A self-adaptive inertial algorithm for split variational inequalities with applications to traffic networks

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Abstract. This paper investigates the split variational inequality problem in real Hilbert spaces. We propose a novel iterative algorithm that combines Tseng's modified forward–backward splitting method with an inertial extrapolation technique and self-adaptive step sizes. The proposed method requires only monotonicity and Lipschitz continuity of the operators, without assuming co-coerciveness. A key feature of the algorithm is that the step sizes are updated at each iteration via simple computations, eliminating the need for prior knowledge of the norm of the transfer operator or the Lipschitz constants. Strong convergence of the generated sequence is established under standard conditions. As an application, we model traffic network equilibrium as a split variational inequality problem and present numerical experiments to demonstrate the efficiency and practical implementability of the proposed method.

1. Introduction

Let C_0 be a nonempty, closed, and convex subset of a real Hilbert space H_0, and $B_0 : C_0 \to H_0$ be an operator. The variational inequality problem on the set C_0 with operator B_0 (VIP(C_0, B_0)) is defined as follows: Find $x^* \in C_0$ such that

$$\langle B_0 x^*, x - x^* \rangle \geq 0, \ \forall x \in C_0. \tag{1.1}$$

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This is a fundamental problem in optimization. The VIP has a wide range of applications in diverse fields, including signal processing, optimal control, image restoration and composite minimization (see [1,12,17,22]). The VIP was later generalized to the following split variational inequality problem (SVIP) by Censor et al. [8]: Find $x^* \in C_0$ such that x^* solves VIP (1.1) and $y^* = Fx^* \in C_1$ solves

$$\langle B_1 y^*, y - y^* \rangle \geq 0, \ \forall y \in C_1, \tag{1.2}$$

where C_1 is a nonempty, closed, and convex subset of real Hilbert space H_1, $B_1 : H_1 \to H_1$ is an operator, and $F : H_0 \to H_1$ is a bounded linear operator. We denote the solution set of SVIP (1.1)–(1.2) by

$$\Gamma := \{x^* \in \mathbf{S}(C_0, B_0) \mid Fx^* \in \mathbf{S}(C_1, B_1)\},$$

where $\mathbf{S}(C_j, B_j)$ denotes the solution set of VIP(C_j, B_j) for each $j = 0, 1$.

The SVIP seeks a solution to the VIP(C_0, B_0) such that its image under the bounded linear operator F also solves the VIP(C_1, B_1). In this sense, the SVIP combines the VIP with the split feasibility problem (SFP) introduced and studied by Censor and Elfving [6]:

$$\text{Find } x \in C_0 \text{ such that } y = Fx \in C_1. \tag{1.3}$$

In particular, when $B_0 = B_1 = 0$, the SVIP reduces to the SFP (1.3).

The SVIP has been widely applied in various fields such as intensity-modulated radiation therapy (IMRT), phase retrieval, medical image reconstruction, signal processing, and network traffic equilibrium (see [3,5,8,15]).

Censor et al. [8] proposed and studied the following iterative method for solving SVIP (1.1)–(1.2): for $x_1 \in H_0$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = U\big(x_n + \tau F^*(V - I)Fx_n\big), \ n \geq 1, \tag{1.4}$$

where $U = P_{C_0}(I - \lambda B_0)$, $V = P_{C_1}(I - \lambda B_1)$ and $\tau \in \big(0, \frac{1}{R}\big)$ with R being the spectral radius of the operator F^*F. They proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution of SVIP (1.1)–(1.2), provided that the solution set of problem SVIP (1.1)–(1.2) is nonempty, B_0, B_1 are α_0-, α_1-co-coercive operators, and $\lambda \in (0, 2\delta]$, where $\delta := \min\{\alpha_0, \alpha_1\}$. However, the co-coercivity assumption imposed on the operators B_0 and B_1 is quite restrictive, which limits the applicability of the algorithm.

To overcome the co-coercivity restriction, researchers have turned to projection-based methods originally developed for the classical VIP. Among these, the extragradient method (EGM), introduced by Korpelevich [14] for finite-dimensional saddle-point problems, stands as the most prevalently employed algorithm. For

a monotone and L_0-Lipschitz continuous operator $B_0 : C_0 \to H_0$, the EGM generates a sequence $\{x_n\}$ via

$$\begin{cases} z_n = P_{C_0}(x_n - \delta B_0 x_n), \\ x_{n+1} = P_{C_0}(x_n - \delta B_0 z_n), \end{cases} \tag{1.5}$$

where $\delta \in (0, 1/L_0)$ is a fixed step size. Under these conditions, the sequence $\{x_n\}$ generated by (1.5) converges weakly to a point in $\mathrm{VIP}(C_0, B_0)$, provided the solution set is nonempty. A notable drawback of the EGM, however, is that it requires two projections onto C_0 at each iteration, which can severely impair the overall computational efficiency when such projections are costly.

To reduce the per-iteration projection cost, Tseng [26] introduced the forward–backward–forward method, commonly referred to as Tseng's extragradient method (TEGM). The iterative scheme reads

$$\begin{cases} z_n = P_{C_0}(x_n - \delta B_0 x_n), \\ x_{n+1} = z_n - \delta(B_0 z_n - B_0 x_n). \end{cases} \tag{1.6}$$

Unlike (1.5), the scheme (1.6) requires only one projection per iteration. Under the same assumptions of monotonicity and L_0-Lipschitz continuity with step size $\delta \in (0, 1/L_0)$, the sequence $\{x_n\}$ generated by (1.6) converges weakly to a solution in $\mathrm{VIP}(C_0, B_0)$.

As an alternative approach, Censor et al. [7] introduced the subgradient extragradient method (SEGM), which replaces the second projection in (1.5) with a projection onto a constructible half-space T_n:

$$\begin{cases} z_n = P_{C_0}(x_n - \delta B_0 x_n), \\ T_n = \{x \in H_0 \mid \langle x_n - \delta B_0 x_n - z_n, x - z_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \delta B_0 z_n). \end{cases} \tag{1.7}$$

Under the conditions that B_0 is L_0-Lipschitz continuous and monotone with step size $\delta \in (0, 1/L_0)$, the scheme (1.7) converges weakly to a solution of monotone variational inequality problem [7].

A shared limitation of the methods (1.5)–(1.7) is their reliance on a *fixed* step size $\delta \in (0, 1/L_0)$, which requires prior knowledge of the Lipschitz constant L_0. Since L_0 is often unavailable in practice, several authors have proposed *self-adaptive* step-size rules that update δ_n at each iteration using only information already computed (see [1,16,18]).

A further acceleration technique that has attracted considerable attention is the *inertial method*, originating from the heavy-ball method of Polyak [20] and the inertial proximal algorithm of Alvarez and Attouch [2]. In the context of variational inequalities, inertial methods augment the standard iterative update

by an extrapolation term $\theta_k(x^k - x^{k-1})$ that exploits the momentum of past iterates to accelerate convergence (see [1,16,18]). In existing inertial algorithms for SVIP, however, the inertial extrapolation and the Halpern-type anchoring are typically treated as separate components applied in successive steps.

Motivated by the SVIP formulation of Censor et al. [8] and the projection-based methods described above, several authors have proposed algorithms for solving SVIP (1.1)–(1.2) under the weaker assumption that B_0 and B_1 are monotone and Lipschitz continuous. To overcome the co-coercivity restriction while retaining strong convergence, Izuchukwu et al. [11] proposed an inertial projection-and-contraction method for SVIP that establishes strong convergence without product-space transformation. However, their algorithm requires the Lipschitz constants L_0 and L_1 to be known in advance to initialize and bound the step sizes, and the inertial parameter is bounded by a fixed constant that itself depends on L_0, L_1.

Motivated by these developments, in this paper we propose a new one-step inertial self-adaptive Tseng's forward–backward splitting algorithm for solving SVIP (1.1)–(1.2) under monotonicity and Lipschitz continuity of B_0, B_1, requiring no prior knowledge of L_0, L_1, or $\|F\|$. A distinguishing feature of the proposed algorithm is that the viscosity anchor $(1-\alpha_k)x^k$ is integrated directly into the inertial extrapolation step, yielding

$$w^k = (1-\alpha_k)x^k + \theta_k(x^k - x^{k-1}), \tag{1.8}$$

so that acceleration and stabilization toward the minimum-norm solution are achieved simultaneously in a single step. This differs from the standard inertial update $w^k = x^k + \theta_k(x^k - x^{k-1})$ used in [1,16,18], where the anchor and inertial components are treated separately. The main contributions of this paper are as follows:

1. We propose a new algorithm that does not require prior knowledge of the Lipschitz constants L_0, L_1 of operator B_0, B_1, respectively, or the operator norm $\|F\|$, thanks to a self-adaptive step size update rule.
2. We prove strong convergence of the generated sequence to the minimum-norm solution of SVIP under standard and mild assumptions.
3. We introduce a two-phase viscosity control parameter $\{\alpha_k\}$ whose decay exponent transitions smoothly between two polynomial rates via a sigmoid function, providing larger step sizes in early iterations for faster progress and smaller ones in later iterations for enhanced stability near the solution.
4. We apply the proposed algorithm to a split variational inequality problem arising from a two-city traffic network equilibrium model, thereby illustrating the practical applicability of the proposed method.

The rest of this paper is organized as follows. Section 2 recalls some necessary preliminaries. Section 3 presents the proposed algorithm and establishes its strong convergence theorem. Section 4 is devoted to the application to the traffic network model. Numerical experiments are provided in Section 5 to illustrate the efficiency of the method.

2. Preliminaries

In this section, we recall some definitions, notations, and lemmas that will be used throughout the paper. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We write $x^n \rightharpoonup x$ and $x^n \to x$ to denote weak and strong convergence of $\{x^n\}$ to x, respectively.

The following identities hold for all $x, y \in H$ and $\lambda \in \mathbb{R}$:

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2, \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Definition 2.1 (see [4]). *Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique element $P_C x \in C$ such that*

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|.$$

The mapping $P_C : H \to C$ is called the metric projection *onto C and satisfies the following characterization properties [4], for all $x \in H$ and for all $y \in C$:*

$$\langle x - P_C x,\, y - P_C x \rangle \leq 0, \tag{2.3}$$

$$\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2. \tag{2.4}$$

Remark 2.1. *Let C be a nonempty closed convex subset of H. The element of minimum norm in C is $u^* = P_C(0)$, which is the unique solution of $\langle u^*, u^* - y \rangle \leq 0$ for all $y \in C$, or equivalently $\langle u^*, y - u^* \rangle \geq 0$ for all $y \in C$.*

Definition 2.2 (see [4]). *An operator $B : H \to H$ is said to be:*

(i) monotone *if $\langle Bx - By,\, x - y \rangle \geq 0$ for all $x, y \in H$;*

(ii) L-Lipschitz continuous *if there exists $L > 0$ such that $\|Bx - By\| \leq L\|x - y\|$ for all $x, y \in H$;*

(iii) α-co-coercive *(or α-inverse strongly monotone) if there exists $\alpha > 0$ such that $\langle Bx - By,\, x - y \rangle \geq \alpha\|Bx - By\|^2$ for all $x, y \in H$.*

We note that every α-co-coercive operator is monotone and $\frac{1}{\alpha}$-Lipschitz continuous, but the converse does not hold in general [16].

Definition 2.3 (see [4]). *Let H_0 and H_1 be two real Hilbert spaces and let $F : H_0 \to H_1$ be a bounded linear operator. The* adjoint operator $F^* : H_1 \to H_0$ *is the unique bounded linear operator satisfying $\langle Fx, y\rangle = \langle x, F^*y\rangle$ for all $x \in H_0$, $y \in H_1$. Moreover, $\|F^*\| = \|F\|$.*

The following lemma characterizes solutions of the split variational inequality problem for monotone operators.

Lemma 2.1 (see [24]). *Let C be a nonempty closed convex subset of H and let $B : H \to H$ be a continuous monotone operator. Then $x^* \in C$ is a solution of the* VIP(C, B) *if and only if*

$$\langle By,\, y - x^*\rangle \geq 0,\ \forall\, y \in C.$$

Lemma 2.2 (see [25]). *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of non-negative real numbers such that $a_{k+1} \leq a_k + b_k$ for all $k \geq 1$. If $\sum_{k=1}^{\infty} b_k < +\infty$, then $\lim_{k\to\infty} a_k$ exists.*

Lemma 2.3 (see [21]). *Let $\{s_k\}$ be a sequence of non-negative real numbers, $\{\alpha_k\} \subset (0, 1)$ with $\sum_{k=1}^{\infty} \alpha_k = \infty$, and $\{d_k\}$ be a sequence of real numbers. Suppose that*

$$s_{k+1} \leq (1 - \alpha_k)s_k + \alpha_k d_k,\ \forall\, k \geq 1.$$

If $\limsup_{l\to\infty} d_{k_l} \leq 0$ for every subsequence $\{s_{k_l}\}$ of $\{s_k\}$ satisfying $\liminf_{l\to\infty}(s_{k_l+1} - s_{k_l}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.4 (see [15]). *Let a, b, c be positive real numbers satisfying $0 < a < \bar{a} < 1$, $0 < b < \bar{b} < 2$, $0 < c < \bar{c} < 1$. Let $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ be non-negative sequences with $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = 0$, and let $\{\lambda_k\}$ be a sequence with $\lim_{k\to\infty} \lambda_k = \lambda^* > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$*

$$\frac{\lambda_k(a_k + a)}{\lambda_{k+1}} \in (0, 1),\ b_k + b \in (0, 2),\ c_k + c \in (0, 1).$$

3. Main Results

Throughout this section, we work under the following standing hypotheses. Let H_0 and H_1 be two real Hilbert spaces; let $C_0 \subseteq H_0$ and $C_1 \subseteq H_1$ be

nonempty closed convex subsets. Let $B_0 : H_0 \to H_0$ and $B_1 : H_1 \to H_1$ be monotone and L_0-, L_1-Lipschitz continuous operators, and let $F : H_0 \to H_1$ be a bounded linear operator. For notational convenience we set $F_1 := F$ and $F_0 := I$ (the identity on H_0), and we assume that the solution set Γ of SVIP (1.1)–(1.2) is nonempty.

3.1. The Proposed Algorithm

We first collect the standing assumptions on the control parameters, which will be in force for the strong convergence analysis.

Assumption 3.1. *The control sequences satisfy the following conditions:*

(C1) $\{\alpha_k\} \subset (0,1)$, $\lim_{k\to\infty} \alpha_k = 0$, *and* $\sum_{k=1}^{\infty} \alpha_k = \infty$.

(C2) $\{\eta_k\} \subset (0,\infty)$ *with* $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$.

(C3) $\{\phi_k\} \subset [a,b] \subset (0,1)$.

(C4) $0 < a_j < \bar{a}_j < 1$, $0 < b_j < \bar{b}_j < 2$, $0 < c_j < \bar{c}_j < 1$ *for each* $j = 0, 1$.

(C5) $a_{j,k}, b_{j,k}, c_{j,k} \geq 0$ *with* $\lim_{k\to\infty} a_{j,k} = \lim_{k\to\infty} b_{j,k} = \lim_{k\to\infty} c_{j,k} = 0$ *for each* $j = 0, 1$.

(C6) $\{\rho_{j,k}\} \subset \mathbb{R}^+$ *and* $\sum_{k=1}^{\infty} \rho_{j,k} < +\infty$ *for each* $j = 0, 1$.

(C7) $0 < \underline{d}_j \leq \xi_j \leq \overline{d}_j < 1$, *for each* $j = 0, 1$ *and* $\xi_0 + \xi_1 = 1$.

For brevity, we set for each $j = 0, 1$

$$a_{j,k} + a_j = \widetilde{a}_{j,k},\ b_{j,k} + b_j = \widetilde{b}_{j,k},\ c_{j,k} + c_j = \widetilde{c}_{j,k}.$$

Algorithm 3.1. One-step inertial self-adaptive algorithm for SVIP

Initialization: Choose $x^0, x^1 \in H_0$ arbitrarily. Let $\lambda_{j,1} = \lambda_1 > 0$ for $j = 0, 1$, a bounded sequence $\{\tau_k\} \subset \mathbb{R}_+$, and parameter sequences satisfying Assumption 3.1.

for $k = 1, 2, \ldots$ **do**

Step 1 (One-step inertial extrapolation). Compute

$$w^k = (1 - \alpha_k)x^k + \theta_k(x^k - x^{k-1}),$$

where

$$\theta_k = \begin{cases} \min\left\{ \dfrac{\eta_k}{\|x^k - x^{k-1}\|},\ \tau_k \right\}, & \text{if } x^k \neq x^{k-1}, \\ \tau_k, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2 (Projection step). For each $j = 0, 1$, set

$$y_j^k = P_{C_j}\left(F_j w^k - \lambda_{j,k} B_j F_j w^k\right),$$

and update the self-adaptive step size

$$\lambda_{j,k+1} = \begin{cases} \min\left\{ \dfrac{\widetilde{a}_{j,k}\|F_j w^k - y_j^k\|}{\|B_j F_j w^k - B_j y_j^k\|},\ \lambda_{j,k} + \rho_{j,k} \right\}, \\ \qquad\qquad \text{if } \|B_j F_j w^k - B_j y_j^k\| \neq 0, \\ \lambda_{j,k} + \rho_{j,k}, \qquad\qquad \text{otherwise.} \end{cases} \tag{3.2}$$

Step 3 (Tseng-type correction). For each $j = 0, 1$, set

$$u_j^k = F_j w^k - \beta_{j,k}\, v_j^k,$$

where

$$v_j^k = F_j w^k - y_j^k - \lambda_{j,k}\big(B_j F_j w^k - B_j y_j^k\big), \tag{3.3}$$

and

$$\beta_{j,k} = \begin{cases} \dfrac{\widetilde{b}_{j,k}\langle F_j w^k - y_j^k,\ v_j^k\rangle}{\|v_j^k\|^2}, & \text{if } v_j^k \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 4 (Convex combination of adjoint corrections).

$$z^k = \sum_{j=0}^{1} \xi_j \left(w^k - \gamma_{j,k} F_j^*(F_j w^k - u_j^k)\right),$$

with

$$\gamma_{j,k} = \begin{cases} \dfrac{\widetilde{c}_{j,k}\|F_j w^k - u_j^k\|^2}{\|F_j^*(F_j w^k - u_j^k)\|^2}, & \text{if } \|F_j^*(F_j w^k - u_j^k)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Step 5 (Update). $x^{k+1} = (1 - \phi_k)\, w^k + \phi_k\, z^k$.

end for

Remark 3.1 (Intuition behind each step)**.**

(i) ***Step 1.*** *The Halpern factor* $(1-\alpha_k)$ *anchors the iterate toward the origin, selecting the minimum-norm solution; the inertial term* $\theta_k(x^k - x^{k-1})$ *extrapolates along recent displacement to accelerate convergence.*

(ii) ***Step 2.*** *The forward–projection pair implements a gradient-descent move onto* C_j*; the self-adaptive rule for* $\lambda_{j,k}$ *implicitly tracks the Lipschitz constant* L_j *without requiring it explicitly.*

(iii) ***Step 3.*** *The Tseng-type correction replaces the costly second projection of the extragradient method by a free forward step, chosen so that* u_j^k *lies no farther from* $F_j u$ *than* $F_j w^k$ *does (cf. inequality* (3.13)*).*

(iv) ***Step 4.*** *The adjoint term* $\gamma_{j,k}F_j^*(F_j w^k - u_j^k)$ *pulls the iterate toward feasibility of the split constraint; the self-adaptive* $\gamma_{j,k}$ *eliminates any dependence on* $\|F\|$*.*

Remark 3.2. *The following is one concrete instance of parameter sequences satisfying Assumption 3.1, serving as an illustration that such choices are readily constructible without any problem-dependent constants:*

$$\lambda_1 = 1, \quad a_j = 0.96, \quad b_j = 1.96, \quad c_j = 0.95, \quad \xi_j = \tfrac{1}{2},$$
$$\alpha_k = \tfrac{1}{k+1}, \quad a_{j,k} = \tfrac{1}{k+1}, \quad b_{j,k} = \tfrac{3}{(2k+1)^3}, \quad c_{j,k} = \tfrac{5}{(2k+1)^5},$$
$$\eta_k = \tfrac{1}{(5k+1)^{1.01}}, \quad \tau_k = \tfrac{k}{k+1}, \quad \phi_k = \tfrac{k+1}{2k+3}, \quad \rho_{j,k} = \tfrac{1}{(k+1)^2}.$$

We emphasize that the convergence theorem applies to any *sequences satisfying Assumption 3.1; the choices above are not privileged. A sigmoid-scheduled variant of* $\{\alpha_k\}$ *with improved early-stage convergence behaviour is introduced in Proposition 3.1 below.*

Proposition 3.1 (Two-Phase Viscosity Control Parameter)**.** *Let* $c > 0$*,* $0 < p_1 \le p_2 < 1$*,* $K_0 \in \mathbb{N}$*, and* $\sigma_0 > 0$*. Define*

$$\alpha_k = (k+c)^{-p(k)}, \qquad p(k) = p_1 + (p_2 - p_1)\,\sigma\left(\frac{k - K_0}{\sigma_0}\right), \tag{3.6}$$

where $\sigma(t) = \left(1 + e^{-t}\right)^{-1}$ *is the sigmoid function. Then* $\{\alpha_k\} \subset (0,1)$*,* $\lim_{k\to\infty} \alpha_k = 0$*, and* $\sum_{k=1}^{\infty} \alpha_k = \infty$*, i.e.,* $\{\alpha_k\}$ *satisfies* (C1) *of Assumption 3.1.*

Proof. Since $\sigma(t) \in (0,1)$ for all $t \in \mathbb{R}$ and $0 < p_1 \leq p_2 < 1$, we have $p_1 \leq p(k) < p_2 < 1$ for all $k \geq 1$. Moreover, since $k \geq 1$ and $c > 0$, the base $k + c > 1$. It follows that $\alpha_k = (k+c)^{-p(k)} \in (0,1)$.

(i) $\alpha_k \to 0$. Since $p(k) \geq p_1 > 0$, we have $0 < \alpha_k \leq (k+c)^{-p_1}$. As $k \to \infty$, $(k+c)^{-p_1} \to 0$, hence $\alpha_k \to 0$.

(ii) $\sum \alpha_k = \infty$. Since $p(k) < p_2$ for all $k \geq 1$, we have:

$$\alpha_k = (k+c)^{-p(k)} > (k+c)^{-p_2}, \quad \forall k \geq 1.$$

Since $p_2 < 1$, the series $\sum_{k=1}^{\infty}(k+c)^{-p_2}$ diverges by the p-series test. By the direct comparison test, $\sum_{k=1}^{\infty} \alpha_k = \infty$.

This construction provides a smooth transition between two polynomial decay regimes: larger step sizes in early iterations (governed by p_1) and smaller ones in later iterations (governed by p_2), which is beneficial for both progress and stability of the iterative process.

Remark 3.3. *Algorithm 3.1 is* free of the Lipschitz constants L_0, L_1 *and* free of the operator norm $\|F\|$. *These quantities are implicitly tracked through the self-adaptive rules* (3.2) *and* (3.5)*, thereby substantially reducing the per-iteration cost.*

Before turning to the convergence analysis, we record three technical facts that will be used repeatedly.

Remark 3.4. (i) *By the assumptions (C1), (C2) and* (3.1)*, we have* $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0$. *Consequently, there exists a constant* $M > 0$ *such that*

$$\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M, \ \forall\, k \geq 1. \tag{3.7}$$

(ii) *Since* B_j *is* L_j*-Lipschitz continuous for each* $j = 0, 1$

$$\|B_j F_j w^k - B_j y_j^k\| \leq L_j \|F_j w^k - y_j^k\|.$$

Hence, when $\|B_j F_j w^k - B_j y_j^k\| \neq 0$,

$$\frac{\widetilde{a}_{j,k}\|F_j w^k - y_j^k\|}{\|B_j F_j w^k - B_j y_j^k\|} \ \geq \ \frac{\widetilde{a}_{j,k}}{L_j} \ \geq \ \frac{a_j}{L_j},$$

which combined with the assumptions (C4)–(C6) and (3.2) *shows that* $\lambda_{j,k} \geq \min\{\frac{a_j}{L_j}, \lambda_1\}$ *and* $\lambda_{j,k} \leq \lambda_1 + s_j$ *with* $s_j := \sum_{k=1}^{\infty}\rho_{j,k}$. *From Lemma 2.2 the limit exists and for each* $j = 0, 1$

$$\lim_{k\to\infty} \lambda_{j,k} = \lambda_j^* \in \left[\min\left\{\frac{a_j}{L_j}, \lambda_1\right\}, \ \lambda_1 + s_j\right]. \tag{3.8}$$

(iii) *From the assumptions (C3), (C4),* (3.8) *and Lemma 2.4, there is an integer* $k_0 \in \mathbb{N}$ *such that for each* $j = 0, 1$ *and every* $k \geq k_0$,

$$\frac{\lambda_{j,k}\,\widetilde{a}_{j,k}}{\lambda_{j,k+1}} \in (0,1),\ \widetilde{b}_{j,k} \in (0,2),\ \widetilde{c}_{j,k} \in (0,1).$$

3.2. Convergence Analysis

The strong convergence of Algorithm 3.1 is proved through a sequence of four lemmas, culminating in Theorem 3.2.

Lemma 3.1. *Let* $\{w^k\}$ *and* $\{z^k\}$ *be generated by Algorithm 3.1 under Assumption 3.1. Then, for every* $u \in \Gamma$ *and every* $k \geq k_0$,

$$\|z^k - u\|^2 \leq \|w^k - u\|^2 - \sum_{j=0}^{1} \xi_j\, \gamma_{j,k}\, (1 - \widetilde{c}_{j,k})\, \|u_j^k - F_j w^k\|^2. \tag{3.9}$$

Proof. Fix $u \in \Gamma$ arbitrarily; then $F_j u \in \mathbf{S}(C_j, B_j) \subset C_j$ for each $j = 0, 1$. By Step 2 and (2.3), for each $j = 0, 1$

$$\langle y_j^k - F_j u,\ y_j^k - F_j w^k + \lambda_{j,k} B_j F_j w^k \rangle \leq 0. \tag{3.10}$$

Using the definition of v_j^k in (3.3), (3.10), and the monotonicity of B_j, we have, for each $j = 0, 1$,

$$\begin{aligned}
\langle y_j^k - F_j u,\ v_j^k \rangle &= \langle y_j^k - F_j u,\ F_j w^k - y_j^k - \lambda_{j,k} B_j F_j w^k \rangle + \lambda_{j,k} \langle y_j^k - F_j u,\ B_j y_j^k \rangle \\
&\geq \lambda_{j,k} \Big[\langle y_j^k - F_j u,\ B_j y_j^k - B_j F_j u \rangle + \langle y_j^k - F_j u,\ B_j F_j u \rangle \Big] \\
&\geq 0. \qquad (3.11)
\end{aligned}$$

From (3.11), we deduce

$$\langle F_j w^k - F_j u,\ v_j^k \rangle \ \geq\ \langle F_j w^k - y_j^k,\ v_j^k \rangle. \tag{3.12}$$

Step 3 gives $\|\beta_{j,k} v_j^k\|^2 = \|u_j^k - F_j w^k\|^2$. By Remark 3.4(iii), $0 < \widetilde{b}_{j,k} < 2$ for all $k \geq k_0$, and, when $v_j^k \neq 0$, (3.4) yields $\beta_{j,k} \|v_j^k\|^2 = \widetilde{b}_{j,k} \langle F_j w^k - y_j^k,\ v_j^k \rangle$. Combining these facts with (2.1) and (3.12), for each $j = 0, 1$ and $k \geq k_0$,

$$\begin{aligned}
\|u_j^k - F_j u\|^2 &= \|F_j w^k - F_j u\|^2 + \beta_{j,k}^2 \|v_j^k\|^2 - 2\beta_{j,k} \langle F_j w^k - F_j u,\ v_j^k \rangle \\
&\leq \|F_j w^k - F_j u\|^2 + \beta_{j,k}^2 \|v_j^k\|^2 - 2\beta_{j,k} \langle F_j w^k - y_j^k,\ v_j^k \rangle \\
&= \|F_j w^k - F_j u\|^2 + \Big(1 - \tfrac{2}{\widetilde{b}_{j,k}}\Big) \|u_j^k - F_j w^k\|^2
\end{aligned}$$

$$\leq \|F_j w^k - F_j u\|^2. \tag{3.13}$$

When $v_j^k = 0$, Step 3 gives $u_j^k = F_j w^k$ and (3.13) trivially holds.

Since $\|\cdot\|^2$ is convex and $\xi_0 + \xi_1 = 1$, Step 4 implies

$$\|z^k - u\|^2 \;\leq\; \sum_{j=0}^{1} \xi_j \left\|w^k - \gamma_{j,k} F_j^*(F_j w^k - u_j^k) - u\right\|^2. \tag{3.14}$$

Using (2.1) and the adjoint identity, together with (3.13),

$$\begin{aligned}
&\left\|w^k - \gamma_{j,k} F_j^*(F_j w^k - u_j^k) - u\right\|^2 \\
&\quad = \|w^k - u\|^2 + \gamma_{j,k}^2 \|F_j^*(F_j w^k - u_j^k)\|^2 \\
&\qquad - 2\gamma_{j,k}\langle F_j w^k - u_j^k,\, F_j(w^k - u)\rangle \\
&\quad = \|w^k - u\|^2 + \gamma_{j,k}^2 \|F_j^*(F_j w^k - u_j^k)\|^2 \\
&\qquad + \gamma_{j,k}\big(\|u_j^k - F_j u\|^2 - \|F_j w^k - F_j u\|^2 - \|u_j^k - F_j w^k\|^2\big) \\
&\quad \leq \|w^k - u\|^2 - \gamma_{j,k}\Big[\|u_j^k - F_j w^k\|^2 - \gamma_{j,k}\|F_j^*(F_j w^k - u_j^k)\|^2\Big]. \qquad (3.15)
\end{aligned}$$

By (3.5), whenever $\|F_j^*(F_j w^k - u_j^k)\| \neq 0$, for all $k \geq k_0$

$$\|u_j^k - F_j w^k\|^2 - \gamma_{j,k}\|F_j^*(F_j w^k - u_j^k)\|^2 = (1 - \widetilde{c}_{j,k})\,\|u_j^k - F_j w^k\|^2. \tag{3.16}$$

Substituting (3.15)–(3.16) into (3.14) yields (3.9). When $\|F_j^*(F_j w^k - u_j^k)\| = 0$, $u_j^k = F_j w^k$ and (3.9) still holds. ■

Lemma 3.2. *Under Assumption 3.1, the sequence $\{x^k\}$ generated by Algorithm 3.1 is bounded.*

Proof. Fix $u \in \Gamma$. From Step 1 and (3.7),

$$\begin{aligned}
\|w^k - u\| &= \big\|(1-\alpha_k)(x^k - u) - \alpha_k u + \theta_k(x^k - x^{k-1})\big\| \\
&\leq (1-\alpha_k)\|x^k - u\| + \alpha_k\|u\| + \theta_k\|x^k - x^{k-1}\| \\
&\leq (1-\alpha_k)\|x^k - u\| + \alpha_k(\|u\| + M). \qquad (3.17)
\end{aligned}$$

By Remark 3.4(iii), $(1 - \widetilde{c}_{j,k})\|u_j^k - F_j w^k\|^2 \geq 0,\ \forall k \geq k_0$, for each $j = 0, 1$ so (3.9) gives

$$\|z^k - u\| \leq \|w^k - u\|,\ \forall\, k \geq k_0. \tag{3.18}$$

Combining (3.17), (3.18) with Step 5 and the convexity of the norm, for every $k \geq k_0$,

$$\|x^{k+1} - u\| \leq (1 - \phi_k)\|w^k - u\| + \phi_k\|z^k - u\|$$

$$\begin{aligned} &\leq \|w^k - u\| \\ &\leq (1-\alpha_k)\|x^k - u\| + \alpha_k(\|u\| + M) \\ &\leq \max\{\|x^k - u\|,\, \|u\| + M\}. \end{aligned} \tag{3.19}$$

A straightforward induction then gives

$$\|x^{k+1} - u\| \leq \max\{\|x^{k_0} - u\|,\, \|u\| + M\},\ \forall k \geq k_0.$$

Hence, $\{x^k\}$ is bounded. The boundedness of $\{w^k\}$ follows from (3.17), and the boundedness of $\{y_j^k\}, \{u_j^k\}, \{v_j^k\}, \{z^k\}, \{F_j w^k\}$ is a consequence of the definitions of y_j^k, u_j^k, v_j^k in Algorithm 3.1, the Lipschitz continuity of B_j and the boundedness of F_j. ■

Lemma 3.3. *Let $\{w^k\}$ and $\{z^k\}$ be two sequences generated by Algorithm 3.1 under Assumption 3.1 with subsequences $\{w^{k_l}\}$ and $\{z^{k_l}\}$, respectively. If $w^{k_l} \rightharpoonup \widehat{u}$ in H_0 and $\lim_{l\to\infty} \|w^{k_l} - z^{k_l}\| = 0$, then $\widehat{u} \in \Gamma$.*

Proof. From (3.9) and (2.1), for $k = k_l \geq k_0$,

$$\begin{aligned} \sum_{j=0}^{1} \xi_j \gamma_{j,k_l}(1 - \widetilde{c}_{j,k_l})\|u_j^{k_l} - F_j w^{k_l}\|^2 &\leq \|w^{k_l} - u\|^2 - \|z^{k_l} - u\|^2 \\ &= \|w^{k_l} - z^{k_l} + z^{k_l} - u\|^2 - \|z^{k_l} - u\|^2 \\ &= \|w^{k_l} - z^{k_l}\|^2 + 2\langle w^{k_l} - z^{k_l},\, z^{k_l} - u\rangle \\ &\leq \|w^{k_l} - z^{k_l}\|^2 + 2\|w^{k_l} - z^{k_l}\|\,\|z^{k_l} - u\|. \end{aligned} \tag{3.20}$$

Since $\|w^{k_l} - z^{k_l}\| \to 0$ and $\{z^{k_l}\}$ is bounded (Lemma 3.2),

$$\sum_{j=0}^{1} \xi_j \gamma_{j,k_l}(1 - \widetilde{c}_{j,k_l})\|u_j^{k_l} - F_j w^{k_l}\|^2 \to 0 \text{ as } l \to \infty,$$

Since each term is non-negative (Remark 3.4(iii)), for each $j = 0, 1$

$$\xi_j \gamma_{j,k_l}(1 - \widetilde{c}_{j,k_l})\|u_j^{k_l} - F_j w^{k_l}\|^2 \to 0 \text{ as } l \to \infty. \tag{3.21}$$

From (3.5) and (3.21), we have, for each $j = 0, 1$,

$$\xi_j\, \widetilde{c}_{j,k_l}(1 - \widetilde{c}_{j,k_l})\, \frac{\|u_j^{k_l} - F_j w^{k_l}\|^4}{\|F_j^*(F_j w^{k_l} - u_j^{k_l})\|^2} \to 0 \text{ as } l \to \infty,$$

which implies

$$\frac{\|u_j^{k_l} - F_j w^{k_l}\|^4}{\|F_j^*(F_j w^{k_l} - u_j^{k_l})\|^2} \to 0 \text{ as } l \to \infty,\ j = 0, 1.$$

From the boundedness of $\{u_j^k\}$ and $\{F_j w^k\}$, the sequence $\{\|F_j^*(F_j w^k - u_j^k)\|^2\}$ is bounded. Hence

$$\|u_j^{k_l} - F_j w^{k_l}\| \to 0 \text{ as } l \to \infty,\ j = 0, 1. \tag{3.22}$$

From (3.2) and the definition (3.3) of v_j^k, when $v_j^{k_l} \neq 0$, for each $j = 0, 1$,

$$\begin{aligned}
\langle F_j w^{k_l} - y_j^{k_l},\, v_j^{k_l}\rangle &= \langle F_j w^{k_l} - y_j^{k_l},\, F_j w^{k_l} - y_j^{k_l} - \lambda_{j,k_l}\big(B_j F_j w^{k_l} - B_j y_j^{k_l}\big)\rangle \\
&= \|F_j w^{k_l} - y_j^{k_l}\|^2 - \lambda_{j,k_l}\langle F_j w^{k_l} - y_j^{k_l},\, B_j F_j w^{k_l} - B_j y_j^{k_l}\rangle \\
&\geq \|F_j w^{k_l} - y_j^{k_l}\|^2 - \lambda_{j,k_l}\|F_j w^{k_l} - y_j^{k_l}\|\,\|B_j F_j w^{k_l} - B_j y_j^{k_l}\| \\
&= \left(1 - \frac{\lambda_{j,k_l}}{\lambda_{j,k_l+1}}\widetilde{a}_{j,k_l}\right)\|F_j w^{k_l} - y_j^{k_l}\|^2. \qquad (3.23)
\end{aligned}$$

Denote $\mu_{j,k_l} := \frac{\lambda_{j,k_l}}{\lambda_{j,k_l+1}}\widetilde{a}_{j,k_l}$. Using Step 3, (3.2), (3.4), (3.23), and the definitions of $\{v_j^{k_l}\}$ and $\{u_j^{k_l}\}$, when $v_j^{k_l} \neq 0$, for each $j = 0, 1$,

$$\begin{aligned}
\|F_j w^{k_l} - y_j^{k_l}\|^2 &\leq \frac{1}{\widetilde{b}_{j,k_l}(1 - \mu_{j,k_l})}\,\beta_{j,k_l}\|v_j^{k_l}\|^2 \\
&\leq \frac{1}{\widetilde{b}_{j,k_l}(1 - \mu_{j,k_l})}\,\beta_{j,k_l}\|v_j^{k_l}\| \\
&\quad \times \Big(\|F_j w^{k_l} - y_j^{k_l}\| + \lambda_{j,k_l}\|B_j F_j w^{k_l} - B_j y_j^{k_l}\|\Big) \\
&\leq \frac{1}{\widetilde{b}_{j,k_l}(1 - \mu_{j,k_l})}\,\beta_{j,k_l}\|v_j^{k_l}\| \\
&\quad \times (1 + \mu_{j,k_l})\,\|F_j w^{k_l} - y_j^{k_l}\| \\
&\leq \frac{1 + \mu_{j,k_l}}{\widetilde{b}_{j,k_l}(1 - \mu_{j,k_l})}\|F_j w^{k_l} - y_j^{k_l}\|\,\|F_j w^{k_l} - u_j^{k_l}\|. \qquad (3.24)
\end{aligned}$$

Consequently, for each $j = 0, 1$,

$$\|F_j w^{k_l} - y_j^{k_l}\| \leq \frac{1 + \mu_{j,k_l}}{\widetilde{b}_{j,k_l}(1 - \mu_{j,k_l})}\|F_j w^{k_l} - u_j^{k_l}\|. \tag{3.25}$$

From Remark 3.4(ii)–(iii), $\lim\limits_{l\to\infty} \lambda_{j,k_l}/\lambda_{j,k_l+1} = 1$, and there exists $k_0 \in \mathbb{N}$ such that $1 - \mu_{j,k_l} > 0,\ \forall k_l \geq k_0$, for each $j = 0, 1$. Therefore, combining (3.22) and (3.25) we obtain

$$\|F_j w^{k_l} - y_j^{k_l}\| \to 0 \text{ as } l \to \infty,\ j = 0, 1. \tag{3.26}$$

If $v_j^{k_l} = 0$, then Step 3 yields $y_j^{k_l} = F_j w^{k_l}$ and (3.26) is immediate.

Now, by the monotonicity of B_j, the projection characterization (2.3), and Step 2, we obtain, for each $j = 0, 1$ and every $u_j \in C_j$,

$$\begin{aligned}
0 &\le \langle y_j^{k_l} - F_j w^{k_l} + \lambda_{j,k_l} B_j F_j w^{k_l},\ u_j - y_j^{k_l} \rangle \\
&= \langle y_j^{k_l} - F_j w^{k_l},\ u_j - y_j^{k_l} \rangle \\
&\quad + \lambda_{j,k_l} \langle B_j F_j w^{k_l},\ F_j w^{k_l} - y_j^{k_l} \rangle + \lambda_{j,k_l} \langle B_j F_j w^{k_l},\ u_j - F_j w^{k_l} \rangle \\
&\le \|y_j^{k_l} - F_j w^{k_l}\|\, \|u_j - y_j^{k_l}\| + \lambda_{j,k_l} \|B_j F_j w^{k_l}\|\, \|F_j w^{k_l} - y_j^{k_l}\| \\
&\quad + \lambda_{j,k_l} \Big(\langle B_j F_j w^{k_l} - B_j u_j,\ u_j - F_j w^{k_l} \rangle + \langle B_j u_j,\ u_j - F_j w^{k_l} \rangle \Big) \\
&\le \|y_j^{k_l} - F_j w^{k_l}\|\, \|u_j - y_j^{k_l}\| + \lambda_{j,k_l} \|B_j F_j w^{k_l}\|\, \|F_j w^{k_l} - y_j^{k_l}\| \\
&\quad + \lambda_{j,k_l} \langle B_j u_j,\ u_j - F_j w^{k_l} \rangle. \qquad (3.27)
\end{aligned}$$

By the L_j-Lipschitz continuity of B_j and (3.26),

$$\|B_j F_j w^{k_l} - B_j y_j^{k_l}\| \to 0 \text{ as } l \to \infty,\ j = 0, 1. \qquad (3.28)$$

Since $w^{k_l} \rightharpoonup \widehat{u}$ and F_j is a bounded linear operator for each $j = 0, 1$, we have $F_j w^{k_l} \rightharpoonup F_j \widehat{u}$, $j = 0, 1$. Hence, by (3.26), $y_j^{k_l} \rightharpoonup F_j \widehat{u}$, $j = 0, 1$. Letting $l \to \infty$ in (3.27) and using (3.26) and (3.28), we obtain, for each $j = 0, 1$,

$$\langle B_j u_j,\ u_j - F_j \widehat{u} \rangle \ge 0,\ \forall\, u_j \in C_j.$$

By Lemma 2.1, $F_j \widehat{u} \in \mathbf{S}(C_j, B_j)$ for each $j = 0, 1$. Therefore, $\widehat{u} \in \Gamma$. ■

Lemma 3.4. *Let $\{x^k\}$ be generated by Algorithm 3.1 under Assumption 3.1. Then, for every $u \in \Gamma$, the following inequality holds:*

$$\|x^{k+1} - u\|^2 \le (1 - \alpha_k)\|x^k - u\|^2 + \alpha_k d_k,\ k \ge k_0, \qquad (3.29)$$

where $\{d_k\}$ is a real sequence.

Proof. Let $u \in \Gamma$. From Step 1, we have

$$\begin{aligned}
\|w^k - u\|^2 &= \big\|(1 - \alpha_k)x^k - (1 - \alpha_k)u - \alpha_k u + \theta_k (x^k - x^{k-1})\big\|^2 \\
&= \|(1 - \alpha_k)(x^k - u) - \alpha_k u\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 \\
&\quad + 2\langle (1 - \alpha_k)(x^k - u) - \alpha_k u,\ \theta_k (x^k - x^{k-1}) \rangle \\
&\le \|(1 - \alpha_k)(x^k - u) - \alpha_k u\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 \\
&\quad + 2\theta_k \big[(1 - \alpha_k)\|x^k - u\| + \alpha_k \|u\|\big]\, \|x^k - x^{k-1}\| \\
&\le \|(1 - \alpha_k)(x^k - u) - \alpha_k u\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 \\
&\quad + 2\theta_k \big[\|x^k - u\| + \alpha_k \|u\|\big]\, \|x^k - x^{k-1}\|,\ k \ge k_0. \qquad (3.30)
\end{aligned}$$

Using (2.1), we observe that

$$\begin{aligned}
&\|(1-\alpha_k)(x^k-u)-\alpha_k u\|^2\\
&\quad=\|(1-\alpha_k)(x^k-u)\|^2+\alpha_k^2\|u\|^2-2\alpha_k(1-\alpha_k)\langle x^k-u,\,u\rangle\\
&\quad=(1-\alpha_k)^2\|x^k-u\|^2+\alpha_k^2\|u\|^2+2\alpha_k^2\langle x^k-u,\,u\rangle-2\alpha_k\langle x^k-u,\,u\rangle\\
&\quad=(1-\alpha_k)^2\|x^k-u\|^2-\alpha_k^2\|u\|^2+2\alpha_k^2\langle x^k,\,u\rangle-2\alpha_k\langle x^k-u,\,u\rangle\\
&\quad\le(1-\alpha_k)\|x^k-u\|^2+2\alpha_k^2\|x^k\|\,\|u\|-2\alpha_k\langle x^k-u,\,u\rangle,\ k\ge k_0. \qquad (3.31)
\end{aligned}$$

From (3.30) and (3.31), we obtain

$$\|w^k-u\|^2\le(1-\alpha_k)\|x^k-u\|^2+\alpha_k d_k,\ k\ge k_0, \qquad (3.32)$$

where

$$\begin{aligned}
d_k&:=2\langle u-x^k,\,u\rangle+2\alpha_k\|x^k\|\,\|u\|+\frac{\theta_k^2}{\alpha_k}\|x^k-x^{k-1}\|^2\\
&\quad+2\,\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|\big[\|x^k-u\|+\alpha_k\|u\|\big],\ k\ge k_0. \qquad (3.33)
\end{aligned}$$

From Step 5, (2.2), Lemma 3.1 and (3.32), we obtain

$$\begin{aligned}
\|x^{k+1}-u\|^2&=\|(1-\phi_k)w^k+\phi_k z^k-(1-\phi_k)u-\phi_k u\|^2\\
&=\|(1-\phi_k)(w^k-u)+\phi_k(z^k-u)\|^2\\
&=(1-\phi_k)\|w^k-u\|^2+\phi_k\|z^k-u\|^2\\
&\quad-\phi_k(1-\phi_k)\|w^k-z^k\|^2\\
&\le(1-\phi_k)\|w^k-u\|^2-\phi_k(1-\phi_k)\|w^k-z^k\|^2\\
&\quad+\phi_k\Big[\|w^k-u\|^2-\sum_{j=0}^{1}\xi_j\gamma_{j,k}(1-\widetilde{c}_{j,k})\|u_j^k-F_jw^k\|^2\Big]\\
&=\|w^k-u\|^2-\phi_k(1-\phi_k)\|w^k-z^k\|^2\\
&\quad-\phi_k\sum_{j=0}^{1}\xi_j\gamma_{j,k}(1-\widetilde{c}_{j,k})\|u_j^k-F_jw^k\|^2\\
&\le(1-\alpha_k)\|x^k-u\|^2+\alpha_k d_k-\phi_k(1-\phi_k)\|w^k-z^k\|^2\\
&\quad-\phi_k\sum_{j=0}^{1}\xi_j\gamma_{j,k}(1-\widetilde{c}_{j,k})\|u_j^k-F_jw^k\|^2,\ k\ge k_0. \qquad (3.34)
\end{aligned}$$

From (3.34), the definitions of $\{\xi_j\}$, $\{\gamma_{j,k}\}$, the inclusion $\{\phi_k\}\subset(0,1)$, and the fact that $(1-\widetilde{c}_{j,k})>0$ for all $k\ge k_0$, we obtain (3.29). ■

Theorem 3.2. *Let $\{x^k\}$ be generated by Algorithm 3.1 under Assumption 3.1. Then $\{x^k\}$ converges strongly to $u^* \in \Gamma$, where $\|u^*\| = \min\{\|u\| \mid u \in \Gamma\}$.*

Proof. Assume $u^* \in \Gamma$. Then $F_j u^* \in \mathbf{S}(C_j, B_j)$ for each $j = 0, 1$. From Lemma 3.4, we obtain

$$\|x^{k+1} - u^*\|^2 \leq (1 - \alpha_k)\|x^k - u^*\|^2 + \alpha_k d_k^*, \tag{3.35}$$

where

$$\begin{aligned} d_k^* &:= 2\langle u^* - x^k,\, u^*\rangle + 2\alpha_k \|x^k\|\,\|u^*\| + \frac{\theta_k^2}{\alpha_k}\|x^k - x^{k-1}\|^2 \\ &\quad + 2\,\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\|\big[\|x^k - u^*\| + \alpha_k\|u^*\|\big],\ k \geq k_0. \end{aligned} \tag{3.36}$$

We now show that the sequence $\{\|x^k - u^*\|\}$ converges to zero. By Lemma 2.3, to complete the proof, it suffices to show that $\limsup_{l\to\infty} d_{k_l}^* \leq 0$ for every subsequence $\{\|x^{k_l} - u^*\|\}$ of $\{\|x^k - u^*\|\}$ satisfying

$$\liminf_{l\to\infty}\big(\|x^{k_l+1} - u^*\| - \|x^{k_l} - u^*\|\big) \geq 0. \tag{3.37}$$

Suppose $\{\|x^{k_l} - u^*\|\}$ is a subsequence of $\{\|x^k - u^*\|\}$ satisfying (3.37). From (3.34), we obtain

$$\begin{aligned} 0 &\leq \phi_{k_l}(1 - \phi_{k_l})\|w^{k_l} - z^{k_l}\|^2 + \phi_{k_l}\sum_{j=0}^{1}\xi_j\gamma_{j,k_l}(1 - \widetilde{c}_{j,k_l})\|u_j^{k_l} - F_j w^{k_l}\|^2 \\ &\leq (1 - \alpha_{k_l})\|x^{k_l} - u^*\|^2 - \|x^{k_l+1} - u^*\|^2 + \alpha_{k_l} d_{k_l}^* \\ &= \big(\|x^{k_l} - u^*\|^2 - \|x^{k_l+1} - u^*\|^2\big) - \alpha_{k_l}\|x^{k_l} - u^*\|^2 + \alpha_{k_l} d_{k_l}^*,\ k_l \geq k_0. \end{aligned}$$

From (3.37) and $\lim_{l\to\infty}\alpha_{k_l} = 0$, letting $l \to \infty$, we obtain

$$\begin{aligned} &\phi_{k_l}(1 - \phi_{k_l})\|w^{k_l} - z^{k_l}\|^2 \\ &\quad + \phi_{k_l}\sum_{j=0}^{1}\xi_j\gamma_{j,k_l}(1 - \widetilde{c}_{j,k_l})\|u_j^{k_l} - F_j w^{k_l}\|^2 \to 0. \end{aligned} \tag{3.38}$$

Therefore, $\|w^{k_l} - z^{k_l}\|^2 \to 0$ and $\|u_j^{k_l} - F_j w^{k_l}\|^2 \to 0$ for each $j = 0, 1$.

From Step 1, we have

$$\begin{aligned} \|w^{k_l} - x^{k_l}\| &= \big\|-\alpha_{k_l}x^{k_l} + \theta_{k_l}(x^{k_l} - x^{k_l-1})\big\| \\ &\leq \alpha_{k_l}\|x^{k_l}\| + \theta_{k_l}\|x^{k_l} - x^{k_l-1}\|, \end{aligned}$$

which, combined with Remark 3.4(i), yields

$$\|w^{k_l} - x^{k_l}\| \to 0 \text{ as } l \to \infty. \tag{3.39}$$

We consider

$$0 \leq \|z^{k_l} - x^{k_l}\| \leq \|z^{k_l} - w^{k_l}\| + \|w^{k_l} - x^{k_l}\|,$$

from which (3.38) and (3.39) imply

$$\|z^{k_l} - x^{k_l}\| \to 0 \text{ as } l \to \infty. \tag{3.40}$$

From Step 5, (3.39), and (3.40), we obtain

$$\|x^{k_l+1} - x^{k_l}\| \leq (1 - \phi_{k_l})\|w^{k_l} - x^{k_l}\| + \phi_{k_l}\|z^{k_l} - x^{k_l}\| \to 0 \text{ as } l \to \infty. \tag{3.41}$$

From (3.39) and (3.41), we deduce

$$\|x^{k_l+1} - w^{k_l}\| \leq \|x^{k_l+1} - x^{k_l}\| + \|x^{k_l} - w^{k_l}\| \to 0 \text{ as } l \to \infty. \tag{3.42}$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ of $\{x^k\}$ such that $x^{k_l} \rightharpoonup \widehat{u}$. From (3.39), $w^{k_l} \rightharpoonup \widehat{u}$. Therefore, by (3.38) and Lemma 3.3, we obtain $\widehat{u} \in \Gamma$.

By the boundedness of $\{x^{k_l}\}$, there exists a subsequence $\{x^{k_{l_m}}\}$ of $\{x^{k_l}\}$ converging weakly to $\widetilde{u} \in H_0$ such that

$$\limsup_{l\to\infty} \langle u^*, u^* - x^{k_l}\rangle = \lim_{m\to\infty} \langle u^*, u^* - x^{k_{l_m}}\rangle = \langle u^*, u^* - \widetilde{u}\rangle. \tag{3.43}$$

Moreover, from (3.38) and Lemma 3.3, we obtain $\widetilde{u} \in \Gamma$. Since, by definition, u^* is the element of minimum norm, Remark 2.1 yields $u^* = P_\Gamma(0)$, and (2.3) gives $\langle u^*, u^* - \widetilde{u}\rangle \leq 0$. Combined with (3.43), this leads to

$$\limsup_{l\to\infty} \langle u^*, u^* - x^{k_l}\rangle \leq 0. \tag{3.44}$$

Now, by Remark 3.4(i), the boundedness of $\{x^{k_l}\}$, and (3.44), it follows from (3.36) that $\limsup_{l\to\infty} d^*_{k_l} \leq 0$. Finally, applying Lemma 2.3 to (3.35), we conclude that the sequence $\{\|x^k - u^*\|\}$ converges to zero, that is, $\{x^k\}$ converges strongly to $u^* \in \Gamma$. ■

4. Application

In this section, we reformulate the equilibrium-flow problem of a time-dependent two-city traffic network as a time-dependent split variational inequality problem (TSVIP) and show how Algorithm 3.1 applies after discretization.

4.1. Traffic Network Model for Two Cities

Let X_0 and X_1 denote two cities. For each $j = 0, 1$, the traffic network of city X_j is described by the tuple

$$G_j = \big(M_j,\ L_j,\ W_j,\ V_j\big),$$

where M_j is the set of nodes, L_j is the set of directed edges, W_j is the set of origin–destination (O–D) pairs, and V_j is the set of routes (paths). Each $w_j = (a_j, b_j) \in W_j$ represents a source node a_j and a destination node b_j, and we denote by $V_j(w_j) \subseteq V_j$ the set of paths from a_j to b_j. The full route set is $V_j = \bigcup_{w_j \in W_j} V_j(w_j)$. Each path carries a flow $x_{r_j} \geq 0$, and each O–D pair w_j is associated with a positive demand $d_{w_j} > 0$.

Feasible sets. For each O–D pair $w_j \in W_j$, the demand-balance constraint

$$\sum_{r_j \in V_j(w_j)} x_{r_j} \ = \ d_{w_j},\ x_{r_j} \geq 0 \ \ \forall\, r_j \in V_j(w_j),$$

defines the feasible set of city X_j

$$C_j \ := \ \Big\{ x_j \in \mathbb{R}^{|V_j|} \ \Big|\ \sum_{r_j \in V_j(w_j)} x_{r_j} = d_{w_j} \ \forall\, w_j \in W_j;\ x_{r_j} \geq 0 \ \forall\, r_j \in V_j \Big\}.$$

Clearly, C_j is a nonempty, convex, closed, and bounded subset of $\mathbb{R}^{|V_j|}$.

Path–edge incidence and edge flow. For each path $r_j \in V_j$ and each edge $e_j \in L_j$, define

$$\phi_{r_j, e_j} = \begin{cases} 1, & \text{if edge } e_j \text{ lies on path } r_j, \\ 0, & \text{otherwise.} \end{cases}$$

Given a path-flow vector $x_j = (x_{r_j})_{r_j \in V_j}$, the induced flow on an edge $e_j \in L_j$ is

$$A_{e_j}(x_j) \ = \ \sum_{r_j \in V_j} \phi_{r_j, e_j}\, x_{r_j} \ = \ \sum_{w_j \in W_j} \sum_{r_j \in V_j(w_j)} \phi_{r_j, e_j}\, x_{r_j}.$$

Edge and path costs. The cost on each edge is a (possibly nonlinear) function of the edge flow: for every edge $e_j \in L_j$,

$$t_{e_j}(x_j) \ = \ C_{e_j}\big(A_{e_j}(x_j)\big),$$

where $C_{e_j} : \mathbb{R}_+ \to \mathbb{R}_+$ is the edge-cost function. The cost of a path r_j is the sum of the edge costs along it:

$$B_{r_j}(x_j) \ = \ \sum_{e_j \in L_j} \phi_{r_j, e_j}\, t_{e_j}(x_j). \tag{4.1}$$

Static Wardrop equilibrium as an SVIP. A feasible flow vector $x_0^* \in C_0$ is said to be in (static) Wardrop equilibrium if, for every O–D pair $w_0 \in W_0$ and every path $r_0 \in V_0(w_0)$,

$$x_{r_0}^* > 0 \implies B_{r_0}(x_0^*) = \min_{p_0 \in V_0(w_0)} B_{p_0}(x_0^*). \tag{4.2}$$

That is, among paths serving the same O–D pair, only those of minimum cost carry positive flow. If, in addition, the two cities are coupled by a bounded linear transfer operator $F : \mathbb{R}^{|V_0|} \to \mathbb{R}^{|V_1|}$, the image $y_0^* := F x_0^* \in C_1$ must satisfy the analogous Wardrop condition in X_1, $\forall r_1 \in V_1(w_1),\ \forall w_1 \in W_1$

$$y_{r_1}^* > 0 \implies B_{r_1}(y_0^*) = \min_{p_1 \in V_1(w_1)} B_{p_1}(y_0^*). \tag{4.3}$$

It was shown in [13] that $x_0^* \in C_0$ satisfies (4.2)–(4.3) if and only if it solves the static SVIP (1.1)–(1.2) with the operators B_0, B_1, F above.

4.2. Time-Dependent Formulation

To model congestion and demand varying over a planning horizon $[0, T]$, all quantities are promoted to time-dependent trajectories. The path-flow vector becomes

$$x_j(t) = \big(x_{r_j}(t)\big)_{r_j \in V_j} \in \mathbb{R}^{|V_j|},$$

with $x_{r_j}(t)$ the flow on r_j at time t. A flow is feasible if, for $j = 0, 1$,

$$\lambda_j(t) \le x_j(t) \le \mu_j(t),\ \Phi_j\, x_j(t) = \rho_j(t),\ \text{a.e. on } [0, T], \tag{4.4}$$

where $\lambda_j, \mu_j \in L^2([0,T]; \mathbb{R}^{|V_j|})$ are capacity bounds, $\rho_j \in L^2([0,T]; \mathbb{R}^{|W_j|})$ is the nonnegative time-varying demand, and $\Phi_j = [\phi_{r_j, w_j}]$ is the pair–route incidence matrix defined by $\phi_{r_j, w_j} = 1$ if r_j connects the pair w_j and 0 otherwise. Under the compatibility condition $\Phi_j \lambda_j(t) \le \rho_j(t) \le \Phi_j \mu_j(t)$, the time-dependent feasible set

$$C_j^{(t)} := \big\{\, x_j(t) \in L^2([0,T]; \mathbb{R}^{|V_j|}) \;\big|\; (4.4) \text{ holds} \,\big\} \tag{4.5}$$

is nonempty, convex, closed, and bounded.

The flow trajectories live in the Hilbert space $L^2([0,T]; \mathbb{R}^{|V_j|})$ equipped with the canonical bilinear form

$$\langle\!\langle x_j,\, y_j \rangle\!\rangle_{X_j} := \int_0^T \langle x_j(t),\, y_j(t) \rangle\, dt, \tag{4.6}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. The cost operator $B_j^{(t)} : C_j^{(t)} \to L^2([0,T]; \mathbb{R}^{|V_j|})$ is the pointwise extension of (4.1), $(B_j^{(t)} x_j)(t) = B_j(x_j(t))$,

and the transfer operator $F^{(t)} : L^2([0,T];\mathbb{R}^{|V_0|}) \to L^2([0,T];\mathbb{R}^{|V_1|})$ is bounded linear. In keeping with Section 3, we set $F_0^{(t)} = I$ and $F_1^{(t)} = F^{(t)}$.

Following [9,13], the TSVIP for two cities reads: find $x^*(t) \in C_0^{(t)}$ such that

$$\big\langle\!\big\langle B_0^{(t)} x^*(t),\ x(t) - x^*(t) \big\rangle\!\big\rangle_{X_0} \geq 0, \quad \forall\, x(t) \in C_0^{(t)}, \tag{4.7}$$

and $y^*(t) := F^{(t)} x^*(t) \in C_1^{(t)}$ solves

$$\big\langle\!\big\langle B_1^{(t)} y^*(t),\ y(t) - y^*(t) \big\rangle\!\big\rangle_{X_1} \geq 0, \quad \forall\, y(t) \in C_1^{(t)}. \tag{4.8}$$

Its solution set is

$$\Gamma^{(t)} := \big\{\, x^*(t) \in \mathbf{S}(C_0^{(t)}, B_0^{(t)}) \,\big|\, F^{(t)} x^*(t) \in \mathbf{S}(C_1^{(t)}, B_1^{(t)}) \,\big\}.$$

Definition 4.1 (see [9,23])**.** *$x^*(t) \in C_0^{(t)}$ is a* time-dependent equilibrium flow *if and only if $x^*(t) \in \Gamma^{(t)}$.*

4.3. Discretization and Algorithmic Scheme

We discretize the horizon as

$$0 = t_0 < t_1 < \cdots < t_n = T. \tag{4.9}$$

At each t_i, problem (4.7)–(4.8) reduces to the static SVIP: find $x^*(t_i) \in C_0^{(t_i)}$ such that $x^*(t_i)$ solves $\mathrm{VIP}(C_0^{(t_i)}, B_0^{(t_i)})$ and $y^*(t_i) := F^{(t_i)} x^*(t_i) \in C_1^{(t_i)}$ solves $\mathrm{VIP}(C_1^{(t_i)}, B_1^{(t_i)})$.

This is exactly the SVIP (1.1)–(1.2) of Section 3 under the identifications $C_j \leftrightarrow C_j^{(t_i)}$, $B_j \leftrightarrow B_j^{(t_i)}$, $F \leftrightarrow F^{(t_i)}$. Hence Algorithm 3.1 can be applied at each t_i, producing the discrete equilibrium curve $\{(t_i, x^*(t_i), y^*(t_i))\}_{i=0}^n$.

Remark 4.1. *If $B_j^{(t_i)}$ is monotone and Lipschitz continuous and $F^{(t_i)}$ is bounded linear, Theorem 3.2 guarantees strong convergence of the inner iterates of Algorithm 3.1 to the minimum-norm equilibrium flow at each t_i.*

5. Numerical illustrations

Here we carry out numerical experiments to demonstrate the applicability of our proposed method. All numerical computations were carried out using Python 3.13.1 on an Apple M1 with an 8-core CPU @ 3.20 GHz and 8 GB unified memory.

Example 5.1. *Let $H_0 = H_1 = L^2([0,1])$ be endowed with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)\, y(t)\, \mathrm{d}t,\ x, y \in L^2([0,1]),$$

and the induced norm $\|x\| = \left(\int_0^1 |x(t)|^2\, \mathrm{d}t\right)^{1/2}$.
Constraint sets. *Let*

$$C_0 = \{x \in L^2([0,1]) : \langle y, x\rangle \le 2\},\ y(t) = t + e^{3t},$$

with metric projection

$$P_{C_0}(x) = \begin{cases} x, & \text{if } \langle y, x\rangle \le 2, \\ x + \dfrac{2 - \langle y, x\rangle}{\|y\|^2}\, y, & \text{otherwise;} \end{cases}$$

and

$$C_1 = \{x \in L^2([0,1]) : \|x - e\| \le 1\},\ e(t) = t,$$

with metric projection

$$P_{C_1}(x) = \begin{cases} x, & \text{if } x \in C_1, \\ e + \dfrac{x - e}{\|x - e\|}, & \text{otherwise.} \end{cases}$$

Operators. *Define $B_0 : L^2([0,1]) \to L^2([0,1])$ by*

$$(B_0 x)(t) = x(t) - c\, t e^t \int_0^1 \tau e^\tau \cos x(\tau)\, \mathrm{d}\tau + c\, t e^t,$$

where $c = \frac{2}{e\sqrt{e^2-1}}$. Then B_0 is monotone and 2-Lipschitz continuous on $L^2([0,1])$ (see [10]). Define $B_1 : L^2([0,1]) \to L^2([0,1])$ by

$$(B_1 x)(t) = \int_0^t x(s)\, \mathrm{d}s,\ x \in L^2([0,1]).$$

Then B_1 is monotone and $\frac{2}{\pi}$-Lipschitz continuous. Define $F : L^2([0,1]) \to L^2([0,1])$ (Fredholm operator) by

$$(Fx)(s) = \int_0^1 e^{-st} x(t)\, \mathrm{d}t,\ x \in L^2([0,1]).$$

Then F is a bounded, self-adjoint linear operator, $F^* x(s) = \int_0^1 e^{-ts} x(t)\, \mathrm{d}t$.

Stopping criterion. We iterate until

$$\begin{aligned}\mathrm{TOL}(k) =& \tfrac{1}{2}\Big[\big\|x^k - P_{C_0}\big((I-\tfrac{1}{2}B_0)\,x^k\big)\big\|^2 \\ &+ \big\|Fx^k - P_{C_1}\big((I-\tfrac{1}{2}B_1)\,Fx^k\big)\big\|^2\Big] < \varepsilon = 10^{-7},\end{aligned} \tag{5.1}$$

or the iteration count reaches a cap of 500. All operators are discretised on a uniform grid of $N = 500$ points using the trapezoidal rule.

Default parameters of the proposed algorithm. Unless otherwise stated we use

$$\lambda_1 = 1.0,\ a_j = 0.96,\ b_j = 1.96,\ c_j = 0.95,\ \xi_j = \frac{1}{2},$$

$$\alpha_k = \frac{1}{k+1},\ a_{j,k} = \frac{1}{k+1},\ b_{j,k} = \frac{3}{(2k+1)^3},\ c_{j,k} = \frac{5}{(2k+1)^5},$$

$$\eta_k = \frac{1}{(5k+1)^{1.01}},\ \tau_k = \frac{k}{k+1},\ \phi_k = \frac{k+1}{2k+3},\ \rho_{j,k} = \frac{1}{(k+1)^2}.$$

We fix $x^1(t) = x^0(t) = 2e^t + t^2$ and vary one parameter sequence at a time, keeping all other sequences at their default values. For every admissible choice we record the number of iterations and the CPU time (in seconds) needed to reach $\mathrm{TOL}(k) < 10^{-7}$.

The sensitivity plots are shown in Figures 1–4. Each figure presents two panels: (left) $\mathrm{TOL}(k)$ vs. iteration count; (right) $\mathrm{TOL}(k)$ vs. cumulative CPU time.

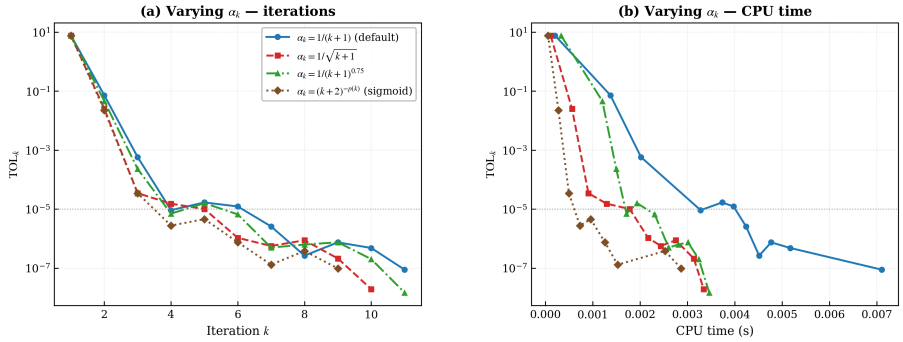


Figure 1: Performance of the algorithm with different choices of α_k.

where $\alpha_k^{\mathrm{sig}} = (k+2)^{-p(k)}$ and $\eta_k^{\mathrm{sig}} = (5k+1)^{-1.01}$, with $p(k) = p_1 + (p_2 - p_1)\,\sigma\Big(\frac{k-K_0}{\sigma_0}\Big)$ and $\sigma(t) = \frac{1}{1+e^{-t}}$, where $p_1 = 0.25$, $p_2 = 0.75 < 1$, $K_0 = 20$, $\sigma_0 = 8$.

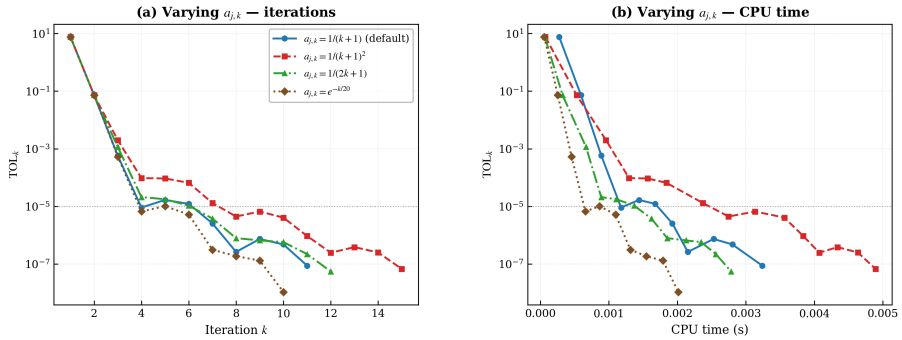


Figure 2: Performance of the algorithm with different choices of $a_{j,k}$.

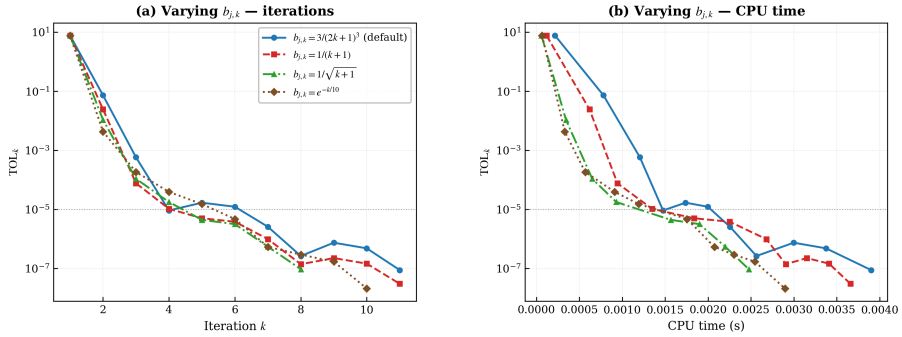


Figure 3: Performance of the algorithm with different choices of $b_{j,k}$.

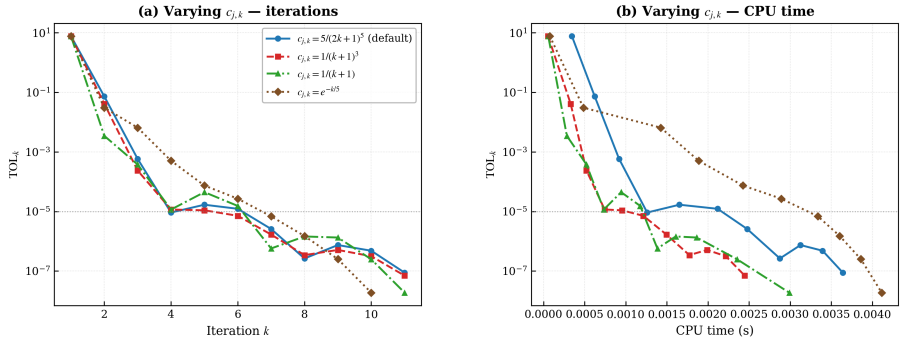


Figure 4: Performance of the algorithm with different choices of $c_{j,k}$.

We compare our algorithm (Our Alg.) with the following two methods:

- **Izuchukwu Alg.** (Algorithm 3.2 in [11]): $\alpha_k = \frac{1}{5k+2}$, $\lambda = 0.4$, $\mu = 1.4$, $\tau = 0.05$, $\theta_k = \frac{1}{2} - \alpha_k$.

- **Pham Alg.** (Algorithm 1 in [19]): $\alpha_k = \frac{1}{5k+2}$, $\lambda_0 = 0.4$, $\mu_0 = 1.4$, $\lambda = \mu = 0.8$, $\delta_k = \frac{k+1}{2k+8}$.

- **Our Alg.** (Algorithm 3.1): $\alpha_k = \frac{1}{5k+2}$, $\lambda_1 = 0.4$, $\eta_k = \frac{1}{(5k+1)^{1.01}}$, $\tau_k = \frac{k}{k+1}$, $a_j = 0.96$, $b_j = 1.96$, $c_j = 0.95$, $a_{j,k} = \frac{1}{k+1}$, $b_{j,k} = \frac{3}{(2k+1)^3}$, $c_{j,k} = \frac{5}{(2k+1)^5}$, $\rho_{j,k} = \frac{1}{(k+1)^2}$, $\phi_k = \frac{k+1}{2k+3}$, $\xi_j = \frac{1}{2}$.

Initial points. We test three choices of initial iterate $x^0 = x^1$:

- Case I: $x^1(t) = \sin(2t)$,

- Case II: $x^1(t) = 2e^t + t^2$,

- Case III: $x^1(t) = t + t^3$.

Stopping criterion. Stop if (5.1) is satisfied or if the iteration count reaches 500.

The numerical results are reported in Figure 5. It can be observed that the proposed algorithm outperforms the methods of Izuchukwu and Pham in terms of both iteration count and CPU time for all tested initial points. Furthermore, when equipped with the adaptive step-size $\{\alpha_k^{\text{sig}}\}$, the proposed algorithm shows a clear improvement over its variant with the conventional choice of $\{\alpha_k\}$.

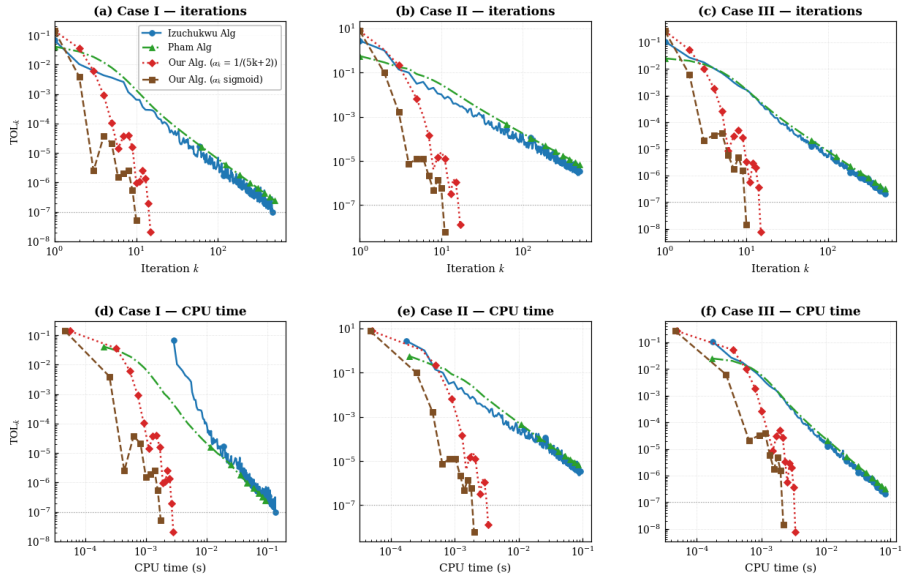


Figure 5: Performances of the three algorithms.

Example 5.2 (see [23]). *The traffic network of city X_0 comprises six nodes and eight links, while city X_1 comprises five nodes and seven links, with a planning horizon $T = 6$. In city X_0, the food company has branches at nodes P_0^1, P_0^2 (sources) that must deliver to nodes P_0^3, P_0^5 (destinations). In city X_1, branches at P_1^1, P_1^4 deliver to P_1^2, P_1^3.*

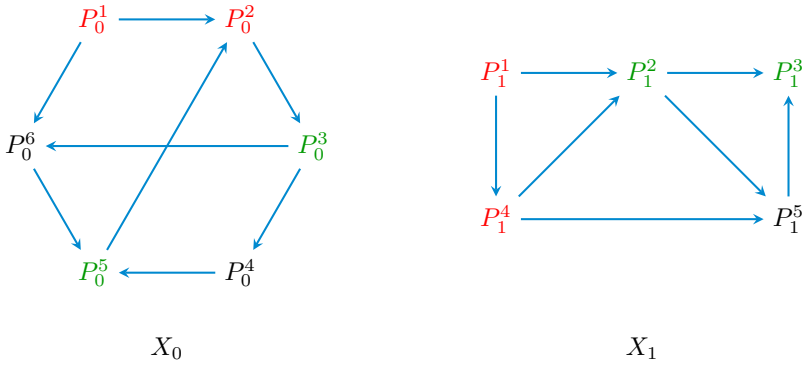


Figure 6: Traffic network of two cities X_0 (6 nodes, 8 links) and X_1 (5 nodes, 7 links). Red nodes: source (supply); green nodes: destination (demand).

Route sets. With $|V_0| = 4$ routes in X_0 and $|V_1| = 5$ routes in X_1, the origin–destination pairs are $w_0^1 = (P_0^1, P_0^3)$, $w_0^2 = (P_0^2, P_0^5)$ and $w_1^1 = (P_1^1, P_1^2)$, $w_1^2 = (P_1^4, P_1^3)$, with the following connecting paths:

$$w_0^1 : \begin{cases} r_0^1 = P_0^1 \to P_0^2 \to P_0^3, \\ r_0^2 = P_0^1 \to P_0^6 \to P_0^5 \to P_0^2 \to P_0^3, \end{cases}$$

$$w_0^2 : \begin{cases} r_0^3 = P_0^2 \to P_0^3 \to P_0^6 \to P_0^5, \\ r_0^4 = P_0^2 \to P_0^3 \to P_0^4 \to P_0^5, \end{cases}$$

$$w_1^1 : \begin{cases} r_1^1 = P_1^1 \to P_1^2, \\ r_1^2 = P_1^1 \to P_1^4 \to P_1^2, \end{cases}$$

$$w_1^2 : \begin{cases} r_1^3 = P_1^4 \to P_1^2 \to P_1^3, \\ r_1^4 = P_1^4 \to P_1^5 \to P_1^3, \\ r_1^5 = P_1^4 \to P_1^2 \to P_1^5 \to P_1^3. \end{cases}$$

Time-dependent feasible sets. The flow trajectories live in $L^2([0,6];\mathbb{R}^{|V_j|})$. The time-dependent feasible sets are

$$C_0^{(t)} = \Big\{ x(t) \in L^2([0,6];\mathbb{R}^4) \mid x_1 + x_2 = t+2,\ x_3 + x_4 = 2t+3 \text{ a.e. in } [0,6] \\ \text{and } (0,0,0,0) \le x(t) \le (t+1,\, t+2,\, 2t+2,\, t+3) \Big\},$$

$$C_1^{(t)} = \Big\{ y(t) \in L^2([0,6];\mathbb{R}^5) \mid y_1 + y_2 = 3t+5, \quad y_3 + y_4 + y_5 = 3t+6 \text{ a.e. in } [0,6] \\ \text{and } (0,0,0,0,0) \le y(t) \le (t+6,\, t+6,\, 2t+2,\, t+4,\, 4t+4) \Big\}.$$

Cost operators. The path-cost operators are defined pointwise:

$$B_0^{(t)} x(t) = x(t),\ B_1^{(t)} y(t) = \big(y_1^2(t),\, y_2^2(t),\, y_3^2(t),\, y_4^2(t),\, y_5^2(t)\big).$$

Here $B_0^{(t)}$ is linear and monotone (identity cost model), while $B_1^{(t)}$ is strictly monotone and models nonlinear congestion in X_1. Both operators are Lipschitz continuous on the respective feasible sets, so the hypotheses of Theorem 3.2 are satisfied.

Transfer operator. The bounded linear operator $F_1^{(t)} : L^2([0,6];\mathbb{R}^4) \to L^2([0,6];\mathbb{R}^5)$ is defined pointwise by $F_1^{(t)} x(t) = F_{\mathrm{mat}}\, x(t)$, where

$$F_{\mathrm{mat}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{5\times 4},$$

and its adjoint satisfies $F_1^{(t)*}y = F_{\text{mat}}^{\top}y$. We set $F_0^{(t)} = I$ (identity on $L^2([0,6];\mathbb{R}^4)$).

Stopping criterion. For a fixed t_i, the static residual at iteration k is

$$\begin{aligned}\text{TOL}(k) =& \frac{1}{2}\Big[\big\|x^k - P_{C_0^{(t_i)}}\big(x^k - \tfrac{1}{2}B_0^{(t_i)}x^k\big)\big\|^2 \\ &+ \big\|F_1^{(t_i)}x^k - P_{C_1^{(t_i)}}\big(F_1^{(t_i)}x^k - \tfrac{1}{2}B_1^{(t_i)}F_1^{(t_i)}x^k\big)\big\|^2\Big].\end{aligned}$$

Algorithm parameters. We discretize $[0,6]$ as $t_i = i/3$, $i = 0,1,\ldots,18$ ($\Delta t = 1/3$) and apply Algorithm 3.1 at each t_i with the same parameters as in Example 5.1. The stopping tolerance is $\varepsilon = 10^{-4}$.

Initial point (Case I). The algorithm is initialized at the upper-capacity midpoint: $x^0 = x^1$ is initialized as a heavily skewed point by assigning up to 85% of the total demand to the first element of each group (within the upper-capacity bound), with the remaining demand distributed equally among the subsequent elements.

Equilibrium results. Tables 1, 2 and Figure 7 report the computed equilibrium flows $x^*(t_i)$ in X_0 and $y^*(t_i)$ in X_1 at every grid point t_i.

Table 1: Equilibrium flow trajectories for city X_0: $x^*(t_i) = (x_1^*, x_2^*, x_3^*, x_4^*)(t_i)$.

t_i	x_1^*	x_2^*	x_3^*	x_4^*	t_i	x_1^*	x_2^*	x_3^*	x_4^*
0	1.0000	1.0000	1.5000	1.5000	3	2.5000	2.5000	4.5000	4.5000
1/3	1.1667	1.1667	1.8333	1.8333	10/3	2.6667	2.6667	4.8333	4.8333
2/3	1.3333	1.3333	2.1667	2.1667	11/3	2.8333	2.8333	5.1667	5.1667
1	1.5000	1.5000	2.5000	2.5000	4	3.0000	3.0000	5.5000	5.5000
4/3	1.6667	1.6667	2.8333	2.8333	13/3	3.1667	3.1667	5.8333	5.8333
5/3	1.8333	1.8333	3.1667	3.1667	14/3	3.3333	3.3333	6.1667	6.1667
2	2.0000	2.0000	3.5000	3.5000	5	3.5000	3.5000	6.5000	6.5000
7/3	2.1667	2.1667	3.8333	3.8333	16/3	3.6667	3.6667	6.8333	6.8333
8/3	2.3333	2.3333	4.1667	4.1667	17/3	3.8333	3.8333	7.1667	7.1667
					6	4.0000	4.0000	7.5000	7.5000

Table 2: Equilibrium flow trajectories for city X_1: $y^*(t_i) = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)(t_i)$.

t_i	y_1^*	y_2^*	y_3^*	y_4^*	y_5^*	t_i	y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	2.5000	2.5000	2.0000	2.0000	2.0000	3	7.0000	7.0000	5.0000	5.0000	5.0000
1/3	3.0000	3.0000	2.3333	2.3333	2.3333	10/3	7.5000	7.5000	5.3333	5.3333	5.3333
2/3	3.5000	3.5000	2.6667	2.6667	2.6667	11/3	8.0000	8.0000	5.6667	5.6667	5.6667
1	4.0000	4.0000	3.0000	3.0000	3.0000	4	8.5000	8.5000	6.0000	6.0000	6.0000
4/3	4.5000	4.5000	3.3333	3.3333	3.3333	13/3	9.0000	9.0000	6.3333	6.3333	6.3333
5/3	5.0000	5.0000	3.6667	3.6667	3.6667	14/3	9.5000	9.5000	6.6667	6.6667	6.6667
2	5.5000	5.5000	4.0000	4.0000	4.0000	5	10.0000	10.0000	7.0000	7.0000	7.0000
7/3	6.0000	6.0000	4.3333	4.3333	4.3333	16/3	10.5000	10.5000	7.3333	7.3333	7.3333
8/3	6.5000	6.5000	4.6667	4.6667	4.6667	17/3	11.0000	11.0000	7.6667	7.6667	7.6667
						6	11.5000	11.5000	8.0000	8.0000	8.0000

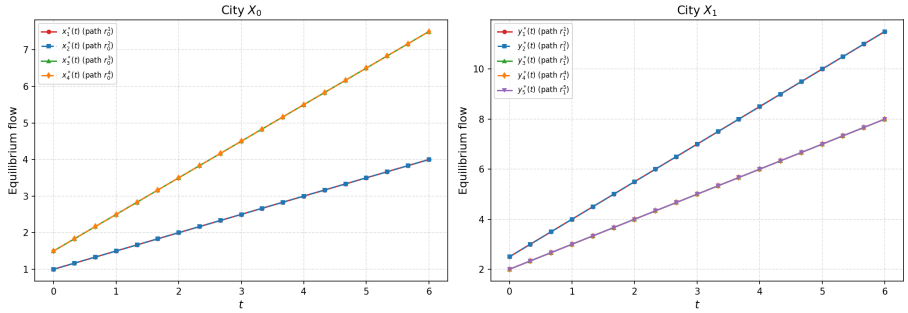


Figure 7: Equilibrium flow curves in city X_0 (left) and city X_1 (right).

6. Conclusion

In this work, we developed a unified framework for solving split variational inequality problems by integrating inertial acceleration with a fully self-adaptive step-size strategy. The proposed approach offers an effective alternative to existing methods by removing the need for parameter tuning based on problem-dependent constants.

The convergence analysis establishes that the algorithm is robust under mild and practically verifiable conditions, while guaranteeing strong convergence to a minimum-norm solution. This result underscores the stability of the method,

particularly for large-scale or implicitly defined problems.

From a practical perspective, the numerical results indicate that the algorithm achieves competitive performance in terms of computational efficiency and accuracy. In particular, its behavior in the traffic network model highlights its applicability to structured equilibrium problems arising in real-world settings.

References

[1] **Alakoya, T. O., Taiwo, A., Mewomo, O. T. and Cho, Y. J.** *An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings.* Annali dell'Università di Ferrara, 67(1):1–31, 2021. doi: 10.1007/s11565-020-00354-2.

[2] **Alvarez, F. and Attouch, H.** *An Inertial Proximal Method for Maximal Monotone Operators via Discretization of a Nonlinear Oscillator with Damping.* Set-Valued Analysis, 9(1–2):3–11, 2001. doi: 10.1023/A:1011253113155.

[3] **Batra, C., Chugh, R., Kumar, R., Suwais, K., Almanasra, S. and Mlaiki, N.** *Strong convergence of split equality variational inequality, variational inclusion, and multiple sets fixed point problems in Hilbert spaces with application.* Journal of Inequalities and Applications, 2024(1):40, 2024. doi: 10.1186/s13660-024-03118-0.

[4] **Bauschke, H. H. and Combettes, P. L.** Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, Cham, 2 edition, 2017. doi: 10.1007/978-3-319-48311-5.

[5] **Byrne, C.** *A unified treatment for some iterative algorithms in signal processing and image reconstruction.* Inverse Problems, 20(1):103–120, 2004. doi: 10.1088/0266-5611/20/1/006.

[6] **Censor, Y. and Elfving, T.** *A multiprojection algorithm using Bregman projections in a product space.* Numerical Algorithms, 8(2):221–239, 1994. doi: 10.1007/BF02142692.

[7] **Censor, Y., Gibali, A. and Reich, S.** *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space.* Optimization Methods and Software, 26(4–5):827–845, 2011. doi: 10.1080/10556788.2010.551536.

[8] **Censor, Y., Gibali, A. and Reich, S.** *Algorithms for the split variational inequality problem.* Numerical Algorithms, 59(2):301–323, 2012. doi: 10.1007/s11075-011-9490-5.

[9] **Daniele, P., Maugeri, A. and Oettli, W.** *Time-dependent traffic equilibria.* Journal of Optimization Theory and Applications, 103(3):543–555, 1999. doi: 10.1023/A:1021779823196.

[10] **Hieu, D., Anh, P. K. and Muu, L. D.** *Modified hybrid projection methods for finding common solutions to variational inequality problems.* Comput. Optim. Appl., 66(1):75–96, 1 2017. doi: https://doi.org/10.1007/s10589-016-9857-6.

[11] **Izuchukwu, C., Mebawondu, A. A. and Mewomo, O. T.** *A new method for solving split variational inequality problems without co-coerciveness.* Journal of Fixed Point Theory and Applications, 22:98, 2020. doi: 10.1007/s11784-020-00834-0.

[12] **Jolaoso, L. O., Sunthrayuth, P., Cholamjiak, P. and Cho, Y. J.** *Inertial projection and contraction methods for solving variational inequalities with applications to image restoration problems.* Carpathian Journal of Mathematics, 39(3):683–704, 2023. doi: 10.37193/CJM.2023.03.09.

[13] **Konnov, I. V.** *Equilibrium Models and Variational Inequalities*, volume 210 of Mathematics in Science and Engineering. Elsevier, Amsterdam, 2007. ISBN 978-0-444-53030-1.

[14] **Korpelevich, G. M.** *The extragradient method for finding saddle points and other problems.* Ekonomika i Matematicheskie Metody, 12(4):747–756, 1976.

[15] **Ly, L. X., Nguyen, T. T. T. and Nguyen, Q. A.** *Relaxed two-step inertial method for solving a class of split variational inequality problems with application to traffic analysis.* Optimization, 75(6):1303–1331, 2025. doi: 10.1080/02331934.2025.2459191.

[16] **Mewomo, O. T., Uzor, V. A. and Gibali, A.** *A strongly convergent algorithm for solving split equality problems beyond monotonicity.* Computational and Applied Mathematics, 43(6):326, 2024. doi: 10.1007/s40314-024-02829-w.

[17] **Nguyen, T. T. T. and Tran, T. T.** *Strong convergence of one-step inertial algorithm for a class of bilevel variational inequalities.* Optimization, 75(5):1101–1133, 2026. doi: 10.1080/02331934.2024.2448737.

[18] **Ogwo, G. N., Izuchukwu, C. and Mewomo, O. T.** *Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity.* Numerical Algorithms, 88(3):1419–1456, 2021. doi: 10.1007/s11075-021-01081-1.

[19] **Pham, V. H., Nguyen, D. H. and Tran, V. A.** *A strongly convergent modified Halpern subgradient extragradient method for solving the split variational inequality problem.* Vietnam Journal of Mathematics, 48 (1):187–204, 2020. ISSN 2305-2228. doi: 10.1007/s10013-019-00378-y.

[20] **Polyak, B. T.** *Some methods of speeding up the convergence of iteration methods.* USSR Computational Mathematics and Mathematical Physics, 4(5):1–17, 1964.

[21] **Saejung, S. and Yotkaew, P.** *Approximation of zeros of inverse strongly monotone operators in Banach spaces.* Nonlinear Analysis: Theory, Methods & Applications, 75(2):742–750, 2012. ISSN 0362-546X. doi: 10.1016/j.na.2011.09.005.

[22] **Sahu, D. R., Yao, J. C., Verma, M. and Shukla, K. K.** *Convergence rate analysis of proximal gradient methods with applications to composite minimization problems.* Optimization, 70(1):75–100, 2021. doi: 10.1080/02331934.2019.1702040.

[23] **Singh, S., Gibali, A. and Qin, X.** *Cooperation in traffic network problems via evolutionary split variational inequalities.* Journal of Industrial and Management Optimization, 18(1):593–611, 2022. doi: 10.3934/jimo.2020170.

[24] **Takahashi, W.** Nonlinear Functional Analysis: Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama, 2000.

[25] **Tan, K. K. and Xu, H. K.** *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process.* Journal of Mathematical Analysis and Applications, 178(2):301–308, 1993. ISSN 0022-247X. doi: 10.1006/jmaa.1993.1309.

[26] **Tseng, P.** *A modified forward-backward splitting method for maximal monotone mappings.* SIAM Journal on Control and Optimization, 38(2): 431–446, 2000. doi: 10.1137/S0363012998338806.

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