J. Math. Math. Sci. (2026) nn–nnn

Weak convergence and solvability of the non-polar part $NP(dd^c\bullet)^n$ of Monge-Ampère measures in the class $\mathcal{D}(\Omega)$

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Dedicated to Professor Ha Huy Khoai on the occasion of his 80 birthday.

(Received July 08, 2026; accepted Oct. 05, 2026)

Abstract. The aim of this note is to prove weak convergence in the class $\mathcal{D}(\Omega)$ and of the corresponding non-polar part $(NP)(dd^c\bullet)^n$ of Monge-Ampère measures in the relationship with the convergence in C_n-capacity on an open subset Ω of $\mathbb{C}^n$. In the end of the note we investigate the solvability of the Monge–Ampère equation for the non-polar part in the class $\mathcal{D}(\Omega)$.

1. Introduction

The study of the continuity of the complex Monge-Ampère measures $(dd^c\bullet)^n$ plays an important role in many problems of pluripotential theory. The original results in the direction of this research stem from the seminal work of Bedford and Taylor in [2]. In this paper, Bedford-Taylor proved that if a sequence of locally bounded plurisubharmonic functions $(u_j) \subset PSH(\Omega) \cap L^\infty_{loc}(\Omega)$,

Key words and phrases: class $\mathcal{D}(\Omega)$, convergence in capacity, weak convergence of the complex Monge-Ampère measures, weak convergence of the non-polar part, solvability of the non-polar part of the complex Monge-Ampère measure.
2020 Mathematics Subject Classification: 31C10, 32U05, 32U15, 32W20

where $PSH(\Omega)$ denotes the set of plurisubharmonic functions on an open subset $\Omega \subset \mathbb{C}^n$ is decreasing to a locally bounded plurisubharmonic function u then $(dd^c u_j)^n$ is weakly convergent to $(dd^c u)^n$. They also established a result about weak convergence of sequence of the complex Monge-Ampère measures $(dd^c u_j)^n$ to $(dd^c u)^n$ under the assumption that (u_j) is an increasing sequence and a.e convergent with respect to the Lebesgue measure dV_{2n} to a locally bounded plurisubharmonic u. In summary, by relying on the results obtained by Bedford-Taylor, we observe that the complex Monge-Ampère measures $(dd^c \bullet)^n$ is continuous in weak*-topology along monotonic sequences in the class of locally bounded plurisubharmonic functions. However, monotone convergence of the sequence of plurisubharmonic functions (u_j) to the plurisubharmonic function u does not always occur. Hence, a natural question is whether the monotone convergence condition can be replaced by a weaker condition, such as convergence in $L^p, 1 \leq p < +\infty$, which still ensures the convergence in the weak*-topology of the sequence of the complex Monge-Ampère measures $(dd^c u_j)^n$ to $(dd^c u)^n$? Unfortunately, that does not happen. Due to a counterexample given in [6], Cegrell showed that there exists a sequence of continuous plurisubharmonic functions (u_j) in $\mathbb{C}^2$ converging in the $L^p, 1 \leq p < +\infty$-topology to a plurisubharmonic function u, but the corresponding sequence of Monge-Ampère measures $(dd^c u_j)^2$ does not converge in the weak*-topology to $(dd^c u)^2$. Thus, convergence in the L^p-topology for $p \geq 1$ does not imply the weak continuity of the sequence of Monge-Ampère measures. At this point, one seeks for a condition weaker than the monotone convergence of sequences of plurisubharmonic functions, yet strong enough to establish the continuity of a sequence of the complex Monge-Ampère measures in the weak*-topology. In 1996, in the paper [19], Xing introduced for the first time the concept of convergence in C_n-capacity for a sequence of plurisubharmonic functions (u_j) to a plurisubharmonic function u, and proved that if a sequence of plurisubharmonic functions (u_j), which is uniformly bounded on an open set Ω in $\mathbb{C}^n$, converges in C_n-capacity to a locally bounded plurisubharmonic function u, then the sequence of Monge-Ampère measures $(dd^c u_j)^n$ converges weakly (i.e., in the weak*-topology) to $(dd^c u)^n$. Also in that paper, based on the concept of convergence in C_n-capacity, Xing obtained several other forms of weak convergence for sequences of Monge-Ampère measures. For more details on these results, we refer readers to [19]. However, in the above mentioned results of Xing we all recognize that there is a requirement for the uniform boundedness of the given sequence of plurisubharmonic functions (u_j). In 2004, in seminal work [8], Cegrell introduced and investigated some classes of unbounded plurisubharmonic functions on a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ on which the complex Monge-Ampère measures $(dd^c \bullet)^n$ is well-defined and it is continuous on decreasing sequences. The most notable result in [8] is the introduction of the class $\mathcal{E}(\Omega)$. For this class of unbounded plurisub-

harmonic functions, Cegrell proved that it is the largest class on which the complex Monge-Ampère measure $(dd^c\bullet)^n$ is well-defined and continuous under decreasing sequences. A natural question that arises is whether, if a sequence $(u_j) \subset \mathcal{E}(\Omega)$ converges in C_n-capacity to a function $u \in \mathcal{E}(\Omega)$, then the sequence of Monge-Ampère measures $(dd^cu_j)^n$ converges *-weakly to $(dd^cu)^n$. The answer to this problem is addressed in [9]. A notable concern regarding the class of plurisubharmonic functions introduced by Cegrell is that these functions need to be defined on a bounded hyperconvex domain. So, one may ask whether it is possible to introduce a class of plurisubharmonic functions on an open set in $\mathbb{C}^n$ such that the complex Monge-Ampère measure is also well-defined and continuous on decreasing sequences. This problem was addressed in Blocki's 2006 work [4]. There, Blocki introduced the class $\mathcal{D}(\Omega)$ of unbounded plurisubharmonic functions on an open set in $\mathbb{C}^n$ such that for every $u \in \mathcal{D}(\Omega)$, the measure $(dd^cu)^n$ is well-defined and this measure is continuous on decreasing sequences. Note that in the case when Ω is a bounded hyperconvex domain of $\mathbb{C}^n$ then $\mathcal{D}(\Omega) \cap PSH^-(\Omega) = \mathcal{E}(\Omega)$ where $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. Therefore, the natural question that arises here is: if a sequence $(u_j) \subset \mathcal{D}(\Omega)$ which converges in C_n-capacity to some function $u \in \mathcal{D}(\Omega)$, does it follow that $(dd^cu_j)^n$ is weakly convergent to $(dd^cu)^n$? So far, we have not seen any results addressing this problem. Therefore, the first problem in this note is to give conditions such that if $(u_j) \subset \mathcal{D}(\Omega)$ which is convergent in C_n-capacity to $u \in \mathcal{D}(\Omega)$ then $(dd^cu_j)^n$ weakly converges to $(dd^cu)^n$. Next, the second problem we are concerned with in this paper is to find conditions under which, if $(u_j) \subset PSH^-(\Omega)$ is convergent in C_n-capacity to $u \in PSH^-(\Omega)$, then their non-polar parts $NP(dd^cu_j)^n$ converge weakly to the non-polar part $NP(dd^cu)^n$ where Ω is an open subset of $\mathbb{C}^n$.

The paper is organized as follows. Besides the introduction, the paper has the other four sections. In Section 2, we recall the definition of the class $\mathcal{D}(\Omega)$ which was introduced and studied by Błocki in [4]. Next, we deal with the results concerning the presentation of the concept of the non-polar part of the Monge-Ampère measure of a plurisubharmonic function on an open set in $\mathbb{C}^n$ introduced and investigated by Bedford-Taylor in [3]. We also provide some results concerning this non-polar part when we consider this non-polar part within the class $\mathcal{D}(\Omega)$. Section 3 is devoted to the proof of some inequalities concerning the non-polar part $(NP)(dd^c\bullet)^n$ including a variant of the Demailly-type inequality in the class $\mathcal{D}(\Omega)$. Section 4 discusses the weak convergence of the Monge-Ampère measures $(dd^cu_j)^n \longrightarrow (dd^cu)^n$ when $u_j, u \in \mathcal{D}(\Omega)$ and $u_j \longrightarrow u$ in C_n-capacity, as well as the weak convergence of the non-polar part $(NP)(dd^cu_j)^n \longrightarrow (NP)(dd^cu)^n$. Finally, in Section 5, we investigate the solvability of the Monge-Ampère equation for the non-polar part in the class $\mathcal{D}(\Omega)$, where Ω is a bounded open subset of $\mathbb{C}^n$. We consider the problem for the case where the Radon measure μ can be bounded by the Monge-Ampère

measures of two functions $u, v \in \mathcal{D}(\Omega)$. This can be viewed as an extended version of a result of Kołodziej in [17] which was proved by Kołodziej in the case where u and v are locally bounded entire plurisubharmonic functions on $\mathbb{C}^n$. Namely, in [17], Kołodziej proved the following result: given two locally bounded entire plurisubharmonic functions u and v on in $\mathbb{C}^n$ satisfying $u \leq v$, $(dd^c v)^n \leq (dd^c u)^n$ and $\lim\limits_{|z|\to\infty}(v(z)-u(z)) = 0$. Then one can solve the Monge-Ampère equation $(dd^c w)^n = \mu$ for any measure μ with $(dd^c v)^n \leq \mu \leq (dd^c u)^n$. Furthermore, the solution w is unique among functions satisfying $u \leq w \leq v$. Our result in Section 5 is the following theorem.

Theorem 5.1. Let Ω be a bounded open subset in $\mathbb{C}^n$ and $u, v \in \mathcal{D}^-(\Omega), u \leq v$ and $(dd^c v)^n \leq (dd^c u)^n$ on Ω. Assume that $(dd^c u)^n(\{u = -\infty\}) = 0$ and $(dd^c v)^n$ is carried by a pluripolar subset in Ω and $P[u] = v$. Then for every positive Radon measure μ on Ω satisfying the condition

$$(dd^c v)^n \leq \mu \leq (dd^c u)^n,$$

there exists a function $w \in \mathcal{D}(\Omega)$ with $u \leq w \leq v$ on Ω and $NP(dd^c w)^n = \mu$ on Ω, $P[w] = v$. Moreover $(dd^c v)^n \leq (dd^c w)^n \leq (dd^c u)^n$.

2. Preliminairies

Let Ω be an open subset of $\mathbb{C}^n$. By $PSH(\Omega)$(resp. $PSH^-(\Omega)$) we denote the set of plurisubharmonic functions on Ω (resp. the set of negative plurisubharmonic functions on Ω). Following [4] we recall the class $\mathcal{D}(\Omega)$ defined as follows.

Definition 2.1. *The class $\mathcal{D}(\Omega)$ consists of plurisubharmonic functions u on Ω such that for each $u \in \mathcal{D}(\Omega)$, there exists a positive Radon measure μ on Ω satisfying the condition if $U \subset \Omega$ is an open subset and $(u_j) \subset PSH \cap C^\infty(U)$ is decreasing to u on U, the sequence of measures $(dd^c u_j)^n$ tends weakly to μ on U.*

The measure μ is denoted by $(dd^c u)^n$ and is called the Monge-Ampère measure of u. Through the paper, by $\mathcal{D}^-(\Omega)$ we denote the set of negative plurisubharmonic functions $u \in \mathcal{D}(\Omega)$.

Remark 2.1. (i) By [4] we note that if $u \in \mathcal{D}(\Omega)$ and $v \in PSH(\Omega)$ then $\max\{u, v\} \in \mathcal{D}(\Omega)$.

(ii) By using Definition 2.1 together with some subtle techniques from pluripotential theory, in particular, those involving the Monge-Ampère operator, we

observe that the class $\mathcal{D}(\Omega)$ possesses the local property. More precisely, if $u \in PSH(\Omega)$, then $u \in \mathcal{D}(\Omega)$ if and only if, for every $a \in \Omega$, there exists a neighborhood U of a such that $u \in \mathcal{D}(U)$.

(iii) By [4] we note that if Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ then $u \in \mathcal{E}(\Omega)$ where $\mathcal{E}(\Omega)$ is the biggest class of negative plurisubharmonic functions on which the complex Monge-Ampère operator $(dd^c.)^n$ is well-defined. It is introduced and investigated in [8].

Because in this note we should use the plurifine topology on an open subset Ω of $\mathbb{C}^n$ then we recall this topology. The plurifine topology was investigated by many authors such as Bedford-Taylor, El Marzguioui, El Kadiri, Fuglede and Wiegerinck. For more details on results concerning to this topology and its applications to pluripotential theory we refer readers to papers: [3], [13], [14], [18]. This topology is defined as the coarsest topology on Ω such that every plurisubharmonic functions on Ω are continuous in this topology. It has a neighborhood basis consisting of sets of the form $\mathbb{B} \cap \{\psi > 0\}$, where $\mathbb{B}$ is an open ball in Ω and $\psi \in PSH(\mathbb{B})$. An open set for this topology is called a plurifinely open set. We recall some results on plurifinely open subset concerning to the complex Monge-Ampère measures.

Proposition 2.1. *([3], Corollary 4.3) Let u and v be two locally bounded plurisubharmonic functions on an open $\Omega \subset \mathbb{C}^n$ and U a plurifinely open set in Ω. If $u = v$ on U, then $(dd^c u)^n|_U = (dd^c v)^n|_U$ in the sense of measures.*

In [15] El Kadiri extended the above result to the class $\mathcal{D}(\Omega)$. Namely, Corollary 3.7 in [15] claims that if $u, v \in \mathcal{D}(\Omega)$ and U is a plurifinely open subset of Ω. Let $u = v$ on U then $(dd^c u)^n|_U = (dd^c v)^n|_U$ in the sense of measures. Next, we discuss the nonpolar part of a Monge-Ampère measure of a plurisubharmonic function u on an open set $\Omega \subset \mathbb{C}^n$ presented in [3].

Definition 2.2. *Let Ω be an open subset in $\mathbb{C}^n$ and $u \in PSH(\Omega)$. The nonpolar part of $(dd^c u)^n$, denoted by $NP(dd^c u)^n$, is a positive Borel measure which is zero on $\{u = -\infty\}$, and for every Borel set $E \subset \{u > -\infty\}$*

$$\int_E (NP)(dd^c u)^n = \lim_{j\to\infty} \int_{E\cap\{u>-j\}} (dd^c \max\{u, -j\})^n.$$

By Proposition 3.3 in [16] we note that if $u \in \mathcal{D}(\Omega)$ then $(NP)(dd^c u)^n = 1_{\{u>-\infty\}}(dd^c u)^n$ where by 1_A we denote the characteristic function of the set A. Hence, $(NP)(dd^c u)^n$ is a positive Radon measure on Ω. We need the following definitions which are quoted from [11].

Definition 2.3. *Let $u \in PSH^-(\Omega)$. Set*

$$P[u] = \Big(\sup\Big\{v \in PSH^-(\Omega) : v \le u + O(1) \text{ on } \Omega, \lim_{\Omega\setminus N \ni z \to \xi_0} \inf(u(z) - v(z)) \ge 0, \forall \xi_0 \in \partial\Omega\Big\}\Big)^*,$$

where $N = \{u = -\infty\}$. A function $u \in PSH^-(\Omega)$ is called model iff $P[u] = u$.

Remark 2.2. As in [11] we note that if Ω is a bounded domain in $\mathbb{C}^n$ and $v \in PSH^-(\Omega)$ then v is model if and only if $NP(dd^c v)^n = 0$.

3. Some results concerning to $(NP)(dd^c\bullet)^n$

The purpose of this section is to present several results related to the measure $NP(dd^c\cdot)^n$ in the class $\mathcal{D}(\Omega)$.

Proposition 3.1. *Let Ω be an open subset of $\mathbb{C}^n$ and $u, v \in \mathcal{D}(\Omega)$. Then the following equality holds*

$$(NP)(dd^c \max\{u,v\})^n|_{\{u>v\}} = (NP)(dd^c u)^n|_{\{u>v\}}. \tag{3.1}$$

Proof. Since $u, v \in \mathcal{D}(\Omega)$ then by [4] we get that $\max\{u,v\} \in \mathcal{D}(\Omega)$. On the other hand $\{u > v\} = \bigcup_{r\in\mathbb{Q}} \{u > r > v\}$, $\mathbb{Q}$ denotes the set of rational numbers, then $\{u > v\}$ is a plurifinely open subset in Ω. By Proposition 3.3 in [16] we have

$$(NP)(dd^c \max\{u,v\})^n = 1_{\{\max\{u,v\}>-\infty\}}(dd^c \max\{u,v\})^n, \tag{3.2}$$

in the sense of measures on Ω. This yields that

$$(NP)(dd^c \max\{u,v\})^n|_{\{u>v\}} = 1_{\{\max\{u,v\}>-\infty\}}(dd^c \max\{u,v\})^n|_{\{u>v\}}. \tag{3.3}$$

We can rewrite the right-hand side of (3.3) in the form

$$1_{\{\max\{u,v\}>-\infty\}}(dd^c \max\{u,v\})^n|_{\{u>v\}} = 1_{\{\max\{u,v\}>-\infty\}} \cdot 1_{\{u>v\}}(dd^c \max\{u,v\})^n. \tag{3.4}$$

However, $\max\{u,v\}|_{\{u>v\}} = u|_{\{u>v\}}$ then by using Corollary 3.7 in [15] we get that

$$1_{\{u>v\}}(dd^c \max\{u,v\})^n = 1_{\{u>v\}}(dd^c u)^n.$$

Hence, we get that

$$\begin{aligned} 1_{\{\max\{u,v\}>-\infty\}}(dd^c\max\{u,v\})^n|_{\{u>v\}} &= 1_{\{\max\{u,v\}>-\infty\}}1_{\{u>v\}}(dd^cu)^n \\ &= 1_{\{u>v\}\cap\{\max\{u,v\}>-\infty\}}(dd^cu)^n. \end{aligned} \tag{3.5}$$

On the other hand, it is clear that $\{u > v\} \cap \{\max\{u, v\} > -\infty\} = \{u > v\} \cap \{u > -\infty\}$. Coupling (3.5), this equality and once more using Proposition 3.3 in [16] we infer that

$$\begin{aligned} 1_{\{u>v\}\cap\{\max\{u,v\}>-\infty\}}(dd^cu)^n &= 1_{\{u>v\}\cap\{u>-\infty\}}(dd^cu)^n \\ &= 1_{\{u>v\}}1_{\{u>-\infty\}}(dd^cu)^n \\ &= 1_{\{u>v\}}(NP)(dd^cu)^n = (NP)(dd^cu)^n|_{\{u>v\}}. \end{aligned} \tag{3.6}$$

Combining (3.3) with (3.5) and (3.6) we arrive that

$$(NP)(dd^c\max\{u,v\})^n|_{\{u>v\}} = (NP)(dd^cu)^n|_{\{u>v\}},$$

and the proof is complete. ■

As well-known in [5], Błocki proved the following inequality of Demailly for locally bounded plurisubharmonic function on an open subset $\Omega \subset \mathbb{C}^n$. Namely, let $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. Then

$$(dd^c\max\{u,v\})^n \geq 1_{\{u\geq v\}}(dd^cu)^n + 1_{\{u<v\}}(dd^cv)^n. \tag{3.7}$$

Now, by virtue of (ii) in Remark 2.1 and Lemma 2.3 in [12], we extend the above Demailly inequality to the non-pluripolar measures $NP(dd^c.)^n$ to the class $\mathcal{D}(\Omega)$. The precise statement is as follows.

Proposition 3.2. *Let Ω be an open subset in $\mathbb{C}^n$ and $u, v \in \mathcal{D}(\Omega)$. Then the following inequality holds*

$$NP(dd^c\max\{u,v\})^n \geq 1_{\{u\geq v\}}NP(dd^cu)^n + 1_{\{u<v\}}NP(dd^cv)^n. \tag{3.8}$$

Proof. Take an arbitrary ball $\mathbb{B} \Subset \Omega$. It suffices to prove that (3.8) holds on $\mathbb{B}$. By (ii) of Remark 2.1 we may assume that $u, v \in \mathcal{D}(\mathbb{B})$ and $u, v \in PSH^-(\mathbb{B})$. Applying Lemma 2.3 in [12] the desired conclusion follows. The proof is complete. ■

From Proposition 3.2 we get the following result.

Corollary 3.1. *Let Ω be an open subset in $\mathbb{C}^n$ and μ be a positive Radon measure on Ω. Assume that $u, v \in \mathcal{D}(\Omega)$. If $(NP)(dd^cu)^n \geq \mu, (NP)(dd^cv)^n \geq \mu$ then*

$$(NP)(dd^c\max\{u,v\})^n \geq \mu. \tag{3.11}$$

Proof. By Proposition 3.2 we have

$$\begin{aligned}(NP)(dd^c\max\{u,v\})^n &\geq 1_{\{u\geq v\}}(NP)(dd^cu)^n + 1_{\{u<v\}}(NP)(dd^cv)^n\\ &\geq 1_{\{u\geq v\}}\mu + 1_{\{u<v\}}\mu = \mu.\end{aligned}$$

The proof is complete. ■

Remark 3.1. In [11] the authors proved a similar result as Corollary 3.1 under the assumptions that Ω is a bounded domain in $\mathbb{C}^n$, $u, v \in PSH^-(\Omega)$, and μ is a positive Borel measure vanishing on pluripolar sets.

4. Weak convergence in the class $\mathcal{D}(\Omega)$ and of the non-polar part

The purpose of this section is to establish the weak convergence of the Monge-Ampère measures $(dd^c.)^n$ and the non-pluripolar Monge-Ampère measures $NP(dd^c.)^n$ for functions in the class $\mathcal{D}(\Omega)$ under convergence in C_n-capacity. This is inspired by a recent result in [16] El Kadiri. In [16], the author investigated the weak convergence of $(dd^cu_j)^n$ to the non-polar part $(NP)(dd^cu)^n$ under the assumption that $(u_j) \subset \mathcal{D}(\Omega)$, (u_j) is decreasing to a plurisubharmonic $u \in PSH(\Omega)$ where $u = \lim\limits_{j\to\infty} u_j$(see Theorem 4.2 in [16]). In this section we give some results on the weak convergence of a sequence $(dd^cu_j)^n$ to $(dd^cu)^n$ under the assumption that (u_j) and $u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ and u_j is convergent to u in C_n-capacity on Ω. Similarly, we also consider when the nonpolar part $(NP)(dd^cu_j)^n \longrightarrow (NP)(dd^cu)^n$ weakly, assuming that $u_j, u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ and $u_j \longrightarrow u$ in C_n-capacity.

First, we recall the following notion introduced and investigated in [2] (see Definition 3.1 in [2]) which plays an important role in many problems of pluripotential theory.

Definition 4.1. *Let Ω be an open subset of $\mathbb{C}^n$ and $E \subset \Omega$ be a Borel subset. We define*

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{\int\limits_E (dd^cu)^n : u \in PSH(\Omega),\quad 0 \leq u \leq 1\Big\}.$$

$C_n(E)$ is called a Monge-Ampère capacity, or Bedford-Taylor capacity of E relative to Ω.

Recently, in [16], El Kadiri gave an another form of the notion of Bedford-Taylor capacity which we will use it in this note.

Definition 4.2. *(see p.5 in [16]) Let Ω be an open subset of $\mathbb{C}^n$ and $E \subset \Omega$ be a Borel subset. The capacity of E to Ω is defined by*

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{\int_\Omega (dd^c u)^n : u \in PSH^-(\Omega), \quad u|_E \leq -1\Big\}.$$

In 1996, Xing in [19] gave the following definition.

Definition 4.3. *Let Ω be an open in $\mathbb{C}^n$ and $(u_j)_{j\geq 1}, u$ be functions defined on Ω. We say that $u_j \longrightarrow u$ in C_n-capacity if for all $K \Subset \Omega$ and $\delta > 0$*

$$\lim_{j\to\infty} C_n\Big(K \cap \{|u_j(z) - u(z)| > \delta\}\Big) = 0.$$

Is is not difficult to see that if $\{u_j\}$ is a monotone sequence of plurisubharmonic functions converging to a plurisubharmonic function u then $u_j \longrightarrow u$ in C_n-capacity (see Lemma 7.13 in [10]). The following result, inspired by Theorem 4.2 in [16], establishes the relationship between the weak convergence of a sequence of Monge-Ampère operators $(dd^c u_k)^n$ to $(dd^c u)^n$, under the assumption that $u_k, u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ and $u_k \longrightarrow u$ in C_n-capacity.

Theorem 4.1. *Let Ω be an open subset in $\mathbb{C}^n$ and $(u_k) \subset \mathcal{D}(\Omega) \cap PSH^-(\Omega)$, $u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$. Assume that $u_k \geq u \geq w$, $w \in PSH^-(\Omega)$ with $\lim_{j\to\infty} j^n C_n\Big(\{w \leq -j\}\Big) = 0$ and $u_k \longrightarrow u$ in C_n-capacity. Then*

$$\lim_{k\to\infty} (dd^c u_k)^n = 1_{\{u>-\infty\}}(dd^c u)^n,$$

in the sense of currents in Ω.

Proof. First we claim that if $u_k \longrightarrow u$ in C_n-capacity then for all $j \geq 1$, so is $\max\{u_k, -j\} \longrightarrow \max\{u, -j\}$ as $k \to \infty$. Indeed, we have

$$\begin{aligned} \max\{u_k, -j\} &= \frac{1}{2}\Big[u_k - j + |u_k + j|\Big] \\ \max\{u, -j\} &= \frac{1}{2}\Big[u - j + |u + j|\Big]. \end{aligned} \tag{4.1}$$

From (4.1) we have

$$\max\{u_k, -j\} - \max\{u, -j\} = \frac{1}{2}\Big[u_k - u + |u_k + j| - |u + j|\Big]. \tag{4.2}$$

It follows that

$$|\max\{u_k, -j\} - \max\{u, -j\}| \leq |u_k - u|. \tag{4.3}$$

Now given $\delta > 0$ and $K \Subset \Omega$, from (4.3) we have

$$\begin{aligned} &C_n\Big(K \cap \{|\max\{u_k, -j\} - \max\{u, -j\}| > \delta\}\Big) \\ &\leq C_n\Big(K \cap \{|u_k - u| > \delta\}\Big) \longrightarrow 0, \end{aligned} \tag{4.4}$$

as $k \to \infty$ as we wanted. Therefore, the assertion is proven. Now take an arbitrary $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$, and $j \geq 1$. By applying Corollary 3.7 in [15] we have

$$\begin{aligned} \int\limits_\Omega \varphi(dd^c u_k)^n &= \int\limits_{\{u_k > -j\}} \varphi(dd^c u_k)^n + \int\limits_{\{u_k \leq -j\}} \varphi(dd^c u_k)^n \\ &= \int\limits_{\{u_k > -j\}} \varphi\Big(dd^c \max\{u_k, -j\}\Big)^n + \int\limits_{\{u_k \leq -j\}} \varphi(dd^c u_k)^n \\ &\leq \int\limits_\Omega \varphi\Big(dd^c \max\{u_k, -j\}\Big)^n + \int\limits_{\{u_k \leq -j\}} \varphi(dd^c u_k)^n. \end{aligned} \tag{4.5}$$

Now by applying the results of Xing in [19], the first term on the right-hand side of (4.5) converges to $\int\limits_\Omega \varphi(dd^c \max\{u, -j\})^n$ as $k \to \infty$. On the other hand, by the hypothesis $w \leq u_k \leq 0$ for all $k \geq 1$ then $\{u_k \leq -j\} \subset \{w \leq -j\}$. Thus it follows that $C_n\Big(\{u_k \leq -j\}\Big) \leq C_n\Big(\{w \leq -j\}\Big)$. We now proceed to estimate the second term in (4.5). We have

$$\begin{aligned} 0 \leq \int\limits_{\{u_k \leq -j\}} \varphi(dd^c u_k)^n &\leq \|\varphi\|_\infty \int\limits_{\{u_k \leq -j\}} (dd^c u_k)^n = \|\varphi\|_\infty j^n \int\limits_{\{u_k \leq -j\}} \Big(dd^c \frac{u_k}{j}\Big)^n \\ &\leq \|\varphi\|_\infty j^n \int\limits_\Omega \Big(dd^c \frac{u_k}{j}\Big)^n \leq \|\varphi\|_\infty j^n C_n\Big(\{u_k \leq -j\}\Big) \leq \|\varphi\|_\infty j^n C_n\Big(\{w \leq -j\}\Big), \end{aligned} \tag{4.6}$$

where $\|\varphi\|_\infty = \sup\{|\varphi(x)| : \ x \in \Omega\}$. Moreover, by the hypothesis we have $u \geq w$ on Ω and by the above mentioned arguments it follows that $C_n\Big(\{u \leq -j\}\Big) \leq C_n\Big(\{w \leq -j\}\Big)$. Next, we give estimations for the term $\int\limits_\Omega \varphi\Big(dd^c \max\{u, -j\}\Big)^n$

as follows. We have

$$\begin{aligned}
&\int_{\Omega}\varphi\Big(dd^c\max\{u,-j\}\Big)^n\\
&\leq \int_{\{u>-j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n+\int_{\{u\leq -j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n\\
&\leq \int_{\{u>-j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n+\|\varphi\|_\infty j^nC_n\Big(\{u\leq -j\}\Big)\\
&\leq \int_{\{u>-j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n+\|\varphi\|_\infty j^nC_n\Big(\{w\leq -j\}\Big). \qquad (4.7)
\end{aligned}$$

Coupling (4.5), (4.6) and (4.7) we conclude that

$$\int_{\Omega}\varphi(dd^cu_k)^n\leq \int_{\{u>-j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n+2\|\varphi\|_\infty j^nC_n\Big(\{w\leq -j\}\Big). \qquad (4.8)$$

From (4.8) we get that

$$\overline{\lim_{k\to\infty}}\int_{\Omega}\varphi(dd^cu_k)^n\leq \int_{\{u>-j\}}\varphi\Big(dd^c\max\{u,-j\}\Big)^n+2\|\varphi\|_\infty j^nC_n\Big(\{w\leq -j\}\Big). \qquad (4.9)$$

Tending $j\to\infty$, using the hypothesis and Definition 4.2, Proposition 3.3 in [16] we obtain the following result

$$\overline{\lim_{k\to\infty}}\int_{\Omega}\varphi(dd^cu_k)^n\leq \int_{\Omega}\varphi(NP)(dd^cu)^n=\int_{\Omega}\varphi 1_{\{u>-\infty\}}(dd^cu)^n. \qquad (4.10)$$

From (4.10) we infer that

$$\overline{\lim_{k\to\infty}}(dd^cu_k)^n\leq 1_{\{u>-\infty\}}(dd^cu)^n,$$

in the sense of currents in Ω. Now, by repeating arguments as in the proof of Theorem 4.2 in [16] and using results in [19] as in the above arguments we obtain

$$\underline{\lim_{k\to\infty}}\int_{\Omega}\varphi(dd^cu_k)^n\geq \int_{\Omega}\varphi(NP)(dd^cu)^n=\int_{\Omega}\varphi 1_{\{u>-\infty\}}(dd^cu)^n. \qquad (4.11)$$

Hence, $\underline{\lim_{k\to\infty}}(dd^cu_k)^n\geq 1_{\{u>-\infty\}}(dd^cu)^n$ in the sense of currents in Ω. By combining the results obtained, we get the desired conclusion. The proof is complete. ■

By using Theorem 4.1, Proposition 3.3 in [16] we give a following result which establishes weak convergence of non-polar parts of the Monge-Ampère measures in the class $\mathcal{D}(\Omega)$.

Theorem 4.2. *Under the same hypotheses of Theorem 4.1 the following statements hold.*
(i) $\overline{\lim\limits_{k\to\infty}}\,(NP)(dd^cu_k)^n \le (NP)(dd^cu)^n$.
(ii) If an extra condition is added $(dd^cu_k)^n(\{u_k=-\infty\})=0$ *for all* $k\ge 1$ *then*

$$\lim_{k\to\infty}(NP)(dd^cu_k)^n=(NP)(dd^cu)^n,$$

in the sense of currents in Ω.

Proof. (i) By Proposition 3.3 in [16] we have

$$(NP)(dd^cu_k)^n=1_{\{u_k>-\infty\}}(dd^cu_k)^n\le(dd^cu_k)^n.$$

Hence, using Theorem 4.1 we get that

$$\begin{aligned}\overline{\lim_{k\to\infty}}\,(NP)(dd^cu_k)^n &\le \overline{\lim_{k\to\infty}}\,(dd^cu_k)^n\\ &\le 1_{\{u>-\infty\}}(dd^cu)^n=(NP)(dd^cu)^n.\end{aligned}\tag{4.12}$$

Hence, (i) is proved.
(ii) We have also by Proposition 3.3 in [16] $(NP)(dd^cu_k)^n=1_{\{u_k>-\infty\}}(dd^cu_k)^n=(dd^cu_k)^n$ because $(dd^cu_k)^n(\{u_k=-\infty\})=0$. By applying Theorem 4.1 we arrive that

$$\begin{aligned}\lim_{k\to\infty}(NP)(dd^cu_k)^n &= \lim_{k\to\infty}(dd^cu_k)^n\\ &=1_{\{u>-\infty\}}(dd^cu)^n=(NP)(dd^cu)^n,\end{aligned}$$

and the desired conclusion follows. The proof is complete. ■

5. The solvability of the Monge-Ampère equation for the non-polar part in the class $\mathcal{D}(\Omega)$

As discussed in the Introduction, in this section we investigate the existence of weak solutions for non-polar parts $NP(dd^c.)^n$ in the class $\mathcal{D}(\Omega)$ where Ω is a bounded open subset in $\mathbb{C}^n$. Namely, we have the following result.

Theorem 5.1. *Let Ω be a bounded open subset in $\mathbb{C}^n$ and $u, v \in \mathcal{D}^-(\Omega), u \leq v$ and $(dd^c v)^n \leq (dd^c u)^n$ on Ω. Assume that $(dd^c u)^n(\{u = -\infty\}) = 0$ and $(dd^c v)^n$ is carried by a pluripolar subset in Ω and $P[u] = v$. Then for every positive Radon measure μ on Ω satisfying the condition*

$$(dd^c v)^n \leq \mu \leq (dd^c u)^n,$$

there exists a function $w \in \mathcal{D}(\Omega)$ with $u \leq w \leq v$ on Ω and $NP(dd^c w)^n = \mu$ on Ω, $P[w] = v$. Moreover $(dd^c v)^n \leq (dd^c w)^n \leq (dd^c u)^n$.

Proof. The proof of this theorem relies on one of the results in [11]. First, by the hypothesis we may assume that $(dd^c v)^n$ is carried by $\{v = -\infty\}$. Hence, we may write $(dd^c v)^n = 1_{\{v=-\infty\}}(dd^c v)^n$. By Proposition 3.3 in [16] we have $(dd^c v)^n = NP(dd^c v)^n + 1_{\{v=-\infty\}}(dd^c v)^n$. Hence, it follows that $NP(dd^c v)^n = 0$. This yields that v is a model function. Set

$$S = \Big\{h \in PSH^-(\Omega) : h \leq v \ \text{ on } \Omega, NP(dd^c h)^n \geq \mu\Big\}. \tag{5.1}$$

First we show that $S \neq \emptyset$. Indeed, we prove $u \in S$. By the hypothesis $u \in PSH^-(\Omega)$ and $u \leq v$ on Ω. Moreover, by the hypothesis and by Proposition 3.3 in [16] we have

$$\mu \leq (dd^c u)^n = 1_{\{u>-\infty\}}(dd^c u)^n + 1_{\{u=-\infty\}}(dd^c u)^n = NP(dd^c u)^n,$$

as we wanted. Next, set

$$w = \Big(\sup\{h \in S\}\Big)^*$$

where g^* denotes the upper semi-continuous regularization of g. It is obvious $u \leq w \leq v$. By [4] it follows that $w \in \mathcal{D}(\Omega)$. By using Theorem 1.2 in [11], we obtain $NP(dd^c w)^n = \mu$ and $P[w] = v$.

Moreover, we have $(dd^c v)^n \leq \mu = NP(dd^c w)^n = 1_{\{w>-\infty\}}(dd^c w)^n \leq (dd^c w)^n$. Next, we prove that $(dd^c w)^n \leq (dd^c u)^n$. Indeed, from $u \leq w \leq v$ then we have $\{w = -\infty\} \subset \{u = -\infty\}$. Now, by using Proposition 3.3 in [16] we have

$$\begin{aligned}(dd^c w)^n &= 1_{\{w>-\infty\}}(dd^c w)^n + 1_{\{w=-\infty\}}(dd^c w)^n \\ &= NP(dd^c w)^n + 1_{\{w=-\infty\}}(dd^c w)^n \\ &= \mu + 1_{\{w=-\infty\}}(dd^c w)^n \leq (dd^c u)^n + 1_{\{w=-\infty\}}(dd^c w)^n \\ &\leq (dd^c u)^n + 1_{\{u=-\infty\}}(dd^c w)^n. \end{aligned} \tag{5.2}$$

We will prove $1_{\{u=-\infty\}}(dd^c w)^n = 0$. Indeed, by Theorem 8.3 in [2] it is enough to argue locally. More precisely, we work on a ball $\mathbb{B} \Subset \Omega$. Let $\mathbb{B}$ be a ball in

Ω. By (iii) of Remark 2.1 we may assume that $u, w \in \mathcal{E}(\mathbb{B}), u \leq w$ on $\mathbb{B}$ and $\{u = -\infty\} \subset \mathbb{B}$. Then using Lemma 4.1 in [1] we infer that

$$\int\limits_{\{u=-\infty\}} (dd^c u)^n \geq \int\limits_{\{u=-\infty\}} (dd^c w)^n. \tag{5.3}$$

By the hypothesis the left-hand side of (5.3) is equal to 0. Hence, it follows that

$$1_{\{u=-\infty\}}(dd^c w)^n = 0$$

and the desired conclusion follows. Substituting the above identity into the right-hand side of (5.2), we obtain $(dd^c w)^n \leq (dd^c u)^n$. The proof is complete. ■

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