J}(y)} \bigcup_{\overline{w}_j \in \Omega_j^N(y)} \overline{\partial}^*_y g_j(y, \overline{w}_j) - \epsilon \mathcal{S}^* \Big\}.$$

Thus, one finds $\overline{\overline{w}}_j \in \Omega_j^N(y)$ $(j \in \hat{J}(y))$ such that

$$\overline{\partial}^*_\epsilon \varphi(y) = \overline{\partial}^*_\epsilon f(y) \bigcup \Big\{ \bigcup_{j\in \hat{J}(y)} \overline{\partial}^*_y g_j(y, \overline{\overline{w}}_j) - \epsilon \mathcal{S}^* \Big\}.$$

Applying Theorem 3 [16], we confirm that $\varphi(y) - \varphi(\overline{x}) - \epsilon\|y - \overline{x}\| \leq 0$, or equivalently,

$$\begin{cases} \varphi(y) = \max\{0, \max_{j\in J} G_j(y)\} \geq 0 \\ \min\Big\{\varphi(y) + f(y) - f(\overline{x}) - \epsilon\|y - \overline{x}\|,\ \varphi(y) - \max_{j\in J} G_j(y)\Big\} \leq 0 \end{cases}.$$

This along with (3.8) ensures that

$$\begin{cases} \varphi(y) + f(y) - f(\overline{x}) - \epsilon\|y - \overline{x}\| > 0, \\ \varphi(y) - \max_{j\in J} G_j(y) \leq 0, \\ \varphi(y) = \max\{0, \max_{j\in J} G_j(y)\}. \end{cases}$$

Two cases can occur as follows:

Case 1: if $\varphi(y) = 0$ then $\max_{j\in J} G_j(y) = 0$, and so, $y \in K$ and $G_j(y) = 0,\ j = \overline{1..m}$.

Case 2: if $\varphi(y) > 0$ then $\max_{j\in J} G_j(y) > 0$, and therefore, there exists $j_0 \in \hat{J}(y)$ such that $G_{j_0}(y) > 0$ and according to the definition of D, $\mu_{j_0} = 0$ for some $j_0 \in \hat{J}(y)$.

All the arguments above lead to a contradiction, and hence, (3.7) is fulfilled, as we have to prove. ■

Theorem 3.3. *(Robust strong $\epsilon-$duality) Suppose that $\epsilon \geq 0$. Let $\overline{x} \in K$ be a robust $\epsilon-$quasi optimal solution of problem (UNMP) and in addition, assuming, that f admits an $\epsilon-$upper convexificator $\overline{\partial}^*_\epsilon f(\overline{x})$ at $\overline{x}$, respectively the continuous functions $g_j(.\,, w_j)$ $(j \in J)$ admit upper convexificators $\overline{\partial}^* g_j(\overline{x}, w_j)$ at $\overline{x}$ for all $w_j \in \Omega_j,\ j \in J$, and $\Omega_j(\overline{x}) = \Omega_j^N(\overline{x})$ for all $j \in J$. Then there exist $\overline{\lambda} \in \mathbb{R}_+$, $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_m) \in \mathbb{R}^m_+$ with $\overline{\lambda} + \sum_{j=1}^m \overline{\mu}_j = 1$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D$. Moreover, if all the hypotheses of Theorem 3.2 are fulfilled, then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a robust $\epsilon-$quasi optimal solution of* (DUNMP) *and the value of the objective functions of (UNMP) and* (DUNMP) *at $\overline{x}$ and $(\overline{x}, \overline{\lambda}, \overline{\mu})$, respectively, are equal.*

Proof. Since $\overline{x} \in K$ is a robust $\epsilon-$quasi optimal solution for (UNMP). Taking in account Theorem 3.1, there exist $(\overline{\lambda}, \overline{\mu}) \in \mathbb{R}_+^{1+m}$ with $\overline{\mu}_j = 0$ for all $j \notin \hat{J}(\overline{x})$, $\overline{\lambda} + \sum_{j=1}^{m} \overline{\mu}_j = 1$ and $\overline{\overline{w}}_j \in \Omega_j(\overline{x})$ such that (3.3) and (3.4) are valid. Because $\Omega_j(\overline{x}) = \Omega_j^N(\overline{x})$ for all $j \in J$, according to the definition of feasible solutions set D, we confirm that $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in D$. Additionally, if all the hypotheses of Theorem 3.2 are fulfilled, then for any feasible solution $(y, \lambda, \mu) \in D$ one can reach from the robust weak duality theorem that $f(y) - f(\overline{x}) - \epsilon\|y - \overline{x}\| \leq 0$. Therefore, $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a robust $\epsilon-$quasi optimal solution of (DUNMP) and the value of the objective functions of (UNMP) and (DUNMP) at $\overline{x}$ and $(\overline{x}, \overline{\lambda}, \overline{\mu})$, respectively, are equal, which the conclusion can be verified. ■

Theorem 3.4. *(Robust converse $\epsilon-$duality) Let $\epsilon \geq 0$ and $\overline{x} \in K$. Assume that $(\overline{x}, \lambda, \mu) \in D$, f admits an $\epsilon-$upper convexificator $\overline{\partial}_\epsilon^* f(\overline{x})$ at $\overline{x}$, respectively the continuous functions $g_j(.\,, w_j)$ $(j \in J)$ admit $\epsilon-$upper regular convexificators $\overline{\partial}^* g_j(\overline{x}, w_j)$ at $\overline{x}$ for all $w_j \in \Omega_j,\ j \in J$. Assume, furthermore, that the objective function f is $\epsilon-$pseudoconvex and the continuous functions $g_j(.\,, w_j)$ $(w_j \in \Omega_j,\ j \in J)$ are pseudoconvex. Then $\overline{x}$ is a robust $\epsilon-$quasi optimal solution of problem (UNMP) and $(\overline{x}, \lambda, \mu)$ is a robust $\epsilon-$quasi optimal solution of problem* (DUNMP).

Proof. Assume to the contrary, that $\overline{x} \in K$ is not an $\epsilon-$quasi optimal solution of problem (UNMP), i.e., there exists at least an element $\widetilde{x} \in C$ satisfying

$$\text{(3.9)} \qquad \begin{cases} G_j(\widetilde{x}) \leq 0, & j \in J, \\ f(\widetilde{x}) - f(\overline{x}) + \epsilon\|\widetilde{x} - \overline{x}\| < 0. \end{cases}$$

Consider the extend-real-valued function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$\varphi(x) = \max\left\{f(x) - f(\overline{x}),\ \max_{j \in J}\{G_j(x)\} - \epsilon\|x - \overline{x}\|\right\},\ \ x \in \mathbb{R}^n.$$

Then, in a similar idea to the proof of Theorem 3.2, the extended-real-valued function φ admits an $\epsilon-$upper regular convexificator at $\overline{x}$ of the following form

$$\overline{\partial}_\epsilon^* \varphi(\overline{x}) = \overline{\partial}_\epsilon^* f(\overline{x}) \bigcup \Big\{ \bigcup_{j \in \hat{J}(\overline{x})} \bigcup_{\overline{w}_j \in \Omega_j^N(\overline{x})} \overline{\partial}_{\overline{x}}^* g_j(\overline{x}, \overline{w}_j) - \epsilon \mathcal{S}^* \Big\},$$

which is equivalent to there exists $\overline{\overline{w}}_j \in \Omega_j^N(\overline{x})$ $(j \in \hat{J}(\overline{x}))$ satisfying

$$\overline{\partial}_\epsilon^* \varphi(\overline{x}) = \overline{\partial}_\epsilon^* f(\overline{x}) \bigcup \Big\{ \bigcup_{j \in \hat{J}(\overline{x})} \overline{\partial}_{\overline{x}}^* g_j(\overline{x}, \overline{\overline{w}}_j) - \epsilon \mathcal{S}^* \Big\}.$$

Applying Theorem 5 [16], one can obtain that $\varphi(\widetilde{x}) - \varphi(\overline{x}) + \epsilon\|\widetilde{x} - \overline{x}\| \geq 0$, i.e.,

$$\max\Big\{f(\widetilde{x}) - f(\overline{x}) + \epsilon\|\widetilde{x} - \overline{x}\|,\ \max_{j \in J}\{G_j(\widetilde{x})\}\Big\} - \max\Big\{0,\ \max_{j \in J}\{G_j(\overline{x})\}\Big\} \geq 0.$$

Since $\overline{x} \in K$, one can reach that $\max\Big\{0, \max_{j\in J}\{G_j(\overline{x})\}\Big\} = 0$, and hence,

$$\max\Big\{f(\widetilde{x}) - f(\overline{x}) + \epsilon\|\widetilde{x} - \overline{x}\|, \max_{j\in J}\{G_j(\widetilde{x})\}\Big\} \geq 0.$$

Under (3.9), we deduce that $\max_{j\in J}\{G_j(\widetilde{x}) \leq 0$, which proves the following inequality

$$f(\widetilde{x}) - f(\overline{x}) + \epsilon\|\widetilde{x} - \overline{x}\| \geq 0,$$

which conflicts with (3.9). Therefore, we have proven that $\overline{x}$ is a robust $\epsilon-$quasi optimal solution for the primal problem (UNMP). By invoking the result achieved of Theorem 3.3, $(\overline{x}, \lambda, \mu)$ is a robust $\epsilon-$quasi optimal solution for (DUNMP) and the value of the objective functions of (UNMP) and (DUNMP) at $\overline{x}$ are equal, which the claim follows. ■

Remark 3.5. *For the case $\epsilon = 0$, the objective and constraint functions are convex on $\mathbb{R}^n$ and locally Lipschitz at given feasible solution, then the subdifferential in the sense of Convex Analysis is an upper regular convexificator (see Remark 2.3). Therefore, the obtained results in this paper coincides with the well-known results in Convex Analysis.*

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