

The algebra generated by z^3 and a continuous function on a compact set in the plane

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Dedicated to Professor Ha Huy Khoai on the occasion of his 80 birthday.

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Abstract. We study uniform approximation on a compact set $K \subset \mathbb{C}$ by the algebra generated by z^3 and a continuous function f . The main geometric point is that, after pulling back by the polynomial map $(z, w) \mapsto (z^3, w)$, the graph of $(z^3, f(z))$ splits into three branches. If the values of $zf(z)$ stay in a sector of aperture strictly smaller than $2\pi/3$, then these three branches can be separated by the polynomial $p(z, w) = zw$. Two applications of Stout's theorem then give density of the polynomial algebra on the pulled-back set. We record an abstract cubic criterion, a smooth version obtained from the standard one-graph theorem, and several concrete examples.

1. Introduction

Let $K \subset \mathbb{C}$ be compact and let $f \in C(K)$. We denote by $C(K)$ the algebra of complex-valued continuous functions on K , endowed with the supremum norm. If $g_1, \dots, g_m \in C(K)$, we write

$$P[g_1, \dots, g_m; K]$$

for the uniform closure on K of polynomials in the functions $g_1, \dots, g_m$.

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Problems of uniform approximation by function algebras generated by one coordinate function and one additional function have been studied for a long time. In particular, algebras of the form

$$P[z, f; K]$$

and, in related settings, algebras associated with quadratic generators such as

$$P[z^2, f; K]$$

have been investigated in a number of earlier works. For compact sets in the plane, function algebras on disks and related approximation phenomena were studied by de Paepe, Nguyen Quang Dieu and their collaborators in [8, 2, 1]. Approximation by polynomials in two functions and related questions were also considered by O'Farrell, Preskenis and Walsh in [6, 7]. These works revealed several important mechanisms governing polynomial convexity and approximation in function algebras generated by two functions.

The purpose of the present paper is to study the cubic algebra

$$P[z^3, f; K].$$

Passing from $[z, f]$ or $[z^2, f]$ to $[z^3, f]$ leads to a substantially more delicate situation. Indeed, after lifting the problem by means of the map

$$\pi(z, w) = (z^3, w),$$

the associated set splits naturally into three branches. Thus, instead of dealing with one graph or with a two-branch configuration, one has to separate and glue together three branches. This makes the geometry of the problem more complicated and requires a more careful use of polynomial convexity techniques.

We also emphasize that the problem considered here is global in nature. More precisely, our aim is to determine when the algebra generated by z^3 and f is dense in $C(K)$ on the whole compact set K . In this respect, the present problem is different from several local approximation questions studied earlier by Nguyen Quang Dieu, de Paepe and O'Farrell, where the emphasis was on local approximation phenomena. The cubic situation considered here requires a global analysis of the lifted set and of the interaction between its three branches.

The main geometric idea of the paper is simple to state. To the pair $(z^3, f(z))$ we associate the compact set

$$X = \{(z^3, f(z)) : z \in K\} \subset \mathbb{C}^2.$$

Then its pullback under π is the union of three graph pieces. If the values of $zf(z)$ stay in a sector of aperture strictly smaller than $2\pi/3$, then these three branches can be separated by the polynomial

$$p(z, w) = zw.$$

Once each individual branch is known to satisfy polynomial approximation, two successive applications of Stout's theorem allow one to pass from the separate branches to their union. To verify the approximation property on each branch, we combine this geometric argument with Izzo's theorem on totally real manifolds [4]. General background on polynomial convexity and several complex variables may be found in [3], while Kallin's lemma and later developments, especially those discussed in [5, 9], provide the underlying separation philosophy.

The paper is organized as follows. In Section 2 we collect the preliminary tools, including the one-graph approximation criterion and the pullback lemma. In Section 3 we formulate and prove an abstract cubic criterion for the equality $P[z^3, f; K] = C(K)$. In Section 4 we derive a smooth version of this result and several concrete consequences. The final section contains examples illustrating the scope of the method.

Throughout the paper we write

$$\omega = e^{2\pi i/3}.$$

The three branches of the cubic pullback will be denoted by L_0, L_1, L_2 .

2. Preliminaries

We first collect the standard tools used in the proof.

Definition 2.1. *A compact set $E \subset \mathbb{C}^n$ is called polynomially convex if for every point $a \in \mathbb{C}^n \setminus E$ there is a polynomial p on $\mathbb{C}^n$ such that*

$$|p(a)| > \max_E |p|.$$

In one complex variable, polynomial convexity is equivalent to connectedness of the complement. We shall use this elementary fact repeatedly when dealing with compact sectors in $\mathbb{C}$.

The first global tool is Stout's refinement of Kallin's lemma.

Theorem 2.1 (Stout). *Let $X_1, X_2 \subset \mathbb{C}^n$ be compact sets such that*

$$P(X_j) = C(X_j), \quad j = 1, 2.$$

Assume that there exist polynomially convex compact sets $Y_1, Y_2 \subset \mathbb{C}$ and a polynomial p on $\mathbb{C}^n$ such that

1. 0 is a boundary point of both Y_1 and Y_2 ,
2. $Y_1 \cap Y_2 = \{0\}$,
3. $p(X_1) \subset Y_1$ and $p(X_2) \subset Y_2$,
4. $p^{-1}(0) \cap (X_1 \cup X_2) = X_1 \cap X_2$.

Then

$$P(X_1 \cup X_2) = C(X_1 \cup X_2).$$

The second tool is Izzo's approximation theorem on totally real manifolds.

Theorem 2.2 (Izzo). *Let M be a C^1 manifold in $\mathbb{C}^n$ and let $X \subset M$ be compact and polynomially convex. Assume that there is a compact set $E \subset X$ such that M is totally real at every point of $X \setminus E$ and*

$$P(E) = C(E).$$

Then

$$P(X) = C(X).$$

The next proposition is the one-graph input needed later.

Proposition 2.3 (one-graph criterion). *Let $E \subset \mathbb{C}$ be compact and polynomially convex, and let g be a C^2 function on a neighborhood of E . Assume that*

$$\left| \frac{\partial g}{\partial \bar{z}} \right|^2 \geq 2 \operatorname{Re} \left(g \frac{\partial^2 g}{\partial z \partial \bar{z}} \right) \quad \text{on } E,$$

and that the set

$$Z_g = \left\{ a \in E : \frac{\partial g}{\partial \bar{z}}(a) = 0 \right\}$$

is finite. Then the graph

$$\operatorname{Graph}(g; E) = \{(z, g(z)) : z \in E\} \subset \mathbb{C}^2$$

is polynomially convex and satisfies

$$P(\operatorname{Graph}(g; E)) = C(\operatorname{Graph}(g; E)).$$

Remark 2.1. *The proof of Proposition 2.4 is the usual combination of a plurisubharmonic graph estimate with Izzo's theorem. Since the genuinely new point of the present note is the cubic branching argument, we take this standard one-graph statement as a preliminary tool.*

We also need a simple rotation invariance.

Lemma 2.1. *Let $u \in \mathbb{C}$ satisfy $|u| = 1$, let $E \subset \mathbb{C}$ be compact, and let g be a C^2 function on a neighborhood of E . Define*

$$g_u(z) = g(u^{-1}z), \quad z \in uE.$$

If g satisfies the assumptions of Proposition 2.4 on E , then g_u satisfies the same assumptions on uE .

Proof. A direct computation gives

$$\frac{\partial g_u}{\partial \bar{z}}(z) = u^{-1} \frac{\partial g}{\partial \bar{z}}(u^{-1}z) \quad \text{and} \quad \frac{\partial^2 g_u}{\partial z \partial \bar{z}}(z) = \frac{\partial^2 g}{\partial z \partial \bar{z}}(u^{-1}z),$$

so the differential inequality is preserved. The exceptional set for g_u is just the rotation uZ_g of the exceptional set for g , hence it is finite. $\blacksquare$

The next lemma is the pullback device that converts the approximation on the cubic preimage into that on the original set.

Lemma 2.2. *Let $K \subset \mathbb{C}$ be compact, let $f \in C(K)$, and define*

$$X = \{(z^3, f(z)) : z \in K\} \subset \mathbb{C}^2, \quad \pi(z, w) = (z^3, w).$$

Assume that z^3 and f separate the points of K . If

$$P(\pi^{-1}(X)) = C(\pi^{-1}(X)),$$

then

$$P[z^3, f; K] = C(K).$$

Proof. Since z^3 and f separate the points of K , the map

$$\Phi : K \longrightarrow X, \quad \Phi(z) = (z^3, f(z))$$

is a homeomorphism. Therefore it is enough to prove that $P(X) = C(X)$.

Take any $h \in C(X)$. Then $h \circ \pi$ is a continuous function on $\pi^{-1}(X)$. By hypothesis there are polynomials $Q_n(z, w)$ on $\mathbb{C}^2$ such that

$$Q_n \longrightarrow h \circ \pi \quad \text{uniformly on } \pi^{-1}(X).$$

Now average Q_n over the cubic rotations in the first variable:

$$\tilde{Q}_n(z, w) = \frac{1}{3}(Q_n(z, w) + Q_n(\omega z, w) + Q_n(\omega^2 z, w)).$$

Because $h \circ \pi$ is invariant under $(z, w) \mapsto (\omega z, w)$, the averaged polynomials still satisfy

$$\tilde{Q}_n \longrightarrow h \circ \pi \quad \text{uniformly on } \pi^{-1}(X).$$

Every monomial $z^m w^\ell$ is sent by the averaging process either to 0 or to $z^{3q} w^\ell$. Hence each $\tilde{Q}_n$ has the form

$$\tilde{Q}_n(z, w) = R_n(z^3, w)$$

for some polynomial R_n on $\mathbb{C}^2$. Therefore

$$R_n \circ \pi = \tilde{Q}_n,$$

and the convergence above shows that

$$R_n \longrightarrow h \quad \text{uniformly on } X.$$

Thus $P(X) = C(X)$. Pulling this back through the homeomorphism Φ gives

$$P[z^3, f; K] = C(K).$$

■

3. An abstract cubic criterion

We now isolate the geometric part of the argument.

Let $K \subset \mathbb{C}$ be compact and let $f \in C(K)$. We define the three cubic branches by

$$L_j = \{(\omega^j z, f(z)) : z \in K\}, \quad j = 0, 1, 2.$$

If

$$X = \{(z^3, f(z)) : z \in K\},$$

then

$$\pi^{-1}(X) = L_0 \cup L_1 \cup L_2, \quad \pi(z, w) = (z^3, w).$$

Theorem 3.1. *Let $K \subset \mathbb{C}$ be compact and let $f \in C(K)$. Assume that the following conditions hold.*

1. z^3 and f separate the points of K .

2. For each $j = 0, 1, 2$,

$$P(L_j) = C(L_j).$$

3. There is a number t with $0 < t < \pi/3$ such that, for every $z \in K \setminus \{0\}$,

$$zf(z) \in \{re^{is} : r > 0, |s| \leq t\}.$$

Then

$$P[z^3, f; K] = C(K).$$

Proof. Let

$$X = \{(z^3, f(z)) : z \in K\} \subset \mathbb{C}^2$$

and let $\pi(z, w) = (z^3, w)$. By Lemma 2.7, it is enough to prove that

$$P(\pi^{-1}(X)) = C(\pi^{-1}(X)).$$

Since

$$\pi^{-1}(X) = L_0 \cup L_1 \cup L_2,$$

we must show that the union of the three branches has dense polynomial algebra.

Step 1. The branches meet only at the point over the origin. Suppose that

$$(\omega^j a, f(a)) = (\omega^k b, f(b))$$

for some $a, b \in K$ and some $j, k \in \{0, 1, 2\}$. Then

$$a^3 = (\omega^j a)^3 = (\omega^k b)^3 = b^3$$

and $f(a) = f(b)$. By the separation assumption, this forces $a = b$. Hence $\omega^j a = \omega^k a$, so either $j = k$ or $a = 0$. Therefore, if $j \neq k$, then

$$L_j \cap L_k = \{(0, f(0))\}$$

whenever $0 \in K$, and the intersection is empty when $0 \notin K$.

Step 2. Define the separating sectors. Set

$$R = \max_{a \in K} |af(a)|$$

and, for $j = 0, 1, 2$, define

$$Y_j = \{re^{is} : 0 \leq r \leq R, |s - 2\pi j/3| \leq t\}.$$

Because $t < \pi/3$, the sectors Y_0, Y_1, Y_2 are pairwise disjoint except for the common boundary point 0. Each Y_j is convex, hence polynomially convex. Moreover, if $(\omega^j z, f(z)) \in L_j$, then

$$p(\omega^j z, f(z)) = \omega^j z f(z) \in Y_j, \quad p(z, w) = zw.$$

So

$$p(L_j) \subset Y_j, \quad j = 0, 1, 2.$$

Since $zf(z) \neq 0$ for $z \neq 0$, we also have

$$p^{-1}(0) \cap (L_j \cup L_k) = L_j \cap L_k \quad (j \neq k).$$

Step 3. Combine the first two branches. Apply Stout's theorem to $X_1 = L_0$ and $X_2 = L_1$ with the polynomial $p(z, w) = zw$ and the target sets Y_0 and Y_1 . The assumptions are satisfied by Step 2 and by condition (2). Hence

$$P(L_0 \cup L_1) = C(L_0 \cup L_1).$$

Step 4. Add the third branch. Set

$$Y_{01} = Y_0 \cup Y_1.$$

The complement of Y_{01} in $\mathbb{C}$ is connected, so Y_{01} is polynomially convex. Also,

$$p(L_0 \cup L_1) \subset Y_{01}, \quad p(L_2) \subset Y_2, \quad Y_{01} \cap Y_2 = \{0\}.$$

From Step 2 we have

$$p^{-1}(0) \cap (L_0 \cup L_1 \cup L_2) = (L_0 \cup L_1) \cap L_2.$$

Therefore a second application of Stout's theorem, now to $X_1 = L_0 \cup L_1$ and $X_2 = L_2$, yields

$$P(L_0 \cup L_1 \cup L_2) = C(L_0 \cup L_1 \cup L_2).$$

That is,

$$P(\pi^{-1}(X)) = C(\pi^{-1}(X)).$$

Lemma 2.7 now finishes the proof. ■

Remark 3.1. *The theorem above separates the genuinely cubic part of the argument from the one-graph input. Once each branch already satisfies polynomial approximation, the only extra work is to split the three branches by sectors and then glue them together with two applications of Stout's theorem.*

4. A smooth cubic theorem

We now combine Theorem 3.1 with Proposition 2.4.

Theorem 4.1. *Let $K \subset \mathbb{C}$ be compact and polynomially convex, and let f be a C^2 function on a neighborhood of K . Assume that the following conditions hold.*

1. The set

$$Z = \left\{ a \in K : \frac{\partial f}{\partial \bar{z}}(a) = 0 \right\}$$

is finite.

2. There is a number t with $0 < t < \pi/3$ such that, for every $z \in K \setminus \{0\}$,

$$zf(z) \in \{re^{is} : r > 0, |s| \leq t\}.$$

3. For every $z \in K$,

$$\left| \frac{\partial f}{\partial \bar{z}}(z) \right|^2 \geq 2 \operatorname{Re} \left(f(z) \frac{\partial^2 f}{\partial z \partial \bar{z}}(z) \right).$$

4. z^3 and f separate the points of K .

Then

$$P[z^3, f; K] = C(K).$$

Proof. For $j = 0, 1, 2$, put

$$K_j = \omega^j K \quad \text{and} \quad g_j(z) = f(\omega^{-j} z), \quad z \in K_j.$$

Then

$$L_j = \{(\omega^j z, f(z)) : z \in K\} = \operatorname{Graph}(g_j; K_j).$$

By Lemma 2.6, each g_j satisfies the assumptions of Proposition 2.4 on K_j . Hence

$$P(L_j) = C(L_j), \quad j = 0, 1, 2.$$

The sector condition in Theorem 3.1 is exactly assumption (2) above, and the separation condition is assumption (4). Therefore Theorem 3.1 applies and gives

$$P[z^3, f; K] = C(K).$$

■

Corollary 4.1. Let $K \subset \mathbb{C}$ be compact and polynomially convex, and let h be holomorphic on a neighborhood of K . Define

$$f(z) = \bar{z} h(z).$$

Assume that the following conditions hold.

1. There is a number t with $0 < t < \pi/3$ such that, for every $z \in K \setminus \{0\}$,

$$h(z) \in \{re^{is} : r > 0, |s| \leq t\}.$$

2. For every $z \in K$,

$$|h(z)|^2 \geq 2 \operatorname{Re}(\bar{z} h(z) h'(z)).$$

3. z^3 and $\bar{z} h(z)$ separate the points of K .

Then

$$P[z^3, \bar{z} h(z); K] = C(K).$$

Proof. Since

$$zf(z) = |z|^2 h(z),$$

condition 1 gives the sector condition in Theorem 4.1. Also,

$$\frac{\partial f}{\partial \bar{z}}(z) = h(z) \quad \text{and} \quad \frac{\partial^2 f}{\partial z \partial \bar{z}}(z) = h'(z),$$

so condition 2 is precisely the differential inequality in Theorem 4.1. Finally, the exceptional set

$$\left\{ z \in K : \frac{\partial f}{\partial \bar{z}}(z) = 0 \right\} = \{ z \in K : h(z) = 0 \}$$

is finite unless h vanishes identically on a neighborhood of a component of K ; the latter possibility is excluded by 1. Thus all hypotheses of Theorem 4.1 are satisfied. $\blacksquare$

Corollary 4.2. *Let $K \subset \mathbb{C}$ be compact and polynomially convex. Then*

$$P[z^3, \bar{z}; K] = C(K).$$

Proof. Apply Corollary 4.2 with $h \equiv 1$. The sector condition is immediate, the differential inequality is trivial, and z^3 together with $\bar{z}$ separate points because $\bar{a} = \bar{b}$ implies $a = b$. $\blacksquare$

5. Examples

We conclude with a few simple examples showing how the theorem can be used.

Example 5.1. *Let $c \in \mathbb{C} \setminus \{0\}$ satisfy $|\arg c| < \pi/3$. For every compact polynomially convex set $K \subset \mathbb{C}$ we have*

$$P[z^3, c\bar{z}; K] = C(K).$$

Indeed,

$$z(c\bar{z}) = c|z|^2,$$

so the sector condition is automatic, and separation follows from the injectivity of $z \mapsto \bar{z}$.

Example 5.2. Fix numbers $0 < \alpha < \pi/3$ and $0 \leq \lambda \leq 1/2$, and let

$$K = \{re^{i\theta} : 0 \leq r \leq 1, |\theta| \leq \alpha\}.$$

Then K is compact and polynomially convex, and z^3 is injective on K because the opening of the sector is less than $2\pi/3$. If

$$f(z) = \bar{z}e^{\lambda z},$$

then

$$P[z^3, \bar{z}e^{\lambda z}; K] = C(K).$$

Indeed, put $h(z) = e^{\lambda z}$. For $z \in K$ we have

$$|\arg h(z)| = |\lambda \operatorname{Im} z| \leq \lambda < \pi/3,$$

so the sector condition holds. Also,

$$|h(z)|^2 = e^{2\lambda \operatorname{Re} z}$$

and

$$2 \operatorname{Re}(\bar{z} h(z) h'(z)) = 2\lambda \operatorname{Re}(z) e^{2\lambda \operatorname{Re} z} \leq e^{2\lambda \operatorname{Re} z} = |h(z)|^2,$$

because $\operatorname{Re} z \leq 1$ on K and $\lambda \leq 1/2$. Therefore Corollary 4.2 applies.

Example 5.3. The smooth theorem also applies to functions of the form

$$f(z) = \bar{z}(1 + az),$$

provided $|a|$ is small enough on the chosen compact set K . Indeed, when $|az| < 1$ on K , the values of $1 + az$ stay in a sector of aperture $< 2\pi/3$ around the positive real axis, and the differential inequality can be checked directly from

$$h(z) = 1 + az, \quad h'(z) = a.$$

This gives a family of concrete perturbations of the basic example $f(z) = \bar{z}$.

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