J. Math. Math. Sci. (2026) nn–nnn

The P_2^1-Margolis homology of the Dickson algebra

Ngo Anh Tuan (Hanoi, Vietnam)

Dedicated to Professor Ha Huy Khoai on the occasion of his 80 birthday.

(Received Apr. 13, 2026; accepted May 19, 2026)

Abstract. Let the mod 2 Steenrod algebra, $\mathcal{A}$, and the general linear group, $GL(n, \mathbb{F}_2)$, act on $\mathbb{F}_2[x_1, \ldots, x_n]$ with $|x_i| = 1$ in the usual manner. The element P_2^1 of the Steenrod algebra $\mathcal{A}$ has the property $(P_2^1)^2 = 0$. Homology of an $\mathcal{A}$-module M with the differential to be the operation P_2^1 is called the P_2^1-Margolis homology of M. In this paper, we compute the P_2^1-Margolis homology of the Dickson algebra $D_n := \mathbb{F}_2[x_1, \ldots, x_n]^{GL(n,\mathbb{F}_2)}$. We also discuss an analog conjecture of Pengelley and Sinha. (See the last remark.)

1. Introduction

Let $\mathcal{A}$ be the mod 2 Steenrod algebra. The Steenrod algebra $\mathcal{A}$ is the graded $\mathbb{F}_2$-algebra generated by the Steenrod squares Sq^i $(i \geq 0)$ with $|Sq^i| = i$ and subject to the Adem relations with $Sq^0 = 1$ (see Steenrod and Epstein [14]).

The action of $\mathcal{A}$ on the polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_n]$ with $|x_i| = 1$ is given by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & otherwise, \end{cases}$$

Key words and phrases: Steenrod algebra, Milnor operation, Margolis homology.
2020 Mathematics Subject Classification: Primary 55S05, 55S10, 55N99

and satisfying to the Cartan formula

$$Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g),$$

for $f, g \in \mathbb{F}_2[x_1, \ldots, x_n]$ (see Steenrod and Epstein [14]).

Milnor described the mod 2 dual Steenrod algebra $\mathcal{A}_*$ as the graded polynomial algebra (see Milnor [12])

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots],$$

where $|\xi_i| = 2^i - 1$. The Steenrod algebra $\mathcal{A}$ has an $\mathbb{F}_2$-basis dual to the monomial basis of $\mathcal{A}_*$. The elements of the $\mathbb{F}_2$-basis of $\mathcal{A}$ which is dual to $\xi_t^{2^s}$ are denoted by P_t^s, and the elements P_t^0 are denoted by Q_{t-1}. When $s < t$, the elements P_t^s has the property $(P_t^s)^2 = 0$ (see Margolis [11, Lemma 4, p.232]). In particular, we have $P_2^1 = Sq^2Sq^4 + Sq^4Sq^2$ and $P_2^1(xy) = P_2^1(x)y + Q_1(x)Q_1(y) + xP_2^1(y)$ for x, y in any $\mathcal{A}$-algebra (see Margolis [11, p.229]). The Milnor operation Q_j can be defined inductively as follows (see Milnor [12, Cor. 2]).

$$Q_0 = Sq^1, Q_{j+1} = Sq^{2^{j+1}}Q_j + Q_jSq^{2^{j+1}}.$$

Further, Q_j is primitive with respect to the coproduct of $\mathcal{A}$ (see Milnor [12, Lemma 9]). Equivalently, Q_j is a derivation on any $\mathcal{A}$-algebra, i.e. $Q_j(xy) = Q_j(x)y + xQ_j(y)$.

Let us study the Dickson algebra of n variables

$$D_n = \mathbb{F}_2[x_1, \ldots, x_n]^{GL(n,\mathbb{F}_2)},$$

where each generator x_i is of degree 1, and the general linear group, $GL(n, \mathbb{F}_2)$, acts canonically on $\mathbb{F}_2[x_1, \ldots, x_n]$. Dickson proved in [3] that D_n is a polynomial algebra on the Dickson invariants

$$D_n = \mathbb{F}_2[c_{n,0}, \ldots, c_{n,n-1}].$$

Since the action of $GL(n, \mathbb{F}_2)$ commutes with the action of the Steenrod algebra, D_n is inherited the structure of an $\mathcal{A}$-module from $\mathbb{F}_2[x_1, \ldots, x_n]$. In this article, when n is fixed, the element $c_{n,s}$ will be denoted by c_s for abbreviation.

Let M be any left $\mathcal{A}$-module and $0 \leq s < t$. The P_t^s-Margolis homology group of M is defined as (see Margolis [11])

$$H_*(M; P_t^s) := KerP_t^s/ImP_t^s.$$

In [5], Hưng computed the Q_j-Margolis homology of the Dickson algebra. In this article, we compute the P_2^1- Margolis homology of the Dickson algebra. Let us recall that P_2^1 is not a (mod 2) derivation, that is in general

$$P_2^1(xy) \neq P_2^1(x)y + xP_2^1(y).$$

In particular, if x and y are some P_2^1-cycles, it does not imply that xy is also a P_2^1-cycle. In other words, $H_*(D_n; P_2^1)$ does not admit any multiplicative structure.

Let $\mathcal{A}(2)$ be the subalgebra of the Steenrod algebra generated by Sq^1, Sq^2, and Sq^4 and let $\wedge(P_2^1)$ be the exterior algebra on the generator P_2^1. According to P. Bhattacharya, I. Bobkova, and B. Thomas (see [2]), there is a map

$$q : Ext^{s,t}_{\mathcal{A}(2)}(D_n, \mathbb{F}_2) \to Ext^{s,t}_{\wedge(P_2^1)}(D_n, \mathbb{F}_2).$$

Note that

$$Ext^{s,t}_{\wedge(P_2^1)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{2,1}],$$

where $|h_{2,1}| = (1, 6)$ and

$$\mathbb{F}_2[h_{2,1}] \otimes H_*(D_n; P_2^1) \subset Ext^{s,t}_{\wedge(P_2^1)}(D_n, \mathbb{F}_2).$$

So, the knowledge of $H_*(D_n; P_2^1)$ is essential for understanding $Ext^{s,t}_{\mathcal{A}(2)}(D_n, \mathbb{F}_2)$. This is the reason why the P_2^1-Margolis homology of the Dickson algebra is interesting.

Let $D_n^4 = \mathbb{F}_2[c_0^4, \ldots, c_{n-1}^4]$. The following theorems are the main results of the paper.

Theorem 1.1. *(a) The P_2^1-Margolis homology of the Dickson algebra D_1 is isomorphic as $\mathbb{F}_2$-vector spaces to*

$$\frac{\mathbb{F}_2[c_0^4]}{(c_0^8)} \oplus \mathbb{F}_2\langle [c_0], [c_0^5]\rangle.$$

(b) The P_2^1-Margolis homology of the Dickson algebra D_2 is isomorphic as $\mathbb{F}_2$-vector spaces to

$$\mathbb{F}_2 \oplus \mathbb{F}_2\langle [c_0c_1]\rangle.$$

(c) The P_2^1-Margolis homology of the Dickson algebra D_3 is isomorphic as $\mathbb{F}_2$-vector spaces to

$$\frac{\mathbb{F}_2[c_0^4, c_1^4, c_2^4]}{(c_0^4, c_1^4)} \oplus \mathbb{F}_2\langle [c_0c_2^{4k_2+1}] \mid k_2 \geq 0\rangle.$$

Theorem 1.2. *For $n \geq 4$, the P_2^1-Margolis homology of the Dickson algebra D_n is isomorphic as $\mathbb{F}_2$-vector spaces to*

$$\frac{D_n^4}{(c_0^4, c_1^4)} \oplus \mathbb{F}_2\langle [c_0 c_2^{4i_2} c_3^{2i_3} c_4^{i_4} \cdots c_{n-1}^{i_{n-1}}], [c_2^{4j_2} c_3^{2j_3} c_4^{j_4} \cdots c_{n-1}^{j_{n-1}}], [c_0^{k_0} c_2^{4k_2+1} c_3^{2k_3} c_4^{k_4} \cdots c_{n-1}^{k_{n-1}}] \rangle.$$

Here, $i_2, \ldots, i_{n-1}, j_2, \ldots, j_{n-1} \geq 0$, at least one of the powers $2j_3, j_4, \ldots, j_{n-1}$ is not divisible by 4, $k_2, \ldots, k_{n-1} \geq 0$ and $0 \leq k_0 \leq 1$.

The paper is divided into 3 sections. In Section 2, we compute the action of the operation P_2^1 on the Dickson invariants. In Sections 3, we completely compute the P_2^1-Margolis homology of the Dickson algebra.

2. The action of the operation P_2^1 on the Dickson invariants

The action of the Milnor operation, Q_j, on the Dickson invariants was compute in [5].

Lemma 2.1. (Hưng [5]) *For $0 \leq s < n$,*

$$Q_1(c_s) = \begin{cases} c_0, & n > 2, s = 2, \\ 0, & n > 2, s \neq 2, \\ c_0 c_s, & n = 2, \\ c_0^4, & n = 1, s = 0. \end{cases}$$

The action of the Steenrod algebra on the Dickson invariants was computed in [4].

Theorem 2.1. (Hưng [4])

$$Sq^i(c_s) = \begin{cases} c_r, & i = 2^s - 2^r, r \leq s, \\ c_r c_t, & i = 2^n - 2^t + 2^s - 2^r, r \leq s < t, \\ c_s^2, & i = 2^n - 2^s, \\ 0, & \textit{otherwise}. \end{cases}$$

Note that D_1 is the polynomial algebra $\mathbb{F}_2[x]$ and $c_0 = x$.

Lemma 2.2. *The action of P_2^1 on D_1 is given by*
$P_2^1(x^{4k}) = P_2^1(x^{4k+1}) = 0$, $P_2^1(x^{4k+2}) = x^{4k+8}, P_2^1(x^{4k+3}) = x^{4k+9}$.

Proof. Since $Q_1(x^{2k}) = 0$, we have

$$P_2^1(x^{4k}) = P_2^1(x^{2k})x^{2k} + (Q_1(x^{2k}))^2 + x^{2k}P_2^1(x^{2k}) = 0.$$

Since $Sq^i(x^j) = \binom{j}{i}x^{i+j}$ (see [14]) and $P_2^1 = Sq^4Sq^2 + Sq^2Sq^4$, we have

$$P_2^1(x) = 0, P_2^1(x^2) = x^8, P_2^1(x^3) = x^9.$$

Therefore, we get

$$\begin{aligned}
P_2^1(x^{4k+1}) &= P_2^1(x^{4k})x + Q_1(x)Q_1(x^{4k}) + x^{4k}P_2^1(x) = 0.\\
P_2^1(x^{4k+2}) &= P_2^1(x^{4k})x^2 + Q_1(x^2)Q_1(x^{4k}) + x^{4k}P_2^1(x^2) = x^{4k+8}.\\
P_2^1(x^{4k+3}) &= P_2^1(x^{4k})x^3 + Q_1(x^3)Q_1(x^{4k}) + x^{4k}P_2^1(x^3) = x^{4k+9}.
\end{aligned}$$

■

Lemma 2.3. *Suppose that i_0 and i_1 are even numbers. The action of P_2^1 on D_2 is given by*

i)

$$P_2^1(c_0^{i_0}c_1^{i_1}) = \begin{cases} c_0^{i_0+2}c_1^{i_1}, & i_0 = 4k_0+2, i_1 = 4k_1,\\ c_0^{i_0+2}c_1^{i_1}, & i_0 = 4k_0, i_1 = 4k_1+2,\\ 0, & i_0 = 4k_0, i_1 = 4k_1,\\ 0, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases}$$

ii)

$$P_2^1(c_0c_0^{i_0}c_1^{i_1}) = \begin{cases} c_0^{i_0+1}c_1^{i_1+3}, & i_0 = 4k_0+2, i_1 = 4k_1,\\ c_0^{i_0+1}c_1^{i_1+3}, & i_0 = 4k_0, i_1 = 4k_1+2,\\ c_0^{i_0+1}c_1^{i_1+3} + c_0^{i_0+3}c_1^{i_1}, & i_0 = 4k_0, i_1 = 4k_1,\\ c_0^{i_0+1}c_1^{i_1+3} + c_0^{i_0+3}c_1^{i_1}, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases}$$

iii)

$$P_2^1(c_1c_0^{i_0}c_1^{i_1}) = \begin{cases} c_0^{i_0}c_1^{i_1+4} + c_0^{i_0+2}c_1^{i_1+1}, & i_0 = 4k_0+2, i_1 = 4k_1,\\ c_0^{i_0}c_1^{i_1+4} + c_0^{i_0+2}c_1^{i_1+1}, & i_0 = 4k_0, i_1 = 4k_1+2,\\ c_0^{i_0}c_1^{i_1+4}, & i_0 = 4k_0, i_1 = 4k_1,\\ c_0^{i_0}c_1^{i_1+4}, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases}$$

iv)

$$P_2^1(c_0c_1c_0^{i_0}c_1^{i_1}) = \begin{cases} c_0^{i_0+3}c_1^{i_1+1}, & i_0 = 4k_0+2, i_1 = 4k_1,\\ c_0^{i_0+3}c_1^{i_1+1}, & i_0 = 4k_0, i_1 = 4k_1+2,\\ 0, & i_0 = 4k_0, i_1 = 4k_1,\\ 0, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases}$$

Proof. By Theorem 2.1, we have

$$P_2^1(c_0) = c_0c_1^3 + c_0^3, P_2^1(c_1) = c_1^4.$$

By Lemma 2.1, we have

$$Q_1(c_0) = c_0^2, Q_1(c_1) = c_0c_1.$$

Therefore, we get

$$P_2^1(c_0c_1) = P_2^1(c_0)c_1 + Q_1(c_0)Q_1(c_1) + c_0P_2^1(c_1) = 0.$$

$$P_2^1(c_0^{2u}) = (Q_1(c_0^u))^2 = u^2c_0^{2u+2}, P_2^1(c_1^{2v}) = (Q_1(c_1^v))^2 = v^2c_0^2c_1^{2v}.$$

i) Since i_0 and i_1 are even numbers, we have $Q_1(c_0^{i_0}) = Q_1(c_1^{i_1}) = 0$. So,

$$\begin{aligned} P_2^1(c_0^{i_0}c_1^{i_1}) &= P_2^1(c_0^{i_0})c_1^{i_1} + Q_1(c_0^{i_0})Q_1(c_1^{i_1}) + P_2^1(c_1^{i_1})c_0^{i_0} \\ &= P_2^1(c_0^{i_0})c_1^{i_1} + P_2^1(c_1^{i_1})c_0^{i_0} \\ &= \begin{cases} c_0^{i_0+2}c_1^{i_1}, & i_0 = 4k_0+2, i_1 = 4k_1, \\ c_0^{i_0+2}c_1^{i_1}, & i_0 = 4k_0, i_1 = 4k_1+2, \\ 0, & i_0 = 4k_0, i_1 = 4k_1, \\ 0, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases} \end{aligned}$$

ii) Since i_0 and i_1 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}) = 0$. By part i) of Lemma 2.3, we get

$$\begin{aligned} P_2^1(c_0c_0^{i_0}c_1^{i_1}) &= P_2^1(c_0)c_0^{i_0}c_1^{i_1} + Q_1(c_0)Q_1(c_0^{i_0}c_1^{i_1}) + c_0P_2^1(c_0^{i_0}c_1^{i_1}) \\ &= P_2^1(c_0)c_0^{i_0}c_1^{i_1} + c_0P_2^1(c_0^{i_0}c_1^{i_1}) \\ &= \begin{cases} c_0^{i_0+1}c_1^{i_1+3}, & i_0 = 4k_0+2, i_1 = 4k_1, \\ c_0^{i_0+1}c_1^{i_1+3}, & i_0 = 4k_0, i_1 = 4k_1+2, \\ c_0^{i_0+1}c_1^{i_1+3} + c_0^{i_0+3}c_1^{i_1}, & i_0 = 4k_0, i_1 = 4k_1, \\ c_0^{i_0+1}c_1^{i_1+3} + c_0^{i_0+3}c_1^{i_1}, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases} \end{aligned}$$

iii) Since i_0 and i_1 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}) = 0$. By part i) of Lemma 2.3, we get

$$\begin{aligned} P_2^1(c_1c_0^{i_0}c_1^{i_1}) &= P_2^1(c_1)c_0^{i_0}c_1^{i_1} + Q_1(c_1)Q_1(c_0^{i_0}c_1^{i_1}) + c_1P_2^1(c_0^{i_0}c_1^{i_1}) \\ &= P_2^1(c_1)c_0^{i_0}c_1^{i_1} + c_1P_2^1(c_0^{i_0}c_1^{i_1}) \\ &= \begin{cases} c_0^{i_0}c_1^{i_1+4} + c_0^{i_0+2}c_1^{i_1+1}, & i_0 = 4k_0+2, i_1 = 4k_1, \\ c_0^{i_0}c_1^{i_1+4} + c_0^{i_0+2}c_1^{i_1+1}, & i_0 = 4k_0, i_1 = 4k_1+2, \\ c_0^{i_0}c_1^{i_1+4}, & i_0 = 4k_0, i_1 = 4k_1, \\ c_0^{i_0}c_1^{i_1+4}, & i_0 = 4k_0+2, i_1 = 4k_1+2. \end{cases} \end{aligned}$$

iv) Since i_0 and i_1 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1})=0$. By part i) of Lemma 2.3, we get

$$\begin{aligned}P_2^1(c_0c_1c_0^{i_0}c_1^{i_1}) &= P_2^1(c_0c_1)c_0^{i_0}c_1^{i_1}+Q_1(c_0c_1)Q_1(c_0^{i_0}c_1^{i_1})+c_0c_1P_2^1(c_0^{i_0}c_1^{i_1})\\ &= c_0c_1P_2^1(c_0^{i_0}c_1^{i_1})\\ &= \begin{cases} c_0^{i_0+3}c_1^{i_1+1}, & i_0=4k_0+2, i_1=4k_1,\\ c_0^{i_0+3}c_1^{i_1+1}, & i_0=4k_0, i_1=4k_1+2,\\ 0, & i_0=4k_0, i_1=4k_1,\\ 0, & i_0=4k_0+2, i_1=4k_1+2.\end{cases}\end{aligned}$$

■

Lemma 2.4. *Suppose that i_0, i_1, and i_2 are even numbers. The action of P_2^1 on D_3 is given by*

i)

$$P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+2}c_1^{i_1}c_2^{i_2-2}, & i_2=4k_2+2,\\ 0, & i_2=4k_2.\end{cases}$$

ii)

$$P_2^1(c_0c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+1}c_1^{i_1+1}c_2^{i_2}+c_0^{i_0+3}c_1^{i_1}c_2^{i_2-2}, & i_2=4k_2+2,\\ c_0^{i_0+1}c_1^{i_1+1}c_2^{i_2}, & i_2=4k_2.\end{cases}$$

iii)

$$P_2^1(c_1c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+2}c_1^{i_1+1}c_2^{i_2-2}+c_0^{i_0}c_1^{i_1+2}c_2^{i_2}, & i_2=4k_2+2,\\ c_0^{i_0}c_1^{i_1+2}c_2^{i_2}, & i_2=4k_2.\end{cases}$$

iv)

$$P_2^1(c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+2}c_1^{i_1}c_2^{i_2-1}+c_0^{i_0}c_1^{i_1+1}c_2^{i_2+1}, & i_2=4k_2+2,\\ c_0^{i_0}c_1^{i_1+1}c_2^{i_2+1}, & i_2=4k_2.\end{cases}$$

v)

$$P_2^1(c_0c_1c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+3}c_1^{i_1+1}c_2^{i_2-2}, & i_2=4k_2+2,\\ 0, & i_2=4k_2.\end{cases}$$

vi)

$$P_2^1(c_0c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+3}c_1^{i_1}c_2^{i_2-1}, & i_2=4k_2+2,\\ 0, & i_2=4k_2.\end{cases}$$

vii)

$$P_2^1(c_1c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+2}c_1^{i_1+1}c_2^{i_2-1}, & i_2 = 4k_2+2, \\ 0, & i_2 = 4k_2. \end{cases}$$

viii)

$$P_2^1(c_0c_1c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) = \begin{cases} c_0^{i_0+1}c_1^{i_1+2}c_2^{i_2+1} + c_0^{i_0+3}c_1^{i_1+1}c_2^{i_2-1}, & i_2 = 4k_2+2, \\ c_0^{i_0+1}c_1^{i_1+2}c_2^{i_2+1}, & i_2 = 4k_2. \end{cases}$$

Proof. By Lemma 2.1, we have

$$Q_1(c_0) = Q_1(c_1) = 0, Q_1(c_2) = c_0.$$

By Theorem 2.1, we have

$$P_2^1(c_0) = c_0c_1, P_2^1(c_1) = c_1^2, P_2^1(c_2) = c_1c_2.$$

So, we get

$$\begin{aligned} P_2^1(c_0c_1) &= P_2^1(c_0)c_1 + Q_1(c_0)Q_1(c_1) + c_0P_2^1(c_1) = 0, \\ P_2^1(c_1c_2) &= P_2^1(c_1)c_2 + Q_1(c_1)Q_1(c_2) + c_1P_2^1(c_2) = 0, \\ P_2^1(c_0c_2) &= P_2^1(c_0)c_2 + Q_1(c_0)Q_1(c_2) + c_0P_2^1(c_2) = 0, \\ P_2^1(c_0c_1c_2) &= P_2^1(c_0c_1)c_2 + Q_1(c_0c_1)Q_1(c_2) + c_0c_1P_2^1(c_2) = c_0c_1^2c_2. \end{aligned}$$

i) From above computations and the fact that $P_2^1(c_0^{i_0}) = P_2^1(c_1^{i_1}) = Q_1(c_0^{i_0}) = Q_1(c_2^{i_2}) = 0$, we have

$$\begin{aligned} P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_0^{i_0}c_1^{i_1})c_2^{i_2} + Q_1(c_0^{i_0}c_1^{i_1})Q_1(c_2^{i_2}) + c_0^{i_0}c_1^{i_1}P_2^1(c_2^{i_2}) \\ &= P_2^1(c_0^{i_0}c_1^{i_1})c_2^{i_2} + c_0^{i_0}c_1^{i_1}P_2^1(c_2^{i_2}) \\ &= c_0^{i_0}c_1^{i_1}P_2^1(c_2^{i_2}) = (i_2/2)^2c_0^{i_0+2}c_1^{i_1}c_2^{i_2-2} \\ &= \begin{cases} c_0^{i_0+2}c_1^{i_1}c_2^{i_2-2}, & i_2 = 4k_2+2, \\ 0, & i_2 = 4k_2. \end{cases} \end{aligned}$$

ii) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned} P_2^1(c_0c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_0)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_0)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_0P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) \\ &= P_2^1(c_0)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_0P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) \\ &= \begin{cases} c_0^{i_0+1}c_1^{i_1+1}c_2^{i_2} + c_0^{i_0+3}c_1^{i_1}c_2^{i_2-2}, & i_2 = 4k_2+2, \\ c_0^{i_0+1}c_1^{i_1+1}c_2^{i_2}, & i_2 = 4k_2. \end{cases} \end{aligned}$$

iii) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}P_2^1(c_1c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_1)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_1)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_1P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= P_2^1(c_1)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_1P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= \begin{cases} c_0^{i_0+2}c_1^{i_1+1}c_2^{i_2-2} + c_0^{i_0}c_1^{i_1+2}c_2^{i_2}, & i_2 = 4k_2+2,\\ c_0^{i_0}c_1^{i_1+2}c_2^{i_2}, & i_2 = 4k_2.\end{cases}\end{aligned}$$

iv) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}P_2^1(c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_2)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= P_2^1(c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= \begin{cases} c_0^{i_0+2}c_1^{i_1}c_2^{i_2-1} + c_0^{i_0}c_1^{i_1+1}c_2^{i_2+1}, & i_2 = 4k_2+2,\\ c_0^{i_0}c_1^{i_1+1}c_2^{i_2+1}, & i_2 = 4k_2.\end{cases}\end{aligned}$$

v) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}P_2^1(c_0c_1c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_0c_1)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_0c_1)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_0c_1P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= P_2^1(c_0c_1)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_0c_1P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= \begin{cases} c_0^{i_0+3}c_1^{i_1+1}c_2^{i_2-2}, & i_2 = 4k_2+2,\\ 0, & i_2 = 4k_2.\end{cases}\end{aligned}$$

vi) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}P_2^1(c_0c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_0c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_0c_2)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_0c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= c_0c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= \begin{cases} c_0^{i_0+3}c_1^{i_1}c_2^{i_2-1}, & i_2 = 4k_2+2,\\ 0, & i_2 = 4k_2.\end{cases}\end{aligned}$$

vii) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}P_2^1(c_1c_2c_0^{i_0}c_1^{i_1}c_2^{i_2}) &= P_2^1(c_1c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_1c_2)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_1c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= P_2^1(c_1c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_1c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\ &= \begin{cases} c_0^{i_0+2}c_1^{i_1+1}c_2^{i_2-1}, & i_2 = 4k_2+2,\\ 0, & i_2 = 4k_2.\end{cases}\end{aligned}$$

viii) Since i_0, i_1, and i_2 are even numbers, we have $Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) = 0$. By part i) of Lemma 2.4, we get

$$\begin{aligned}
&P_2^1(c_0c_1c_2c_0^{i_0}c_1^{i_1}c_2^{i_2})\\
&= P_2^1(c_0c_1c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + Q_1(c_0c_1c_2)Q_1(c_0^{i_0}c_1^{i_1}c_2^{i_2}) + c_0c_1c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\
&= P_2^1(c_0c_1c_2)c_0^{i_0}c_1^{i_1}c_2^{i_2} + c_0c_1c_2P_2^1(c_0^{i_0}c_1^{i_1}c_2^{i_2})\\
&= \begin{cases} c_0^{i_0+1}c_1^{i_1+2}c_2^{i_2+1} + c_0^{i_0+3}c_1^{i_1+1}c_2^{i_2-1}, & i_2 = 4k_2+2,\\ c_0^{i_0+1}c_1^{i_1+2}c_2^{i_2+1}, & i_2 = 4k_2. \end{cases}
\end{aligned}$$

■

Lemma 2.5. *For $n \geq 4$, a monomial f in D_n, and $c_2, c_3 \nmid f$, the action of P_2^1 on D_n is given by*

$$P_2^1(c_2^{i_2}c_3^{i_3}f) = \begin{cases} 0, & i_2 = 4k_2, i_3 = 2k_3,\\ c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2, i_3 = 2k_3+1,\\ 0, & i_2 = 4k_2+1, i_3 = 2k_3,\\ c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+1, i_3 = 2k_3+1,\\ c_0^2c_2^{i_2-2}c_3^{i_3}f, & i_2 = 4k_2+2, i_3 = 2k_3,\\ c_0^2c_2^{i_2-2}c_3^{i_3}f + c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+2, i_3 = 2k_3+1,\\ c_0^2c_2^{i_2-2}c_3^{i_3}f, & i_2 = 4k_2+3, i_3 = 2k_3,\\ c_0^2c_2^{i_2-2}c_3^{i_3}f + c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+3, i_3 = 2k_3+1. \end{cases}$$

Proof. By Theorem 2.1, we have

$$P_2^1(c_s) = (Sq^4Sq^2 + Sq^2Sq^4)(c_s) = \begin{cases} c_1, & s = 3,\\ 0, & s \neq 3. \end{cases}$$

By Lemma 2.1, we have

$$Q_1(c_s) = \begin{cases} c_0, & s = 2,\\ 0, & s \neq 2. \end{cases}$$

So, we obtain

$$P_2^1(c_2^{i_2}) = \begin{cases} 0, & i_2 = 4k_2,\\ 0, & i_2 = 4k_2+1,\\ c_0^2c_2^{i_2-2}, & i_2 = 4k_2+2\\ c_0^2c_2^{i_2-2}, & i_2 = 4k_2+3. \end{cases}$$

$$P_2^1(c_3^{i_3}) = \begin{cases} 0, & i_3 = 2k_3,\\ c_1c_3^{i_3-1}, & i_3 = 2k_3+1. \end{cases}$$

From the above computations and the fact that $Q_1(f) = P_2^1(f) = 0$, we get

$$\begin{aligned}
P_2^1(c_2^{i_2}c_3^{i_3}f) &= c_3^{i_3}P_2^1(c_2^{i_2}f) + Q_1(c_3^{i_3})Q_1(c_2^{i_2}f) + c_2^{i_2}fP_2^1(c_3^{i_3}) \\
&= c_3^{i_3}(P_2^1(c_2^{i_2})f + Q_1(c_2^{i_2})Q_1(f) + c_2^{i_2}P_2^1(f)) + c_2^{i_2}fP_2^1(c_3^{i_3}) \\
&= c_3^{i_3}P_2^1(c_2^{i_2})f + c_2^{i_2}fP_2^1(c_3^{i_3}) \\
&= \begin{cases}
0, & i_2 = 4k_2, i_3 = 2k_3, \\
c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2, i_3 = 2k_3+1, \\
0, & i_2 = 4k_2+1, i_3 = 2k_3, \\
c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+1, i_3 = 2k_3+1, \\
c_0^2c_2^{i_2-2}c_3^{i_3}f, & i_2 = 4k_2+2, i_3 = 2k_3, \\
c_0^2c_2^{i_2-2}c_3^{i_3}f + c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+2, i_3 = 2k_3+1, \\
c_0^2c_2^{i_2-2}c_3^{i_3}f, & i_2 = 4k_2+3, i_3 = 2k_3, \\
c_0^2c_2^{i_2-2}c_3^{i_3}f + c_1c_2^{i_2}c_3^{i_3-1}f, & i_2 = 4k_2+3, i_3 = 2k_3+1.
\end{cases}
\end{aligned}$$

■

3. The P_2^1-Margolis homology of the Dickson algebra

In this section, we will compute the P_2^1-Margolis homology of the Dickson algebra D_n.

The following theorem is also numbered as Theorem 1.1 in the introduction.

Theorem 3.1. *(a) The P_2^1-Margolis homology of the Dickson algebra D_1 is isomorphic as $\mathbb{F}_2$-vector spaces to*

$$\frac{\mathbb{F}_2[c_0^4]}{(c_0^8)} \oplus \mathbb{F}_2\langle [c_0], [c_0^5]\rangle.$$

(b) The P_2^1-Margolis homology of the Dickson algebra D_2 is isomorphic as $\mathbb{F}_2$-vector spaces to

$$\mathbb{F}_2 \oplus \mathbb{F}_2\langle [c_0c_1]\rangle.$$

(c) The P_2^1-Margolis homology of the Dickson algebra D_3 is isomorphic as $\mathbb{F}_2$-vector spaces to

$$\frac{\mathbb{F}_2[c_0^4, c_1^4, c_2^4]}{(c_0^4, c_1^4)} \oplus \mathbb{F}_2\langle [c_0c_2^{4k_2+1}] \mid k_2 \geq 0\rangle.$$

Proof. (a) By Lemma 2.2, $KerP_2^1$ is generated by x^{4i} and x^{4i+1} for $i \geq 0$.

By Lemma 2.2, we have

$$x^{4i} = P_2^1(x^{4i-6}) \text{ for } i \geq 2 \text{ and } x^{4i+1} = P_2^1(x^{4i-5}) \text{ for } i \geq 2.$$

So, we get $[x^{4i}] = [x^{4i+1}] = 0$ in $H_*(D_1; P_2^1)$ for $i \geq 2$.

Let W_1 be the $\mathbb{F}_2$-subspace of D_1 generated by $c_0^{i_0}$, with i_0 not divisible by 4. The evident equality $D_1 = D_1^4 \oplus W_1$ and Lemma 2.2 imply that

$$\begin{aligned} KerP_2^1 &= D_1^4 \oplus (W_1 \cap KerP_2^1), \\ ImP_2^1 &= (D_1^4 \cap ImP_2^1) \oplus (W_1 \cap ImP_2^1). \end{aligned}$$

By Lemma 2.2, we have $D_1^4 \cap ImP_2^1 = \mathbb{F}_2\langle c_0^{4k+8} \mid k \geq 0\rangle = (c_0^8)D_1^4$.

From Lemma 2.2 and the above computations, we have

$$\frac{W_1 \cap KerP_2^1}{W_1 \cap ImP_2^1} = \mathbb{F}_2\langle [c_0], [c_0^5]\rangle.$$

So, we obtain

$$\begin{aligned} H_*(D_1; P_2^1) :&= KerP_2^1/ImP_2^1 \cong \frac{D_1^4}{D_1^4 \cap ImP_2^1} \oplus \frac{W_1 \cap KerP_2^1}{W_1 \cap ImP_2^1} \\ &= \frac{\mathbb{F}_2[c_0^4]}{(c_0^8)} \oplus \mathbb{F}_2\langle [c_0], [c_0^5]\rangle. \end{aligned}$$

(b) By Lemma 2.3, $KerP_2^1$ is generated by the following elements (1) through (8).

1. $c_0^{4k_0}c_1^{4k_1}$ for $k_0, k_1 \geq 0$.
2. $c_0^{4k_0+2}c_1^{4k_1+2}$ for $k_0, k_1 \geq 0$.
3. $c_0^{4k_0+1}c_1^{4k_1+1}$ for $k_0, k_1 \geq 0$.
4. $c_0^{4k_0+3}c_1^{4k_1+3}$ for $k_0, k_1 \geq 0$.
5. $c_0^{4k_0+3}c_1^{4k_1} + c_0^{4k_0+1}c_1^{4k_1+3}$ for $k_0 \geq 0, k_1 \geq 0$.
6. $c_0^{4k_0+1}c_1^{4k_1+2} + c_0^{4k_0-1}c_1^{4k_1+5}$ for $k_0 \geq 1, k_1 \geq 0$.
7. $c_0^{4k_0+2}c_1^{4k_1} + c_0^{4k_0+4}c_1^{4k_1-3}$ for $k_0 \geq 0, k_1 \geq 1$.
8. $c_0^{4k_0}c_1^{4k_1+2} + c_0^{4k_0+2}c_1^{4k_1-1}$ for $k_0 \geq 0, k_1 \geq 1$.

By Lemma 2.3, we have

$$P_2^1(c_0^{4k_0-2}c_1^{4k_1}) = c_0^{4k_0}c_1^{4k_1} \text{ for } k_0 \geq 1, k_1 \geq 0.$$
$$P_2^1(c_0^{4k_0}c_1^{4k_1+1}) = c_0^{4k_0}c_1^{4k_1+4} \text{ for } k_0, k_1 \geq 0.$$
$$P_2^1(c_0^{4k_0}c_1^{4k_1+2}) = c_0^{4k_0+2}c_1^{4k_1+2} \text{ for } k_0, k_1 \geq 0.$$
$$P_2^1(c_0^{4k_0-1}c_1^{4k_1+1}) = c_0^{4k_0+1}c_1^{4k_1+1} \text{ for } k_0 \geq 1, k_1 \geq 0.$$
$$P_2^1(c_0c_1^{4k_1-2}) = c_0c_1^{4k_1+1} \text{ for } k_1 \geq 1.$$
$$P_2^1(c_0^{4k_0+1}c_1^{4k_1+3}) = c_0^{4k_0+3}c_1^{4k_1+3} \text{ for } k_0, k_1 \geq 0.$$
$$P_2^1(c_0c_0^{4k_0}c_1^{4k_1}) = c_0^{4k_0+3}c_1^{4k_1} + c_0^{4k_0+1}c_1^{4k_1+3} \text{ for } k_0, k_1 \geq 0.$$
$$P_2^1(c_0c_0^{4k_0-2}c_1^{4k_1+2}) = c_0^{4k_0+1}c_1^{4k_1+2} + c_0^{4k_0-1}c_1^{4k_1+5} \text{ for } k_0 \geq 1, k_1 \geq 0.$$
$$P_2^1(c_0^{4k_0+2}c_1^{4k_1-3}) = c_0^{4k_0+2}c_1^{4k_1} + c_0^{4k_0+4}c_1^{4k_1-3} \text{ for } k_0 \geq 0, k_1 \geq 1.$$
$$P_2^1(c_0^{4k_0}c_1^{4k_1-1}) = c_0^{4k_0}c_1^{4k_1+2} + c_0^{4k_0+2}c_1^{4k_1-1} \text{ for } k_0 \geq 0, k_1 \geq 1.$$

From the above computations, we get

$$[c_0^{4k_0}c_1^{4k_1}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0^2 + k_1^2 \neq 0.$$
$$[c_0^{4k_0+2}c_1^{4k_1+2}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0, k_1 \geq 0.$$
$$[c_0^{4k_0+1}c_1^{4k_1+1}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0 \geq 1, k_1 \geq 0.$$
$$[c_0^{4k_0+3}c_1^{4k_1+3}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0, k_1 \geq 0.$$
$$[c_0^{4k_0+3}c_1^{4k_1} + c_0^{4k_0+1}c_1^{4k_1+3}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0, k_1 \geq 0.$$
$$[c_0^{4k_0+1}c_1^{4k_1+2} + c_0^{4k_0-1}c_1^{4k_1+5}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0 \geq 1, k_1 \geq 0.$$
$$[c_0^{4k_0+2}c_1^{4k_1} + c_0^{4k_0+4}c_1^{4k_1-3}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0 \geq 0, k_1 \geq 1.$$
$$[c_0^{4k_0}c_1^{4k_1+2} + c_0^{4k_0+2}c_1^{4k_1-1}] = 0 \text{ in } H_*(D_2; P_2^1) \text{ for } k_0 \geq 0, k_1 \geq 1.$$

By Lemma 2.3, we see that c_0c_1 is not in the image of P_2^1. So, $[c_0c_1] \neq 0$ in $H_*(D_2; P_2^1)$.

Let W_2 be the $\mathbb{F}_2$-subspace of D_2 generated by $c_0^{i_0}c_1^{i_1}$ with at least one of the powers i_0, i_1 not divisible by 4. The evident equality $D_2 = D_2^4 \oplus W_2$ and Lemma 2.3 imply that

$$Ker P_2^1 = D_2^4 \oplus (W_2 \cap Ker P_2^1),$$
$$Im P_2^1 = (D_2^4 \cap Im P_2^1) \oplus (W_2 \cap Im P_2^1).$$

By Lemma 2.3, we have $D_2^4 \cap Im P_2^1 = \mathbb{F}_2\langle c_0^{4k_0+4}c_1^{4k_1}, c_0^{4i_0}c_1^{4i_1+4} \mid k_0, k_1, i_0, i_1 \geq 0\rangle = (c_0^4, c_1^4)D_2^4$.

From Lemma 2.3 and the above computations, we have

$$\frac{W_2 \cap KerP_2^1}{W_2 \cap ImP_2^1} = \mathbb{F}_2\langle [c_0c_1]\rangle.$$

So, we obtain

$$\begin{aligned} H_*(D_2; P_2^1) &:= KerP_2^1/ImP_2^1 \cong \frac{D_2^4}{D_2^4 \cap ImP_2^1} \oplus \frac{W_2 \cap KerP_2^1}{W_2 \cap ImP_2^1} \\ &= \frac{\mathbb{F}_2[c_0^4, c_1^4]}{(c_0^4, c_1^4)} \oplus \mathbb{F}_2\langle [c_0c_1]\rangle \\ &= \mathbb{F}_2 \oplus \mathbb{F}_2\langle [c_0c_1]\rangle. \end{aligned}$$

(c) By Lemma 2.4, $KerP_2^1$ is generated by the following elements (1) through (6).

1. $c_0^{2k_0}c_1^{2k_1}c_2^{4k_2}$ for $k_0, k_1, k_2 \geq 0$.
2. $c_0^{2k_0+1}c_1^{2k_1+1}c_2^{4k_2}$ for $k_0, k_1, k_2 \geq 0$.
3. $c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+1}$ for $k_0, k_1, k_2 \geq 0$.
4. $c_0^{2k_0}c_1^{2k_1+1}c_2^{4k_2+1}$ for $k_0, k_1, k_2 \geq 0$.
5. $c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+2} + c_0^{2k_0+2}c_1^{2k_1-1}c_2^{4k_2}$ for $k_0, k_2 \geq 0, k_1 \geq 1$.
6. $c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2} + c_0^{2k_0-1}c_1^{2k_1+1}c_2^{4k_2+2}$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
7. $c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+1} + c_0^{2k_0-2}c_1^{2k_1+1}c_2^{4k_2+3}$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
8. $c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+3} + c_0^{2k_0+3}c_1^{2k_1-1}c_2^{4k_2+1}$ for $k_0, k_2 \geq 0, k_1 \geq 1$.

By Lemma 2.4, we have

$$\begin{aligned} &P_2^1(c_0^{2k_0-2}c_1^{2k_1}c_2^{4k_2+2}) = c_0^{2k_0}c_1^{2k_1}c_2^{4k_2} \text{ for } k_0 \geq 1, k_1, k_2 \geq 0. \\ &P_2^1(c_0^{2k_0}c_1^{2k_1-1}c_2^{4k_2}) = c_0^{2k_0}c_1^{2k_1}c_2^{4k_2} \text{ for } k_0, k_2 \geq 0, k_1 \geq 1. \end{aligned}$$

So, we get $[c_0^{2k_0}c_1^{2k_1}c_2^{4k_2}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0^2 + k_1^2 \neq 0, k_2 \geq 0$. By Lemma 2.4, we see that $c_2^{4k_2}(k_2 \geq 0)$ is not in the image of P_2^1, so $[c_2^{4k_2}] \neq 0$ in $H_*(D_3; P_2^1)$.

By Lemma 2.4, we have

$$\begin{aligned} &P_2^1(c_0^{2k_0-1}c_1^{2k_1}c_2^{4k_2+3}) = c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+1} \text{ for } k_0 \geq 1, k_1, k_2 \geq 0. \\ &P_2^1(c_0c_1^{2k_1-1}c_2^{4k_2+1}) = c_0c_1^{2k_1}c_2^{4k_2+1} \text{ for } k_0, k_2 \geq 0, k_1 \geq 1. \end{aligned}$$

So, we get $[c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+1}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0^2 + k_1^2 \neq 0, k_2 \geq 0$. By Lemma 2.4, we see that $c_0c_2^{4k_2+1}(k_2 \geq 0)$ is not in the image of P_2^1, so $[c_0c_2^{4k_2+1}] \neq 0$ in $H_*(D_3; P_2^1)$.

By Lemma 2.4, we have

$P_2^1(c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2}) = c_0^{2k_0+1}c_1^{2k_1+1}c_2^{4k_2}$ for $k_0, k_1, k_2 \geq 0$.
$P_2^1(c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+1}) = c_0^{2k_0}c_1^{2k_1+1}c_2^{4k_2+1}$ for $k_0, k_1, k_2 \geq 0$.
$P_2^1(c_0^{2k_0}c_1^{2k_1-2}c_2^{4k_2+2}) = c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+2} + c_0^{2k_0+2}c_1^{2k_1-1}c_2^{4k_2}$ for $k_0, k_2 \geq 0, k_1 \geq 1$.
$P_2^1(c_0^{2k_0-1}c_1^{2k_1}c_2^{4k_2+2}) = c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2} + c_0^{2k_0-1}c_1^{2k_1+1}c_2^{4k_2+2}$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
$P_2^1(c_0^{2k_0-2}c_1^{2k_1}c_2^{4k_2+3}) = c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+1} + c_0^{2k_0-2}c_1^{2k_1+1}c_2^{4k_2+3}$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
$P_2^1(c_0^{2k_0+1}c_1^{2k_1-1}c_2^{4k_2+3}) = c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+3} + c_0^{2k_0+3}c_1^{2k_1-1}c_2^{4k_2+1}$ for $k_0, k_2 \geq 0, k_1 \geq 1$.

From the above computations, we get

$[c_0^{2k_0+1}c_1^{2k_1+1}c_2^{4k_2}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0, k_1, k_2 \geq 0$.
$[c_0^{2k_0}c_1^{2k_1+1}c_2^{4k_2+1}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0, k_1, k_2 \geq 0$.
$[c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+2} + c_0^{2k_0+2}c_1^{2k_1-1}c_2^{4k_2}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0, k_2 \geq 0, k_1 \geq 1$.
$[c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2} + c_0^{2k_0-1}c_1^{2k_1+1}c_2^{4k_2+2}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
$[c_0^{2k_0}c_1^{2k_1}c_2^{4k_2+1} + c_0^{2k_0-2}c_1^{2k_1+1}c_2^{4k_2+3}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0 \geq 1, k_1, k_2 \geq 0$.
$[c_0^{2k_0+1}c_1^{2k_1}c_2^{4k_2+3} + c_0^{2k_0+3}c_1^{2k_1-1}c_2^{4k_2+1}] = 0$ in $H_*(D_3; P_2^1)$ for $k_0, k_2 \geq 0, k_1 \geq 1$.

Let W_3 be the $\mathbb{F}_2$-subspace of D_3 generated by $c_0^{i_0}c_1^{i_1}c_2^{i_2}$ with at least one of the powers i_0, i_1, i_2 not divisible by 4. The evident equality $D_3 = D_3^4 \oplus W_3$ and Lemma 2.4 imply that

$$\begin{aligned} Ker P_2^1 &= D_3^4 \oplus (W_3 \cap Ker P_2^1), \\ Im P_2^1 &= (D_3^4 \cap Im P_2^1) \oplus (W_3 \cap Im P_2^1). \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} &D_3^4 \cap Im P_2^1 \\ &= \mathbb{F}_2\langle c_0^{4k_0+4}c_1^{4k_1}c_2^{4k_2}, c_0^{4i_0}c_1^{4i_1+4}c_2^{4i_2} \mid k_0, k_1, k_2, i_0, i_1, i_2 \geq 0\rangle = (c_0^4, c_1^4)D_3^4. \end{aligned}$$

From Lemma 2.4 and the above computations, we have

$$\frac{W_3 \cap Ker P_2^1}{W_3 \cap Im P_2^1} = \mathbb{F}_2\langle [c_0c_2^{4k_2+1}] \mid k_2 \geq 0\rangle.$$

So, we obtain

$$H_*(D_3;P_2^1) := KerP_2^1/ImP_2^1 \cong \frac{D_3^4}{D_3^4\cap ImP_2^1}\oplus\frac{W_3\cap KerP_2^1}{W_3\cap ImP_2^1}$$
$$= \frac{\mathbb{F}_2[c_0^4,c_1^4,c_2^4]}{(c_0^4,c_1^4)}\oplus\mathbb{F}_2\langle[c_0c_2^{4k_2+1}]\mid k_2\geq 0\rangle.$$

■

The following theorem is also numbered as Theorem 1.2 in the introduction.

Theorem 3.2. *For $n\geq 4$, the P_2^1-Margolis homology of the Dickson algebra D_n is isomorphic as $\mathbb{F}_2$-vector spaces to*

$$\frac{D_n^4}{(c_0^4,c_1^4)}\oplus\mathbb{F}_2\langle[c_0c_2^{4i_2}c_3^{2i_3}c_4^{i_4}\cdots c_{n-1}^{i_{n-1}}],[c_2^{4j_2}c_3^{2j_3}c_4^{j_4}\cdots c_{n-1}^{j_{n-1}}],[c_0^{k_0}c_2^{4k_2+1}c_3^{2k_3}c_4^{k_4}\cdots c_{n-1}^{k_{n-1}}]\rangle.$$

Here, $i_2,\ldots,i_{n-1},j_2,\ldots,j_{n-1}\geq 0$, at least one of the powers $2j_3,j_4,\ldots,j_{n-1}$ is not divisible by 4, $k_2,\ldots,k_{n-1}\geq 0$ and $0\leq k_0\leq 1$.

Proof. By Lemma 2.5, $KerP_2^1$ is generated by the following elements (1) through (4).

1. $c_2^{4k_2}c_3^{2k_3}f$ for $k_2,k_3\geq 0$, a monomial f in D_n and $c_2,c_3\nmid f$.
2. $c_2^{4k_2+1}c_3^{2k_3}f$ for $k_2,k_3\geq 0$, a monomial f in D_n and $c_2,c_3\nmid f$.
3. $c_0^2c_2^{4k_2}c_3^{2k_3+1}f+c_1c_2^{4k_2+2}c_3^{2k_3}f$ for $k_2,k_3\geq 0$, a monomial f in D_n and $c_2,c_3\nmid f$.
4. $c_1c_2^{4k_2+3}c_3^{2k_3}f+c_0^2c_2^{4k_2+1}c_3^{2k_3+1}f$ for $k_2,k_3\geq 0$, a monomial f in D_n and $c_2,c_3\nmid f$.

If $c_0^2\mid f$ and $c_2,c_3\nmid f$ then $P_2^1(c_2^{4k_2+2}c_3^{2k_3}\frac{f}{c_0^2})=c_2^{4k_2}c_3^{2k_3}f$ (by Lemma 2.5).

If $c_1\mid f$ and $c_2,c_3\nmid f$ then $P_2^1(c_2^{4k_2}c_3^{2k_3+1}\frac{f}{c_1})=c_2^{4k_2}c_3^{2k_3}f$ (by Lemma 2.5).

If $c_0^2,c_1,c_2,c_3\nmid f$ then $c_2^{4k_2}c_3^{2k_3}f$ is not in the image of P_2^1 (by Lemma 2.5). So $[c_2^{4k_2}c_3^{2k_3}f]\neq 0$ in $H_*(D_n;P_2^1)$.

If $c_0^2\mid f$ and $c_2,c_3\nmid f$ then $P_2^1(c_2^{4k_2+3}c_3^{2k_3}\frac{f}{c_0^2})=c_2^{4k_2+1}c_3^{2k_3}f$ (by Lemma 2.5).

If $c_1\mid f$ and $c_2,c_3\nmid f$ then $P_2^1(c_2^{4k_2+1}c_3^{2k_3+1}\frac{f}{c_1})=c_2^{4k_2+1}c_3^{2k_3}f$ (by Lemma 2.5).

If $c_0^2,c_1,c_2,c_3\nmid f$ then $c_2^{4k_2+1}c_3^{2k_3}f$ is not in the image of P_2^1 (by Lemma 2.5). So $[c_2^{4k_2+1}c_3^{2k_3}f]\neq 0$ in $H_*(D_n;P_2^1)$.

By Lemma 2.5, for $c_2,c_3,\nmid f$ we have

$$P_2^1(c_2^{4k_2+3}c_3^{2k_3+1}f)=c_1c_2^{4k_2+3}c_3^{2k_3}f+c_0^2c_2^{4k_2+1}c_3^{2k_3+1}f,$$
$$P_2^1(c_2^{4k_2+2}c_3^{2k_3+1}f)=c_1c_2^{4k_2+2}c_3^{2k_3}f+c_0^2c_2^{4k_2}c_3^{2k_3+1}f.$$

So $[c_1c_2^{4k_2+3}c_3^{2k_3}f + c_0^2c_2^{4k_2+1}c_3^{2k_3+1}f] = [c_1c_2^{4k_2+2}c_3^{2k_3}f + c_0^2c_2^{4k_2}c_3^{2k_3+1}f] = 0$ in $H_*(D_n; P_2^1)$.

Let W_n be the $\mathbb{F}_2$-subspace of D_n generated by $c_0^{i_0}\cdots c_{n-1}^{i_{n-1}}$ with at least one of the powers $i_0, \ldots, i_{n-1}$ not divisible by 4. The evident equality $D_n = D_n^4 \oplus W_n$ and Lemma 2.5 imply that

$$\begin{aligned} KerP_2^1 &= D_n^4 \oplus (W_n \cap KerP_2^1),\\ ImP_2^1 &= (D_n^4 \cap ImP_2^1) \oplus (W_n \cap ImP_2^1). \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned} &D_4^4 \cap ImP_2^1\\ &= \mathbb{F}_2\langle c_0^{4k_0+4}c_1^{4k_1}\cdots c_{n-1}^{4k_{n-1}}, c_0^{4i_0}c_1^{4i_1+4}c_2^{4i_2}\cdots c_{n-1}^{4i_{n-1}} \mid k_0, \ldots, k_{n-1}, i_0, \ldots, i_{n-1} \geq 0\rangle\\ &= (c_0^4, c_1^4)D_n^4. \end{aligned}$$

From Lemma 2.5 and the above computations, we have

$$\begin{aligned} &\frac{W_n \cap KerP_2^1}{W_n \cap ImP_2^1}\\ &= \mathbb{F}_2\langle [c_0c_2^{4i_2}c_3^{2i_3}c_4^{i_4}\cdots c_{n-1}^{i_{n-1}}], [c_2^{4j_2}c_3^{2j_3}c_4^{j_4}\cdots c_{n-1}^{j_{n-1}}], [c_0^{k_0}c_2^{4k_2+1}c_3^{2k_3}c_4^{k_4}\cdots c_{n-1}^{k_{n-1}}]\rangle, \end{aligned}$$

with $i_2, \ldots, i_{n-1}, j_2, \ldots, j_{n-1} \geq 0$, at least one of the powers $2j_3, j_4, \ldots, j_{n-1}$ not divisible by 4, $k_2, \ldots, k_{n-1} \geq 0$ and $0 \leq k_0 \leq 1$.

So, we obtain

$$\begin{aligned} H_*(D_n; P_2^1) &:= KerP_2^1/ImP_2^1 \cong \frac{D_n^4}{D_n^4 \cap ImP_2^1} \oplus \frac{W_n \cap KerP_2^1}{W_n \cap ImP_2^1}\\ &= \frac{D_n^4}{(c_0^4, c_1^4)}\\ &\oplus \mathbb{F}_2\langle [c_0c_2^{4i_2}c_3^{2i_3}c_4^{i_4}\cdots c_{n-1}^{i_{n-1}}], [c_2^{4j_2}c_3^{2j_3}c_4^{j_4}\cdots c_{n-1}^{j_{n-1}}], [c_0^{k_0}c_2^{4k_2+1}c_3^{2k_3}c_4^{k_4}\cdots c_{n-1}^{k_{n-1}}]\rangle, \end{aligned}$$

with $i_2, \ldots, i_{n-1}, j_2, \ldots, j_{n-1} \geq 0$, at least one of the powers $2j_3, j_4, \ldots, j_{n-1}$ not divisible by 4, $k_2, \ldots, k_{n-1} \geq 0$ and $0 \leq k_0 \leq 1$.

The theorem is proved. ■

Remark 3.1. (a) Pengelley and Sinha stated their conjecture only for the Q_j-Margolis homology of the Dickson algebra (see [13]). However, following their idea, an analog conjecture for the P_2^1-Margolis homology of the Dickson algebra should be as follows: any representative of a homology class in $H_*(D_n; P_2^1)$ can be taken in D_n^4.

(b) Let $E^n = (\mathbb{Z}/p)^n$ be the elementary abelian group. N. H. V. Hưng and N. H. V. Hưng - N. Đ. Ngà proved that the (analog) Pengelley - Sinha conjecture is invalid. More precisely, for

(i) $G = S_{2^n}$, the Margolis homology of the image for the restriction $ImRes(G, E^n)$ is not only generated by D_n^2, (see [5, 6]),

(ii) $G = S_{p^n}$ and p an odd prime, the Margolis homology of the image $ImRes(G, E^n)$ is not only generated by the Dickson algebra D_n, (see [7, 8]),

(iii) $G = A_{p^n}$ and p an odd prime, the Margolis homology of the image $ImRes(G, E^n)$ is not only generated by

$$\widetilde{D}_n = \mathbb{F}_p[\widetilde{L}_n, c_1, \ldots, c_{n-1}],$$

where $\widetilde{L}_n^2 = c_0$, (see [9]),

(iv) $G = S_{2^n,2}$, the Margolis homology of the image $ImRes(G, E^n)$ is not only generated by M_n^2, where $M_n = ImRes(S_{2^n,2}, E^n)$ denotes the Mùi algebra of rank n, (see [10]).

Here S_m, A_m, and $S_{2^n,2}$ denote respectively the symmetric group on m letters, the alternating group on m letter, and the 2-Sylow subgroup of S_{2^n}. In all the above cases, the "new" generators of the Margolis homology for the image $ImRes(G, E^n)$ are called the *critical elements*.

Theorem 3.1 and Theorem 3.2 show that the analog Pengelley - Sinha conjecture is not correct for $n \geq 1$ because there are critical elements in $H_*(D_n; P_2^1)$ with $n \geq 1$. For $n = 1$, the critical elements are $[c_0]$ and $[c_0^5]$. For $n = 2$, the critical element is $[c_0c_1]$. For $n = 3$, the critical elements are $[c_0c_2^{4k_2+1}]$ with $k_2 \geq 0$. For $n \geq 4$, the critical elements are

$$[c_0c_2^{4i_2}c_3^{2i_3}c_4^{i_4}\cdots c_{n-1}^{i_{n-1}}], [c_2^{4j_2}c_3^{2j_3}c_4^{j_4}\cdots c_{n-1}^{j_{n-1}}], [c_0^{k_0}c_2^{4k_2+1}c_3^{2k_3}c_4^{k_4}\cdots c_{n-1}^{k_{n-1}}]$$

with $i_2, \ldots, i_{n-1}, j_2, \ldots, j_{n-1} \geq 0$, at least one of the powers $2j_3, j_4, \ldots, j_{n-1}$ not divisible by 4, $k_2, \ldots, k_{n-1} \geq 0$ and $0 \leq k_0 \leq 1$.

Acknowledgment. I would like to express my gratitude to Prof. Nguyễn H. V. Hưng for giving several criticisms. The author was partially funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2023.04.

References

[1] **J. F. Adams and H. Margolis**, *Modules over the Steenrod algebra*, Topology **10** (1971), 271–282.

[2] **P. Bhattacharya, I. Bobkova, and B. Thomas**, *The P_2^1-Margolis homology of connective topological modular forms*, Homology, Homotopy and Applications **23(2)** (2021), 379–402.

[3] **L. E. Dickson**, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75–98.

[4] **Nguyễn H. V. Hưng**, *The action of the Steenrod squares on the modular invariants of linear groups*, Proc. Amer. Math. Soc. **113** (1991), 1097–1104.

[5] **Nguyễn H. V. Hưng**, *The mod 2 Margolis homology of the Dickson algebra*, C. R. Math. Acad. Sci. Paris **358** (2020), 505–510.

[6] **Nguyễn H. V. Hưng**, *The Margolis homology of the Dickson algebra and Pengelley-Sinha's Conjecture*, submitted.

[7] **Nguyễn H. V. Hưng**, *The mod p Margolis homology of the Dickson-Mùi algebra*, C. R. Math. Acad. Sci. Paris **359** (2021), 229–236.

[8] **Nguyễn H. V. Hưng**, *The Margolis homology of the Dickson-Mùi algebra*, submitted.

[9] **Nguyễn H. V. Hưng**, *The Margolis homology of the restriction image from the alternating group to its generic maximal elementary abelian subgroup*, Homology, Homotopy and Applications (2026), to appear.

[10] **Nguyễn H. V. Hưng and Nguyễn Đ. Ngà**, *The mod 2 Margolis homology of the invariants under the Sylow subgroup of the general linear group*, Homology, Homotopy and Applications **28(2)** (2026), 79-105.

[11] **H. R. Margolis**, *Spectra and the Steenrod algebra*, volume 29 of North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, (1983). Modules over the Steenrod algebra and the stable homotopy category.

[12] **J. W. Milnor**, *The Steenrod algebra and its dual*, Ann. Math. **67** (1958), 150–171.

[13] **D. P. Sinha**, *Cohomology of symmetric groups*, Lecture on the Vietnam-US Mathematical Joint Meeting, Quynhon, June 10–13, (2019). https://pages.uoregon.edu/dps/VNUS2019/symmetric6.pdf

[14] **N. E. Steenrod and D. B. A. Epstein**, Cohomology operations, Ann. of Math. Stud. Vol. 50, Princeton Univ. Press, Princeton, N. J. (1962).

Ngo Anh Tuan
Department of Mathematics, University of Science, Vietnam National University, Hanoi
Hanoi
Vietnam
ngoanhtuan@vnu.edu.vn