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A SURVEY ON A FEW RECENT PAPERS IN P-ADIC VALUE DISTRIBUTION

Alain Escassut (Clermaont-Ferrand, France)

In memory of Professor Wolfgang Tutschke

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Abstract. In this article, we propose to present several recent results: a new proof of the p-adic Hermite-Lindemann Theorem, a new proof of the p-adic Gel'fond-Schneider Theorem, exceptional values of meromorphic functions and derivatives and the p-adic Nevanlinna theory applied to small functions. We first have to recall the definitions of the p-adic logarithm and exponential.

1 Logarithm and exponential in a p-adic field

Notations: We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the p-adic absolute value and by $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$, which is known to be algebraically closed [7]. In general, we denote by $\mathbb{K}$ an algebraically closed field of characteristic 0 complete with respect to an ultrametric absolute value, such as $\mathbb{C}_p$. The ultrametric absolute value of $\mathbb{K}$ is denoted $|\ .\ |$ while the archimedean absolute value of $\mathbb{C}$ is denoted $|\ .\ |_\infty$.

Let $a \in \mathbb{K}$ and let $R \in \mathbb{R}_+$. We denote by $d(a, R)$ the "closed " disk $\{x \in \mathbb{K} \mid |x - a| \le R\}$ and by $d(a, R^-)$ the "open" disk $\{x \in \mathbb{K} \mid |x - a| < R\}$.

We denote by $\mathcal{A}(\mathbb{K})$ the algebra of power series converging in all $\mathbb{K}$. Given $a \in \mathbb{K}$ and $R > 0$, we denote by $\mathcal{A}(d(a, R^-))$ the algebra of power series $\sum_{j=0}^{\infty} a_n(x-a)^n$ converging in $d(a, R^-)$ and by $\mathcal{A}_b(d(a, R^-))$ the subalgebra of

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functions $f(x) \in \mathcal{A}_b(d(a,R^-))$ that are bounded in $(d(a,R^-))$ and we put $\mathcal{A}_u(d(a,R^-)) = \mathcal{A}(d(a,R^-)) \setminus \mathcal{A}_b(d(a,R^-))$.

Moreover we denote by $H(d(a,r)$ the algebra of power series $\sum_{j=0}^{\infty} a_n(x-a)^n$ converging in $d(a,R)$ called *analytic elements in* $d(a,R)$. Given an element f of $H(d(0,R))$ we put $|f|(r) = \sup_{x\in d(0,R)} |f(x)|$.

We will define the p-adic logarithm and the p-adic exponential and will shortly study them, in connection with the study of the roots of 1. Here, as in [7], we compute the radius of convergence of the p-adic exponential by using results on injectivity.

The following lemma 1.a is easy:

Lemma 1.a: *$\mathbb{K}$ is supposed to have residue characteristic $p \neq 0$. Let $r \in]0,1[$ and for each $n \in \mathbb{N}$, let $h_n(x) = (1+x)^{p^n}$. The sequence h_n converges to 1 with respect to the uniform convergence on $d(0,r)$.*

Notations: We denote by log the real logarithm function of base e. Given a power series $\sum_{j=0}^{\infty} a_j x^j$ converging in $d(0,R^-)$ and given a number $\mu < \log(R)$ we denote by $\nu^+(f,\mu)$ the biggest integer q such that $\sup_{j\geq 0} \log(|a_j)| + j\mu = \log(|a_q)| + q\mu$.

For each $q \in \mathbb{N}^*$ we denote by R_q the positive number such that $\log_p(R_q) = -\dfrac{1}{p^{q-1}(p-1)}$. We denote by $g(x)$ the series $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$.

The following lemma 1.b is well known (Theorem B.13.7 in [7]):

Lemma 1.b: *Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$ be converging in $d(0,R^-)$ and let $r < R$. Then $\nu^+(f,\log(r))$ is the number of zeros of f in $d(0,r)$, taken multiplicity into account.*

Theorem 1.1: *g has a radius of convergence equal to 1. If the residue characteristic of $\mathbb{K}$ is $p \neq 0$, then g is unbounded in $d(0,1^-)$. If the residue characteristic is zero, then $|g(x)|$ is bounded by 1 in $d(0,1^-)$. The function defined in $d(1,1^-)$ as $Log(x) = g(x-1)$ has a derivative equal to $\dfrac{1}{x}$ and satisfies $Log(ab) = Log(a) + Log(b)$ whenever $a,\ b \in d(1,1^-)$.*

Proof. It is clearly seen that the radius of g is 1, because $|n| \geq \dfrac{1}{n}$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. As in the Archimedean context, the property $Log(ab) =$

$Log(a) + Log(b)$ comes from the fact that both Log and the function h_a defined as $h_a(x) = Log(ax)$ have the same derivative. The other statements are immediate.

Notation: When $\mathbb{K}$ has residue characteristic $p \neq 0$, we introduce the group W of the p^s-th roots of 1, i.e., the set of the $u \in \mathbb{K}$ satisfying $u^{p^s} = 1$ for some $s \in \mathbb{N}$.

Recall that analytic elements were defined by M. Krasner and are defined in [7].

Theorem 1.2: *$\mathbb{K}$ is supposed to have residue characteristic $p \neq 0$ (resp. 0). All zeros of Log are of order 1. The set of zeros of the function Log is equal to W, (resp. 1 is the only zero of Log). The restriction of Log to the disk $d(1,(R_1)^-)$ (resp. $d(1,1^-)$) is injective and is a bijection from $d(1,(R_1)^-)$ onto $d(0,(R_1)^-)$ (resp. from $d(1,1^-)$ onto $d(0,1^-)$).*

Proof. It is obvious that the zeros of Log are of order 1 because the derivative of Log has no zero. First, we suppose $\mathbb{K}$ to have residue characteristic $p \neq 0$. Each root of 1 in $d(1,1^-)$ is a zero of Log. Moreover, by Theorem A.6.8 of [7], we know that the only roots of 1 in $d(1,1^-)$ are the p^n-th roots. Now we can check that Log admits no zero other than the roots of 1. Indeed, suppose that a is a zero of Log but is not a root of 1, and for each $n \in \mathbb{N}$, let $b_n = a^{p^n}$. Since b_n belongs to $d(1,1^-)$, by Lemma B.16.1 of [7] we have $\lim\limits_{n\to\infty} b_n = 1$. But obviously $Log(b_n) = 0$ for every $n \in \mathbb{N}$, hence this contradicts the fact that 1 is an isolated zero of Log.

Thus, Log has no zero in the disk $d(1,(R_1)^-)$, except 1 and therefore, by Lemma 1.b the series $f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ satisfies $\nu^+(f, \log r) = 1$ for every $r \in]0, R_1[$, hence $r > \frac{r^n}{|n|}$ for all $r \in]0, R_1[$, for every $n \in \mathbb{N}^*$. Therefore, by Corollary B.14.10 of [7] it is injective in $d(0, R_1^-)$. Then, by Corollary B.13.10 of [7], we see that $Log(d(1, R_1^-)) = d(0, R_1^-)$.

Now we suppose that $\mathbb{K}$ has residue characteristic zero. Then, the function $f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ satisfies $\nu^+(f, \log r) = 1$ for every $r \in]0, 1[$, hence $r > \frac{r^n}{n}$ for all $r \in]0, 1[$, for every $n \in \mathbb{N}^*$. Therefore, f has no zero different from 1 in $d(0,1^-)$ and, by Corollary B.14.10 of [7], is injective in $d(0,1^-)$. Then by Corollary B.13.10 of [7] we see that $Log(d(1,1^-)) = d(0,1^-)$. This ends the proof.

Corollary 1.A: *$\mathbb{K}$ is supposed to have residue characteristic* 0*. There is no root of* 1 *in* $d(1,1^-)$*, except* 1*.* **Proof.** Indeed any root of 1 should be a zero of *Log* in $d(1,1^-)$.

Notations: If $\mathbb{K}$ has residue characteristic $p \neq 0$, we first denote by *exp* the inverse (or reciprocal) function of the restriction of *Log* to $d(1,R_1^-)$, which obviously is a function defined in $d(0,R_1^-)$, with values in $d(1,R_1^-)$. If $\mathbb{K}$ has residue characteristic 0 we denote by *exp* the inverse function of *Log*, which is obviously defined in $d(0,1^-)$ and takes values in $d(1,1^-)$.

Theorem 1.3: *$\mathbb{K}$ is supposed to have residue characteristic $p \neq 0$ (resp. $p = 0$). The function exp belongs to $\mathcal{A}_b(d(0,R_1^-))$ (resp. $\mathcal{A}_b(d(0,1^-))$), is a bijection from $d(0,R_1^-)$ onto $d(1,R_1^-)$ (resp. from $d(0,1^-)$ onto $d(1,1^-)$), and satisfies $exp(x) = exp'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ whenever $x \in d(0,R_1^-)$ (resp. $x \in d(0,1^-)$). Moreover, the disk of convergence of its series is equal to $d(0,R_1^-)$ (resp. $d(0,1^-)$). Further, if $p \neq 0$, then exp is not an analytic element on $d(0,R_1^-)$.*

Proof. By Corollary B.14.15 of [7] we know that the function *exp* belongs to $\mathcal{A}_b(d(0,R_1^-))$ (resp. $\mathcal{A}_b(d(0,1^-))$) and is obviously a bijection from $d(0,R_1^-)$ onto $d(1,R_1^-)$ (resp. from $d(0,1^-)$ onto $d(1,1^-)$). As it is the reciprocal of *Log*, it must satisfy $exp(x) = exp'(x)$ for all $x \in d(0,R_1^-)$ (resp. $x \in d(0,1^-)$) and, therefore, $exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ whenever $x \in d(0,R_1^-)$ (resp. $x \in d(0,1^-)$). Thus the radius of convergence r is at least R_1 (resp. 1). If the residue characteristic is 0, it is obviously seen that the series cannot converge for $|x| = 1$, hence the disk of convergence is $d(0,1^-)$.

Now we suppose that the residue characteristic is $p \neq 0$. Suppose that the power series of *exp* converges in $d(0,R_1)$. Then *exp* has continuation to an analytic element element on $d(0,R_1)$. On the other hand, since $\nu(f,\log r) = 1$ for all $r \in]0,R_1[$, we have $\nu^-(f,\log R_1) = 1$ and then by Theorem B.13.9 of [7] $Log(d(1,R_1))$ is equal to $d(0,R_1)$. Hence, we can consider $exp(Log(x))$ in all the disk $d(0,R_1)$. By Corollary B.3.3 of [7] this is an analytic element element on $d(1,R_1)$. But this element is equal to the identity in all of $d(1,R_1^-)$ and, therefore, in all of $d(1,R_1)$. Of course this contradicts the fact that *Log* is not injective in the circle $C(1,R_1)$. This finishes proving that the disk of convergence of *exp* is just $d(0,R_1^-)$.

Notations: Henceforth, we put $e^x = exp(x)$.

Theorem 1.4: *$\mathbb{K}$ is supposed to have residue characteristic $p \neq 0$. Let $x \in d(0, R_1^-)$. Then e^x is algebraic over $\mathbb{Q}_p$ if and only if so is x. Let $u \in d(0, 1^-)$. Then $\log(1+u)$ is algebraic over $\mathbb{Q}_p$ if and only if so is u.*

Proof. By Theorem B.5.24 of [7], if x is algebraic over $\mathbb{Q}_p$, so is e^x. Similarly, if u is algebraic over $\mathbb{Q}_p$, so is $\log(1+u)$. Consequently, suppose that e^x is algebraic over $\mathbb{Q}_p$. Then e^x is of the form $1+t$ with $|t| < 1$, hence $\log(1+t)$ is algebraic over $\mathbb{Q}_p$. But then, $\log(1+t) = \log(e^x) = x$, hence x is algebraic over $\mathbb{Q}_p$. Now, more generally, suppose $\log(1+u)$ is algebraic over $\mathbb{Q}_p$, with $|u| < 1$. Take $q \in \mathbb{N}$ such that $|p^q \log(1+u)| < R_1$. We have $p^q \log(1+u) = \log((1+u)^{p^q})$. Since $|p^q \log(1+u)| < R_1$, we have $|\log((1+u)^{p^q})| < R_1$, hence $exp\big(\log((1+u)^{p^q})\big) = (1+u)^{p^q}$. Consequently, $(1+u)^{p^q}$ is algebraic over $\mathbb{Q}_p$ and hence so is u.

We can show a similar result when $p = 0$.

Theorem 1.5: *$\mathbb{K}$ is supposed to have residue characteristic 0. Let $x \in d(0, 1^-)$. Then e^x is algebraic over $\mathbb{Q}_p$ if and only if so is x. Let $u \in d(0, 1^-)$. Then $\log(1+u)$ is algebraic over $\mathbb{Q}_p$ if and only if so is u.*

The following proposition 1.6 will be used in the poof of Theorem 2.3 and is proven by induction, similarly as (1.4.2) in [16].

Proposition 1.6: *Let $P_1, ..., P_q \in \mathbb{K}[X]$ different from 0 and let $w_1, ..., w_q \in \mathbb{K}$ be pairwise distinct. Let $F(x) = \sum_{j=1}^{q} P_j(x) e^{w_j x}$. Then F is not identically zero.*

2 Hermite-Lindemann's and Gel'fond-Schneider's Theorems in ultrametric fields

We will use the following classical notation:

Notation: We will denote by $\mathcal{K}$ an algebraically closed complete ultrametric extension of $\mathbb{Q}$ of residue characteristic 0.

We will denote by U the disk $d(0, 1)$ and by D_0 the disk $d(0, 1^-)$ in the field $\mathbb{K}$ no matter what the residue characteristic.

If the residue characteristic of $\mathbb{K}$ is $p > 0$ we put $R_1 = p^{\frac{-1}{p-1}}$ and denote by D_1 the disk $d(0, R_1^-)$.

Given an algebraic number $a \in \mathbb{C}_p$ (resp. $a \in \mathcal{K}$) and $a_1, a_2, ..., a_q$ its conjugates over $\mathbb{Q}$ (with $a_1 = a$), we put $\overline{|a|} = \max_{1 \leq j \leq q} |a_j|$ and we denote by

$den(a)$ its *smallest denominator*, i.e. the smallest positive integer q such that qa is an algebraic integer. Then we put $s(a) = \max(\log \overline{|a|}, \log(den(a)))$ and $s(a)$ is called *the size* of a. More generaly we call denominator of a number a all positive integer multiple of its smallest denominator.

Given a polynomial $P(X_1, ..., X_q) \in \mathbb{Z}[X_1, ..., X_q]$, we denote by $H(P)$ the supremum of the archimedean absolute values of its coefficients.

Given a positive real number a, we denote by $[a]$ the largest integer n such that $n \le a$.

Hermite-Lindemann's theorem is well known in complex analysis. The same holds in p-adic analysis. The first proof was presented in 1930 by K. Malher [13]. This proof given in [13] is written in German and uses symbols which are not currently known. Here we present a new proof using classical methods in transcendental processes that are maybe easier to understand.

We will need Siegel's Lemma in all the following theorems of this chapter. We will choose a particular form of this famous lemma [16] whose formulation is due to M. Mignotte:

Lemma 2.a (Siegel): *Let E be a finite extension of $\mathbb{Q}$ of degree q and let $\lambda_{i,j}$ $1 \le i \le m$, $1 \le j \le n$ be elements of E integral over $\mathbb{Z}$. Let $M = \max(\overline{|\lambda_{i,j}|}\ 1 \le i \le m,\ |\ 1 \le j \le n)$ and let $(\mathcal{S})$ be the linear system $\{\sum_{j=1}^{n} \lambda_{i,j} x_j = 0,\ 1 \le i \le m\}$. There exists solutions $(x_1, ..., x_n)$ of $(\mathcal{S})$ such that $x_j \in \mathbb{Z}$ $\forall j = 1, ..., n$ and*

$$\log(|x_j|_\infty) \le \log(M)\frac{qm}{n - qm} + \frac{\log(2)}{2}\ \forall j = 1, ..., n.$$

Lemma 2.b will be necessary in the proof of Theorem 2.4 and is easily proven in [16] since its proof implies no change in the field $\mathbb{K}$ since it only concerns algebraic numbers

Lemma 2.b: *Let $a_1, ..., a_q \in \mathbb{K}$ be algebraic over $\mathbb{Q}$, let $P(X_1, ..., X_q) \in \mathbb{Z}[X_1, ..., x_q]$ be such that $\deg_{X_j}(P) \le r_j$ $1 \le j \le q$ and let $\beta = P(a_1 ... a_q)$. Then β is algebraic over $\mathbb{Q}$, $d(a_1)^{r_1} ... d(a_q)^{r_q}$ is a multiple of $den(\beta)$ and we have*

$$s(\beta) \le \log H(P) + \sum_{j=1}^{q}(r_j s(a_j) + \log(r_j) + 1)$$

Theorem 2.1 (Hermite-Lindemann): *Suppose that $\mathbb{K}$ has residue characteristic $p > 0$. Let $\alpha \in D_1$ be algebraic. Then e^α is transcendental.*

Proof. We suppose that α and e^{α} are algebraic. Let $h = |\alpha|$. Let E be the field $\mathbb{Q}[\alpha, e^{\alpha}]$, let $q = [E : \mathbb{Q}]$ and let w be a common denominator of α and e^{α}. We will construct a sequence of polynomials $(P_N(X,Y))_{N\in\mathbb{N}}$ in two variables such that $\deg_X(P_N) = [\frac{N}{\log(N)}]$, $\deg_Y(P_N) = [(\log N)^3]$ and such that the function $F_N(x) = P_N(x, e^x)$ satisfiy further, for every $s = 0, ..., N-1$ and for every $j = 0, ..., [\log(N)]$

$$\frac{d^s}{dx^s} F_N(j\alpha) = 0.$$

According to formal computations in the proof of Hermite Lindemann's Theorem in the complex context, (Theorem 3.1.1 in [16]) we have

$$\frac{d^M F_N(\gamma_N)}{dx^M} = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \Big(\frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!}\Big)\Big(\frac{l!}{(u_1(N)-\sigma)!}\Big).$$
$$m^{u_1(N)-\sigma}(1) j^{u_1(N)-\sigma}.(\alpha)^{u_1(N)-\sigma}.(e^{\alpha})^{ju_2(N)}.$$

We put $u_1(N) = \deg_X(P_N)$, $u_2(N) = \deg_Y(P_N)$. We will solve the system

$$w^{u_1(N)+u_2(N)} \frac{d^s}{dx^s} F_N(j\alpha) = 0, \quad 0 \le s \le N-1, \ j = 0, ..., [\log(N)]$$

where the undeterminates are the coefficients $b_{l,m,N}$ of P_N. We then write the system under the form

$$\sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{\min(s,l)} \Big(\frac{s!}{\sigma!(s-\sigma)!}\Big)\Big(\frac{l!}{(l-\sigma)!}\Big) m^{s-\sigma}.j^{l-\sigma}.$$

$$(w\alpha)^{l-\sigma}(we^{\alpha})^{jm}.w^{u_1(N)-(l-\sigma)+u_2(N)-jm} = 0. \tag{2}$$

That represents a system of $N[\log(N)]$ equations of at least $N([\log(N)])^2$ undeterminates, with coefficients in E, integral over $\mathbb{Z}$.

According to formal computations of Hermite-Lindemann's Theorem in the complex context (Theorem 3.1.1 in [16]), it appears that in the system (2), each factor $\Big(\frac{s!}{\sigma!(s-\sigma)!}\Big)$, $\Big(\frac{l!}{(l-\sigma)!}\Big)$, $m^{s-\sigma}$, $j^{l-\sigma}$, $(w\alpha)^{l-\sigma}$, $(we^{\alpha})^{jm}$, $w^{u_1(N)-(l-\sigma)+u_2(N)-jm}$ admits a bounding of the form $SN(\log(\log(N))$ when N goes to $+\infty$. On one hand $w^{u_1(N)+u_2(N)}$ is a common denominator and we have

$$\log(w^{u_1(N)+u_2(N)}) \le \log(\omega)\Big(\frac{N}{\log(N)} + (\log(N)^3\Big)$$

and hence we have a constant $T > 0$ such that

$$\log(w^{u_1(N)+u_2(N)}) \le \frac{TM}{\log M}. \tag{3}$$

Next we notice that

$$\log\Big(\frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!}\Big) \le u_1(N)\log(u_1(N)) \le \frac{N}{\log(N)}\log(\frac{N}{\log(N)}) \le N \tag{4}$$

and similarly,

$$\log\big(\frac{l!}{(u_1(N)-\sigma)!}\big) \le u_1(N)\log(u_1(N)) \le N. \tag{5}$$

and

$$\log(m^{u_1(N)-\sigma}) \le \frac{3N}{\log(N)}\log(\log(N)). \tag{6}$$

Now, we check that

$$\log\Big(j^{u_1(N)-\sigma}.(|\overline{\alpha}|)^{u_1(N)-\sigma}.(|\overline{e^{\alpha}}|)^{ju_2(N)}\Big) \le N + \frac{N}{\log(N)}\log(|\overline{\alpha}|)+ \log(N)(\log(N))^3\log(|\overline{e^{\alpha}}|)$$

and hence there exists a constant $L > 0$ such that

$$\log\Big(j^{u_1(N)-\sigma}.(|\overline{\alpha}|)^{u_1(N)-\sigma}.(|\overline{e^{\alpha}}|)^{ju_2(N)}\Big) \le LN. \tag{7}$$

Therefore by (2), (3), (4), (5), (6) and (7) we have a constant $C > 0$ such that each coefficient a of the system satisfies

$$s(a) \le CN(\log(\log(N)). \tag{8}$$

By Siegel's Lemma 2.a and by (8) there exist integers $b_{l,m,N}$, $0 \le l \le u_1(N)$, $0 \le m \le u_2(N)$ in $\mathbb{Z}$ such that

(9)

$$0 < \max_{l\le u_1(N),\ m\le u_2(N)} \log(|b_{l,m,N}|_\infty) \le \frac{qN\log(N)}{N(\log(N))^2 - qN\log(N)}(CN\log(\log(N))$$

and such that the function

$$F_N(x) = \sum_{l=0}^{u_1(N)}\sum_{m=0}^{u_2(N)} b_{l,m;N}x^l e^{mx} \tag{10}$$

satisfies

$$\frac{d^s}{dx^s}F_N(j\alpha) = 0,\ 0 \le s \le N-1,\ j = 0, 1, ..., [\log(N)].$$

Now, by (9), we can check that there exists a constant $G > 0$ such that

$$\max_{l \le u_1(N),\ m \le u_2(N)} (\log(|b_{l,m,N}|_\infty) \le \frac{GN\log(\log(N))}{\log(N)}. \tag{11}$$

The function F_N defined in (10) belongs to $\mathcal{A}(D_1)$ and is not identically zero, hence at least one of the numbers $\frac{d^s}{dx^s}F_N(0)$ is not null. Let M be the biggest of the integers such that $\frac{d^s}{dx^s}F_N(j\alpha) = 0\ \forall s = 0, ..., M-1,\ j = 0, 1, 2, ..., [\log(N)]$. Thus we have $M \ge N$ and there exists $j_0 \in \{0, 1, ..., [\log(N)]\}$ such that $\frac{d^M}{dx^M}F_N(j_0\alpha) \ne 0$. We put $\gamma_N = \frac{d^M}{dx^M}F_N(j_0\alpha)$.

Let us now give an upper bound of $s(\gamma_N)$. On one hand $w^{u_1(N)+u_2(N)}$ is a common denominator and by (2) we have a constant $T > 0$ such that

$$\log(w^{u_1(N)+u_2(N)}) \le \frac{TM}{\log M}.$$

On the other hand, by (1) we have

$$\frac{d^M F_N(\gamma_N)}{dx^M} = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \Big(\frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!}\Big)\Big(\frac{l!}{(u_1(N)-\sigma)!}\Big).$$
$$m^{u_1(N)-\sigma}.j^{u_1(N)-\sigma}.(\alpha)^{u_1(N)-\sigma}.(e^{\alpha})^{ju_2(N)}.$$

Now, by (2), (3), (6), (7), (8), (10) and taking into account that the number of terms is bounded by $N(\log N)^2$, we can check that there exists a constant B such that

$$s(\gamma_N) \le BN. \tag{12}$$

Let us now give an upper bound of $|\gamma_N|$. For convenience, we first suppose that $j_0 = 0$, hence $\frac{d^M}{dx^M}F_N(0) \ne 0$. Set $h = |\alpha|$. Then by Theorem B.9.1 of [7] we have $|\gamma_N| \le \frac{|F_N|(h)}{h^M}$. Moreover, we notice that F_N admits at least $M[\log(M)]$ zeros in $d(0, h)$ and therefore by Corollary B.13.30 of [7] we have $|F_N|(h) \le \Big(\frac{h}{R_1}\Big)^{M[\log(M)]}$ because $|F_N|(r) \le 1\ \forall r < R_1$. Consequently, $|\gamma_N| \le \frac{h^{M(\log(M-1)}}{(R_1)^{M\log M}}$ and hence

$$\log(|\gamma_N|) \le M(\log(M)-1)(\log(h)) - M\log(M)(\log(R_1))).$$

Let $\lambda = \log(h) - \log(R_1)$. Then $\lambda < 0$. And we have $\log(|\gamma_N|) \leq \lambda M \log(M) - M \log(h)$, therefore there exists a constant $A > 0$ such that

$$\log(|\gamma_N|) \leq -AM \log(M). \tag{13}$$

Let us now stop assuming that $j_0 = 0$. Putting $z = x - j\alpha$ and $g(z) = f(x)$, since all points $j\alpha$ belong to $d(0, h)$, it is immediate to go back to the case $j_0 = 0$, which confirms (13) in the general case. But now, by Lemma A.8.10 in [7], relations (12) and (13) make a contradiction to the relation $-2qs(\gamma_N) \leq \log(|\gamma_N|)$ satisfied by algebraic numbers and show that γ_N is transcendental. But then, so is e^α.

Example: Let $Q(x) \in \mathbb{Z}[x]$. Then $e^{pQ(p)}$ is transcendental. Moreover, if Q is monic, and if α is a zero of Q, then $|p\alpha| \leq \frac{1}{p}$ because Q is monic and obviously $p\alpha$ is algebraic, hence $e^{p\alpha}$ is transcendental.

In the field of characteristic 0, $\mathcal{K}$ such as Levi-Civita's field [15], we have a similar version:

Theorem 2.2: *Let $\alpha \in \mathcal{K}$ be algebraic, such that $|\alpha| < 1$. Then e^α is transcendental over $\mathbb{Q}$.*

Proof. Everything works in $\mathcal{K}$ as in a field of residue characteristic $p \neq 0$ up to Relation (8) in the proof of Theorem 2.1. Here we can replace R_1 by 1 and therefore the conclusion is the same as in Theorem 2.1.

Similarly as Hermite-Lindermann's Theorem, Gelfond-Schneider's Theorem is well known in the field $\mathbb{C}$ and has an analogue in an ultrametric field.

In the proof of Theorem 2.4 we will need the following theorem:

Theorem 2.3: *Let $b_1, ..., b_n \in D_1$ (resp. in D_0). the functions $x,\ e^{b_1 x}, ..., e^{b_n x}$ are algebraically independant over $\mathbb{K}$ (resp. over $\mathcal{K}$) if and only if $b_1, ..., b_n$ are $\mathbb{Q}$-linearly independant.*

Theorem 2.4 (Gel'fond-Schneider): *$\mathbb{K}$ is supposed to have residue characteristic $p \neq 0$. Let $\ell \in D_1,\ \ell \neq 0$, and let $b \notin \mathbb{Q}$ belong to $\mathbb{K}$ be such that $b\ell \in D_1$. Then at least one of the three numbers $a = e^\ell,\ b,\ e^{b\ell}$ is transcendental.*

Proof. A large part of the proof does not involve the topology of the feld $\mathbb{K}$ and hence is similar to the proof in the field $\mathbb{C}$ [16] where we can copy many technical relations. We suppose that $a = e^\ell$, b and $e^{b\ell}$ are algebraic over $\mathbb{Q}$. Let $L = \mathbb{Q}[e^\ell,\ b,\ e^{b\ell}]$ and let $\delta = [L : \mathbb{Q}]$ and let d be a common denominator of $b, e^\ell, e^{b\ell}$.

Put $S = \max(1, |b|)$, $T \in]S, \frac{R_1}{|\ell|}[$, $\sigma = \log(\frac{T}{S})$, $\tau = \log T$, $\Lambda = d(0,S)$ and $\Delta = d(0,T)$. We will consider integers N of the form q^2, with $q \in \mathbb{N}$ and we will first show that there exists a non-identically zero polynomial $P_N(X,Y) \in \mathbb{Z}[X,Y]$ such that $\deg_X(P_N) \le N^{\frac{3}{2}}$, and $\deg_Y(P_N) \le 2\delta N^{\frac{1}{2}}$ such that the function $F_N(x)$ defined in Δ by $F_N(x) = P_N(x, e^{\ell x})$ satisfy

$$F_N(i + jb) = 0 \ \forall i = 1, ..., N, \ \forall j = 1, ..., N.$$

In order to find P_N, let us write it

$$\sum_{h=0}^{N^{\frac{3}{2}}-1} \sum_{k=0}^{2\delta N^{\frac{1}{2}}-1} C_{h,k}(N) X^h Y^k$$

with $C_{h,k}(N) \in \mathbb{Z}$ and consider the system of equations where the $C_{h,k}(N)$ are the undeterminates:

$$d^{(4\delta+1)N^{\frac{3}{2}}}.F_N(i + jb) = 0 \ (1 \le i \le N; 1 \le j \le N).$$

Thus, we obtain a system of N^2 equations of $2\delta N^2$ undeterminates in $\mathbb{Z}$, with coefficients in L. By Lemma 2.b, these coefficients have size bounded by

$$N^{\frac{3}{2}} \log(N) + N^{\frac{3}{2}}(8\delta + 2)\log(d) + \log(1 + \overline{|b|}) + 2\delta \log(\overline{|e^{\ell + b\ell}|}) \le \frac{3}{2} N^{\frac{3}{2}} \log(N).$$

By Lemma 2.a we can find in $\mathbb{Z}$ a family of integers not all equal to zero, $(C_{h,k}(N), \ 0 \le N^{\frac{3}{2}} - 1, \ 0 \le k \le 2\delta N^{\frac{1}{2}} - 1)$ satisfying

$$\log\big(\max_{h,k} |C_{h,k}(N)|_\infty\big) \le 2N^{\frac{3}{2}} \log N \Big(\frac{\delta N^2}{2\delta N^2 - \delta N^2}\Big) = 2N^{\frac{3}{2}} \log N$$

such that the function F_N defined by $F_N(x) = P_N(x, e^{\ell x})$ satisfies $F_N(i+jb) = 0 \ \forall i = 1, ..., N, \ j = 1, ..., N$.

Now we can check the function F_N is an analytic element in every disk of the form $d(0,r)$ such that $r|\ell| < R_1$ and hence in $\Delta = d(0,T)$ [7]. Since the power of x in the various terms is at most $N^{\frac{3}{2}}$ and since all coefficients are integers, we can check that $\log(|F_N|(T)) \le \tau N^{\frac{3}{2}}$. On the other hand, since the polynomial P_N is not identically zero, by Proposition 1.6 F_N is not identically zero and then, by classical results [7], the function F_N has finitely many zeros in Λ. Particularly, there exists a point of the form $i+jb$ such that $F_N(i+jb) \ne 0$. Consequently there exists $M \ge N$ such that $F_N(i + jb) = 0 \ \forall i \le M, \ \forall j \le M$ and there exists a point γ_N of the form $i_0 + j_0 b$ such that $F_N(\gamma_N) \ne 0$ with $M < i_0 \le M + 1, \ M < j_0 \le M + 1$. Consequently the number of zeros of F_N

in Λ is at least M^2. Then by Corollary B.13.30 in [7] we have $\log(|F_N(\gamma_N)|) \leq \tau N^{\frac{3}{2}} - \sigma M^2$, hence there exists $\lambda > 0$ such that

$$\log(|F_N(\gamma_N)|) \leq -\lambda M^2 \ \forall N \in \mathbb{N}. \tag{1}$$

By definition neither σ nor τ depend on N, hence neither does λ.

On the other hand, by Lemma 2.b we can check that $s\big(F_N(\gamma_N)\big)$ satisfies an inequality of the form $s\big(F_N(\gamma_N)\big) \leq AM^{\frac{3}{2}}\log(M)$ which by (1) contradicts the inequality $-2\delta s\big(F_N(\gamma_N)\big) \leq \log\big(|F_N(\gamma_N)|\big)$ and this ends the proof.

Example: Let $\ell = pe^p$ and let let $b \notin \mathbb{Q}$ be such that $|b| \leq 1$. Then at least one of the 3 numbers $\ell,\ b,\ e^{b\ell}$ is transcendental.

Theorem 2.5 (Gel'fond-Schneider in zero residue characteristic): *Let $\mathcal{K}$ be an algebraically closed complete ultrametric field whose residue characteristic is 0. Let $\ell \in D_0,\ \ell \neq 0$, and let $b \notin \mathbb{Q}$ belong to $\mathcal{K}$ and be such that $b\ell \in D_0$. Then at least one of the three numbers $a = e^{\ell},\ b,\ e^{b\ell}$ is transcendental.*

Proof. The proof is identical to the proof of Theorem 2.4 except that T now belongs to $]S, \dfrac{1}{|\ell|}[$.

3 Nevanlinna Theory in $\mathbb{K}$ and in an open disk

Notations: We denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$. Let $d(a, R^-)$ be a disk in $\mathbb{K}$. We denote by $\mathcal{M}(d(a, R^-))$ the field of fractions $\mathcal{A}(d(a, R^-))$ and by $\mathcal{M}_b(d(a, R^-))$ the field of fractions $\mathcal{A}_b(d(a, R^-))$. Finally we put $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Given two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ or $f,\ g \in \mathcal{M}(d(a, R^-))$ $(a \in \mathbb{K},\ R > 0)$, we will denote by $W(f, g)$ the Wronskian of f and g: $f'g - fg'$.

Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. Let $f \in \mathcal{M}_u(d(\alpha, R^-)))$. A value $b \in \mathbb{K}$ will be called *a quasi-exceptional value for f* if $f - b$ has finitely many zeros in $\mathbb{K}$ (resp. in $(\alpha, R^-)))$ and it will be called *an exceptional value for f* if $f - b$ has no zero in $\mathbb{K}$ (resp. in $d(\alpha, R^-)$).

We have the follwing result:

Theorem 3.1: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp.$f \in \mathcal{M}_u(d(a, R^-))$). Then f amits at most one quasi-exceptional value. Moreover, if $f \in \mathcal{A}(\mathbb{K})$ (resp.$f \in \mathcal{A}_u(d(a, R^-))$) then f amits no quasi-exceptional value*

The Nevanlinna Theory was made by Rolf Nevanlinna on complex functions [14], and widely used by many specialists of complex functions, particularly Walter Hayman [10]. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f-b,\ b\in\mathbb{C}$.

A similar theory for functions in a p-adic field was constructed and correctly proved by A. Boutabaa [5] in the field $\mathbb{K}$, after some previous work by Ha Huy Khoai [9]. See also [11]. In [6] the theory was extended to functions in $\mathcal{M}(d(0,R^-))$ by taking into account Lazard's problem [12]. A new extension to functions out of a hole was made in [7] but we won't describe it because we would miss place. Here we will only give an abstract of the ultrametric Nevanlinna Theory in order to give the new theorems on q small functions.

Notations: Recall that given three functions $\phi,\ \psi,\ \zeta$ defined in an interval $J=]a,+\infty[$ (resp. $J=]a,R[$), with values in $[0,+\infty[$, we shall write $\phi(r)\leq\psi(r)+O(\zeta(r))$ if there exists a constant $b\in\mathbb{R}$ such that $\phi(r)\leq\psi(r)+b\zeta(r)$. We shall write $\phi(r)=\psi(r)+O(\zeta(r))$ if $|\psi(r)-\phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r)\leq\psi(r)+o(\zeta(r))$ if there exists a function h from $J=]a,+\infty[$ (resp. from $J=]a,R[$) to $\mathbb{R}$ such that $\lim\limits_{r\to+\infty}\dfrac{h(r)}{\zeta(r)}=0$ (resp. $\lim\limits_{r\to R}\dfrac{h(r)}{\zeta(r)}=0$) and such that $\phi(r)\leq\psi(r)+h(r)$. And we shall write $\phi(r)=\psi(r)+o(\zeta(r))$ if there exists a function h from $J=]a,+\infty[$ (resp. from $J=]a,R[$) to $\mathbb{R}$ such that $\lim\limits_{r\to+\infty}\dfrac{h(r)}{\zeta(r)}=0$ (resp. $\lim\limits_{r\to R}\dfrac{h(r)}{\zeta(r)}=0$) and such that $\phi(r)=\psi(r)+h(r)$.

Throughout the next paragraphs, we will denote by I the interval $[t,+\infty[$ and by J an interval of the form $[t,R[$ with $t>0$.

We have to introduce the counting function of zeros and poles of f, counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: Next, let $f=\dfrac{h}{l}\in\mathcal{M}(\mathbb{K})$ (resp. $f=\dfrac{h}{l}\in\mathcal{M}(d(a,R^-))$). The order of a zero α of f will be denoted by $\omega_\alpha(f)$. Next, given any point $\alpha\in\mathbb{K}$ resp. $\alpha\in d(a,R^-)$), the number $\omega_\alpha(h)-\omega_\alpha(l)$ does not depend on the functions $h,\ l$ chosed to make $f=\dfrac{h}{l}$. Thus, we can generalize the notation by setting $\omega_\alpha(f)=\omega_\alpha(h)-\omega_\alpha(l)$. We then denote by $Z(r,f)$ the counting function of zeros of f in $d(0,r)$ in the following way.

Let $(a_n),\ 1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n.

We set $Z(r,f) = \max(\omega_0(f), 0)\log r + \sum_{n=1}^{\sigma(r)} s_n(\log r - \log|a_n|)$ and so, $Z(r,f)$ is called *the counting function of zeros of f in $d(0,r)$, counting multiplicity.*

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f without multiplicity:

$\overline{Z}(r,f) = \overline{\omega_0}(f)\log r + \sum_{n=1}^{\sigma(r)} (\log r - \log|a_n|)$ and so, $\overline{Z}(r,f)$ is called *the counting function of zeros of f in $d(0,r)$ ignoring multiplicity.*

In the same way, considering the finite sequence $(b_n),\ 1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n, we put

$N(r,f) = \max(-\omega_0(f), 0)\log r + \sum_{n=1}^{\tau(r)} t_n(\log r - \log|b_n|)$ and then $N(r,f)$ is called *the counting function of the poles of f, counting multiplicity*

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega_0}}(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega_0}}(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$\overline{N}(r,f) = \overline{\overline{\omega_0}}(f)\log r + \sum_{n=1}^{\tau(r)} (\log r - \log|b_n|)$ and then $\overline{N}(r,f)$ is called *the counting function of the poles of f, ignoring multiplicity*

Now we can define the the Nevanlinna function $T(r,f)$ in I or J as

$T(r,f) = \max(Z(r,f), N(r,f))$ and the function $T(r,f)$ is called *characteristic function of f or Nevanlinna function of f.*

Finally, if S is a subset of $\mathbb{K}$ we will denote by $Z_0^S(r,f')$ the counting function of zeros of f', excluding those which are zeros of $f - a$ for any $a \in S$.

Remark: If we change the origin, the functions $Z,\ N,\ T$ are not changed, up to an additive constant.

In a p-adic field such as $\mathbb{K}$, the first Main Theorem is almost immediate.

Theorem 3.2: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0,R^-))$) have no zero and no pole at 0. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r,f) - N(r,f)$.*

Then we can derive Theorem 3.3 (Theorem C.4.3 in [7])

Theorem 3.3: *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{M}(d(0,R^-))$*). Then* $Z(r,fg) \leq Z(r,f)+Z(r,g),\ N(r,fg) \leq N(r,f)+N(r,g),\ T(r,fg) \leq T(r,f)+T(r,g),\ \ T(r,f+g) \leq T(r,f)+T(r,g)+O(1),\ T(r,cf) = T(r,f)\ \forall c \in \mathbb{K}^*,$ $T(r,\frac{1}{f}) = T(r,f)),\ \ T(r,\frac{f}{g}) \leq T(r,f)) + T(r,g).$

Suppose now $f,g \in \mathcal{A}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{A}(d(0,R^-))$*). Then* $Z(r,fg) = Z(r,f)+Z(r,g),\ T(r,f) = Z(r,f)),\ T(r,fg) = T(r,f)+T(r,g)+O(1)$ *and* $T(r,f+g) \leq \max(T(r,f),T(r,g))$*. Moreover, if* $\lim\limits_{r\to+\infty} T(r,f)-T(r,g) = +\infty$ *then* $T(r,f+g) = T(r,f)$ *when* r *is big enough.*

Corollary 3.A: *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}(d(0,R^-))$*). Then*

$$Z(r,\frac{f'}{f}) - N(r,\frac{f'}{f}) \leq -\log r + O(1).$$

Thus we have Theorem 3.4 (Theorem C.4.8 in [7])

Theorem 3.4 (First Main Fundamental Theorem): *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp. let* $f,\ g \in \mathcal{M}(d(0,R^-))$*). Then* $T(r,f+b) = T(r,f)+O(1)$*. Let* h *be a Moebius function. Then* $T(r,f) = T(r,h\circ f)+O(1)$*. Let* $P(X) \in \mathbb{K}[X]$*. Then* $T(r,P(f)) = \deg(P)T(r,f)+O(1)$ *and* $T(r,f'P(f) \geq T(r,P(f))$*.*

Suppose now $f,g \in \mathcal{A}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{A}(d(0,R^-))$*). Then* $Z(r,fg) = Z(r,f)+Z(r,g),\ T(r,f) = Z(r,f)),\ T(r,fg) = T(r,f)+T(r,g)+O(1)$ *and* $T(r,f+g) \leq \max(T(r,f),T(r,g))$*. Moreover, if* $\lim\limits_{r\to+\infty} T(r,f)-T(r,g) = +\infty$ *then* $T(r,f+g) = T(r,f)$ *when* r *is big enough.*

The following Theorem 3.5 is a good way to obtain the famous Second Main Theorem (Theorem C.4.24 in [7]).

Theorem 3.5: *Let* $f \in \mathcal{M}(\mathbb{K})$ *and let* $a_1,...,a_q \in \mathbb{K}$ *be distinct. Then*

$$(q-1)T(r,f) \leq \max_{1\leq k\leq q}\Big(\sum_{j=1,j\neq k}^{q} Z(r,f-a_j)\Big) + O(1).$$

Theorem 3.6 (Second Main Theorem, Theorem C.4.24 in [7]): *Let* $\alpha_1,...,\alpha_q \in \mathbb{K}$*, with* $q \geq 2$*, let* $S = \{\alpha_1,...,\alpha_q\}$ *and let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}_u(d(0,R^-))$*). Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r, f-\alpha_j) + \overline{N}(r,f) - Z_0^S(r,f') - \log r + O(1) \quad \forall r \in I$$

(resp. $\forall r \in J$*).*

Now we can easily deduce the following corollaries:

Corollary 3.B: *Let* $a_1, a_2 \in \mathbb{K}$ *(*$a_1 \neq a_2$*) and let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2$*). Then* $f = g$.

Remark: Corollary 3.B does not hold in complex analysis. Indeed, let $f(z) = e^z,\ g(z) = e^{-z}$, let $a_1 = 1,\ a_2 = -1$. Then $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ $(i = 1,\ 2)$, though $f \neq g$.

Corollary 3.C: *Let* $a_1,\ a_2,\ a_3 \in \mathbb{K}$ *(*$a_i \neq a_j\ \forall i \neq j$*) and let* $f,\ g \in \mathcal{A}_u(d(a, R^-))$ *(resp.*$f,\ g \in \mathcal{A}_u(D)$ *) satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3$*). Then* $f = g$.

Corollary 3.D: *Let* $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{K}$ *(*$a_i \neq a_j\ \forall i \neq j$*) and let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3,\ 4$*). Then* $f = g$.

Corollary 3.E: *Let* $a_1,\ a_2,\ a_3,\ a_4,\ a_5 \in \mathbb{K}$ *(*$a_i \neq a_j\ \forall i \neq j$*) and let* $f,\ g \in \mathcal{M}_u(d(a, R^-)))$ *(resp.* $f,\ g \in \mathcal{M}_u(D)$ *satisfy* $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ *(*$i = 1,\ 2,\ 3,\ 4,\ 5$*). Then* $f = g$.

Remark: Let $f(x) = \dfrac{x}{3x-1},\ g(x) = \dfrac{x^2}{x^2+2x-1}$. Let $a_0 = 0,\ a_1 = 1,\ a_2 = \dfrac{1}{2}$. Then we can check that $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\}),\ i = 1,\ 2,\ 3$. So, Corollary 3.D is sharp.

4 Exceptional values of meromorphic functions and derivatives

The paragraph is aimed at studying various properties of derivatives of meromorphic functions, particularly their sets of zeros [2], [3], [4]. Many important results are due to Jean-Paul Bézivin [1], [2].

We will first notice a general property concerning quasi-exceptional values of meromorphic functions and derivatives.

Theorem 4.1: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. Let $f \in \mathcal{M}_u(d(\alpha, R^-))$). If f admits a quasi-exceptional value, then f' has no quasi-exceptional value different from* 0. **Proof.** Without loss of generality, we may assume $\alpha = 0$ and that f has no zero and no pole at 0. Let $b \in \mathbb{K}$ and suppose that b is a quasi-exceptional value of f. There exist $P \in \mathbb{K}[x]$ and $l \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. and $l \in \mathcal{A}_u(d(0, R^-))$) without common zeros, such that $f = b + \dfrac{P}{l}$.

Let $c \in \mathbb{K}^*$. Remark that $f' - c = \dfrac{P'l - Pl' - cl^2}{l^2}$. Let $a \in \mathbb{K}$ (resp. let $a \in d(0, R^-)$). If a is a pole of f, it is a pole of $f' - c$ and we can check that

(1) $\omega_a(P'l - Pl' - cl^2) = \omega_a(l') = \omega_a(l) - 1$

because a is not a zero of P.

Now suppose that a is not a pole of f. Then

(2) $\omega_a(f' - c) = \omega_a(P'l - Pl' - cl^2)$

Consequently, $Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0)$. But, by (1) we have

(3) $Z(r, (P'l - Pl' - cl^2) \mid l(x) = 0) < Z(r, l)$.

and therefore by (2) and (3) we obtain

(4) $Z(r, f'-c) = Z(r, (P'l-Pl'-cl^2) \mid l(x) \neq 0) > Z(r, P'l-Pl'-cl^2) - Z(r, l)$

Now, let us examine $Z(r, P'l - Pl' - cl^2)$. Let $r \in]0, +\infty[$ $\big($resp. let $r \in]0, R[\big)$. Since $l \in \mathcal{A}(\mathbb{K})$ is transcendental (resp. since $l \in \mathcal{A}_u(d(0, R^-))$), we can check that when r is big enough, we have $|Pl'|(r) < |c|\big(|l|(r)\big)^2$ and $|Pl|(r) < |c|\big(|l|(r)\big)^2$, hence clearly $|P'l - Pl'|(r) < |c|\big(|l|(r)\big)^2$ and hence $|P'l - Pl' - cl^2|(r) = |c|\big(|l|(r)\big)^2$. Consequently, when r is big enough, by Theorem C.4.2 in [7] we have $Z(r, P'l - Pl' - cl^2) = Z(r, l^2) + O(1)$. But $Z(r, l^2) = 2Z(r, l)$, hence $Z(r, P'l - Pl' - cl^2) = 2Z(r, l) + O(1)$ and therefore by (4) we check that when r is big enough, we obtain

(5) $Z(r, f' - c) > Z(r, l)$.

Now, if $l \in \mathcal{A}(\mathbb{K})$, since l is transcendental, by (5), for every $q \in \mathbb{N}$, we have $Z(r, f' - c) > Z(r, l) > q \log r$, when r is big enough, hence $f' - c$ has infinitely many zeros in $\mathbb{K}$. And similarly if $l \in \mathcal{A}_u(d(0, R^-))$, then by (5), $Z(r, f' - c)$ is unbounded when r tends to R, hence $f' - c$ has infinitely many zeros in $d(0, R^-)$.

We will now notice a property of differential equations of the form $y^{(n)} - \psi y = 0$ that is almost classical.

The problem of a constant Wronskian is involved in several questions.

Theorem 4.2: *Let $h,\ l \in \mathcal{A}(\mathbb{K})$ (resp. $h,\ l \in \mathcal{A}(d(\alpha, R^-))$) and satisfy $h'l - hl' = c \in \mathbb{K}$, with h non-affine. If $h,\ l$ belong to $\mathcal{A}(\mathbb{K})$, then $c = 0$ and $\frac{h}{l}$ is a constant. If $c \neq 0$ and if $h,\ l \in \mathcal{A}(d(\alpha, R^-))$, there exists $\phi \in \mathcal{A}(d(\alpha, R^-))$ such that $h'' = \phi h,\ l'' = \phi l$.* **Proof.** Suppose $c \neq 0$. If $h(a) = 0$, then $l(a) \neq 0$. Next, h and l satisfy

(1) $$\frac{h''}{h} = \frac{l''}{l}.$$

Remark first that since h is not affine, h'' is not identically zero. Next, every zero of h or l of order ≥ 2 is a trivial zero of $h'l - hl'$, which contradicts $c \neq 0$. So we can assume that all zeros of h and l are of order 1.

Now suppose that a zero a of h is not a zero of h''. Since a is a zero of h of order 1, $\frac{h''}{h}$ has a pole of order 1 at a and so does $\frac{l''}{l}$, hence $l(a) = 0$, a contradiction. Consequently, each zero of h is a zero of order 1 of h and is a zero of h'' and hence, $\frac{h''}{h}$ is an element ϕ of $\mathcal{M}(\mathbb{K})$ (resp. of $\mathcal{M}(d(\alpha, R^-))$)) that has no pole in $\mathbb{K}$ (resp. in $d(\alpha, R^-)$). Therefore ϕ lies in $\mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(d(\alpha, R^-))$).

The same holds for l and so, l'' is of the form ψl with $\psi \in \mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(d(\alpha, R^-))$). But since $\frac{h''}{h} = \frac{l''}{l}$, we have $\phi = \psi$.

Now, suppose $h,\ l$ belong to $\mathcal{A}(\mathbb{K})$. Since h'' is of the form ϕh with $\phi \in \mathcal{A}(\mathbb{K})$, we have $|h''|(r) = |\phi|(r)|h|(r)$. But by Theorem C.2.10 in [7], we know that $|h''|(r) \leq \frac{1}{r^2}|h|(r)$, a contradiction when r tends to $+\infty$. Consequently, $c = 0$. But then $h'l - hl' = 0$ implies that the derivative of $\frac{h}{l}$ is identically zero, hence $\frac{h}{l}$ is constant.

Corollary 4.A : *Let $h,\ l \in \mathcal{A}(\mathbb{K})$ with coefficients in $\mathbb{Q}$, also be entire functions in $\mathbb{C}$, with h non-affine. If $h'l - hl'$ is a constant c, then $c = 0$.*

Theorem 4.3: *Let $\psi \in \mathcal{M}(\mathbb{K})$ (resp. let $\psi \in \mathcal{M}_u(d(\alpha, R^-))$) and let $(\mathcal{E})$ be the differential equations $y'' - \psi y = 0$. Let E be the sub-vector space of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(\alpha, R^-))$) of the solutions of $(\mathcal{E})$. Then, the dimension of E is 0 or 1.* **Proof.** Suppose E is not $\{0\}$. Let $h,\ l \in E$ be non-identically zero. Then $h''l - hl'' = 0$ and therefore $h'l - hl'$ is a constant c. On the other hand, since $h,\ l$ are not identically zero, neither are $h'',\ l''$. Therefore, $h,\ l$ are not affine functions.

Suppose ψ belongs to $\mathcal{M}(\mathbb{K})$ and that $h,\ l$ belong to $\mathcal{A}(\mathbb{K})$. By Theorem 4..2, we have $c = 0$ and hence $\frac{h}{l}$ is a constant, which proves that E is of dimension 1.

Suppose now that ψ lies in $\mathcal{M}_u(d(\alpha, R^-))$ and that $h,\ l$ belong to $\mathcal{A}(d(\alpha, R^-))$. If ψ lies in $\mathcal{A}(d(\alpha, R^-))$, then by Theorem 4.1, $E = \{0\}$. Finally, suppose that ψ lies in $\mathcal{M}_u(d(\alpha, R^-)) \setminus \mathcal{A}(d(\alpha, R^-))$. If $c \neq 0$, by Theorem 4.2, there exists $\phi \in \mathcal{A}(d(\alpha, R^-))$ such that $h'' = \phi h,\ l'' = \phi l$. Consequently, $\phi = \psi$, hence $\psi \in \mathcal{A}(\mathbb{K})$ and therefore $c = 0$. Hence $h'l - hl' = 0$ again and hence $\dfrac{h}{l}$ is a constant. Thus, we see that E is at most of dimension 1.

Remark: The hypothesis ψ *unbounded in* $d(\alpha, R^-)$ is indispensable to show that the space E is of dimension 0 or 1, as shows the example given again by the p-adic hyperbolic functions $h(x) = \cosh(x)$ and $l(x) = \sinh(x)$. The radius of convergence of both $h,\ l$ is $p^{\frac{-1}{p-1}}$ when $\mathbb{K}$ has residue characteristic p and is 1 when $\mathbb{K}$ has residue characteristic 0. Of course, both functions are solutions of $y'' - y = 0$ but they are bounded.

The following Theorem 4.4 is an improvement of Theorem 4.2. It follows previous results [1].

Theorem 4.4 [2]: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $W(f, g)$ is a non-identically zero polynomial. Then both $f,\ g$ are polynomials.* **Proof.** First, by Theorem 4.2 we check that the claim is satisfied when $W(f, g)$ is a polynomial of degree 0. Now, suppose the claim holds when $W(f, g)$ is a polynomial of certain degree n. We will show it for $n + 1$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $W(f, g)$ is a non-identically zero polynomial P of degree $n + 1$

Thus, by hypothesis, we have $f'g - fg' = P$, hence $f"g - fg" = P'$. We can extract g' and get $g' = \frac{(f'g-P)}{f}$. Now consider the function $Q = f"g' - f'g"$ and replace g' by what we just found: we can get $Q = f'(\frac{(f"g-fg")}{f}) - \frac{Pf"}{f}$.

Now, we can replace $f"g - fg"$ by P' and obtain $Q = \frac{(f'P'-Pf")}{f}$. Thus, in that expression of Q, we can write $|Q|(R) \leq \dfrac{|f|(R)|P|(R)}{R^2|f|(R)}$, hence $|Q|(R) \leq \frac{|P|(R)}{R^2}$ $\forall R > 0$. But by definition, Q belongs to $\mathcal{A}(\mathbb{K})$. Consequently, Q is a polynomial of degree $t \leq n - 1$.

Now, suppose Q is not identically zero. Since $Q = W(f', g')$ and since $\deg(Q) < n$, by the induction hypothesis f' and g' are polynomials and so are f, g. Finally, suppose $Q = 0$. Then $P'f' - Pf" = 0$ and therefore f', P are two solutions of the differential equation of order 1 for meromorphic functions in $\mathbb{K}:\ (\mathcal{E})\ y' = \psi y$ with $\psi = \frac{P'}{P}$, whereas y belongs to $\mathcal{A}(\mathbb{K})$. By Theorem 4.3, the space of solutions of $(\mathcal{E})$ is of dimension 0 or 1. Consequently, there exists $\lambda \in \mathbb{K}$ such that $f' = \lambda P$, hence f is a polynomial. The same holds for g.

Here we can find again the following result that is known and may be proved without ultrametric properties:

Let F be an algebraically closed field and let $P,\ Q \in F[x]$ be such that $PQ' - P'Q$ is a constant c, with $\deg(P) \geq 2$. Then $c = 0$.

Notation: Let $f \in \mathcal{A}(\mathbb{K})$. We can factorize f in the form $\overline{f}\widetilde{f}$ where the zeros of $\overline{f}$ are the distinct zeros of f each with order 1. Moreover, if $f(0) \neq 0$ we will take $\overline{f}(0) = 1$.

Lemma 4.a: *Let $U, V \in \mathcal{A}(\mathbb{K})$ have no common zero and let $f = \dfrac{U}{V}$. If f' has finitely many zeros, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\widetilde{V}$* **Proof.** If V is a constant, the statement is obvious. So, we assume that V is not a constant. Now $\widetilde{V}$ divides V' and hence V' factorizes in the way $V' = \widetilde{V}Y$ with $Y \in \mathcal{A}(\mathbb{K})$. Then no zero of Y can be a zero of V. Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\overline{V} - UY}{\overline{V}^2\widetilde{V}}.$$

The two functions $U'\overline{V} - UY$ and $\overline{V}^2\widetilde{V}$ have no common zero since neither have U and V. So, the zeros of f' are those of $U'\overline{V} - UY$ which therefore has finitely many zeros and consequently is a polynomial.

Theorem 4.5: *Let $f \in \mathcal{M}(\mathbb{K})$ have finitely many multiple poles, such that for certain $b \in \mathbb{K}$, $f' - b$ has finitely many zeros. Then f belongs to $\mathbb{K}(x)$.*

Proof. Suppose first $b = 0$. Let us write $f = \dfrac{U}{V}$ with $U,\ V \in \mathcal{A}(\mathbb{K})$, having no common zeros. By Lemma 4.a, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\widetilde{V}$. Since f has finitely many multiple poles, $\widetilde{V}$ is a polynomial, hence so is $U'V - UV'$. But then by Theorem 4.4, both $U,\ V$ are polynomials, which ends the proof when $b = 0$. Consider now the general case. $f' - b$ is the derivative of $f - bx$ that satisfies the same hypothesis, so the conclusion is immediate.

Notation: For each $n \in \mathbb{N}^*$, we set $\lambda_n = \max\{\frac{1}{|k|},\ 1 \leq k \leq n\}$. Given positive integers $n,\ q$, we denote by C_n^q the combination $\dfrac{n!}{q!(n-q)!}$. Let us recall that log is the Neperian logarithm, we denote by e the number such that $\log(e) = 1$ and Exp is the real exponential function.

Remark: For every $n \in \mathbb{N}^*$, we have $\lambda_n \leq n$ because $k|k| \geq 1\ \forall k \in \mathbb{N}$. The equality holds for all n of the form p^h.

Lemmas 4.b and 4.c are due to Jean-Paul Bézivin [1]:

Lemma 4.b: *Let $U,\ V \in \mathcal{A}(d(0,R^-))$. Then for all $r \in]0,R[$ and $n \geq 1$ we have*

$$|U^{(n)}V - UV^{(n)}|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

More generally, given $j,\ l \in \mathbb{N}$, we have

$$|U^{(j)}V^{(l)} - U^{(l)}V^{(j)}|(r) \leq |(j!)(l!)|\lambda_{j+l} \frac{|U'V - UV'|(r)}{r^{j+l-1}}$$

Lemma 4.c: *Let $U,V \in \mathcal{A}(\mathbb{K})$ and let $r,\ R \in]0,+\infty[$ satisfy $r < R$. For all $x,y \in \mathbb{K}$ with $|x| \leq R$ and $|y| \leq r$, we have the inequality:*

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}$$

Notation: Let $f \in \mathcal{M}(d(0,R^-))$. For each $r \in]0,R[$, we denote by $\zeta(r,f)$ the number of zeros of f in $d(0,r)$, taking multiplicity into account and set $\xi(r,f) = \zeta(r,\frac{1}{f})$. Similarly, we denote by $\beta(r,f)$ the number of multiple zeros of f in $d(0,r)$, each counted with its multiplicity and we set $\gamma(r,f) = \beta(r,\frac{1}{f})$.

Theorem 4.6 [2] *Let $f \in \mathcal{M}(\mathbb{K})$ be such that for some $c,q \in]0,+\infty[$, $\gamma(r,f)$ satisfies $\gamma(r,f) \leq cr^q$ in $[1,+\infty[$. If f' has finitely many zeros, then $f \in \mathbb{K}(x)$.*

Proof. Suppose f' has finitely many zeros and set $f = \dfrac{U}{V}$. If V is a constant, the statement is immediate. So, we suppose V is not a constant and hence it admits at least one zero a. By Lemma 4.a, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\widetilde{V}$. Next, we take $r,R \in [1,+\infty[$ such that $|a| < r < R$ and $x \in d(0,R),\ y \in d(0,r)$. By Lemma 4.c we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}.$$

Notice that $U(a) \neq 0$ because U and V have no common zero. Now set $l = \max(1,|a|)$ and take $r \geq l$. Setting $c_1 = \dfrac{1}{e|U(a)|}$, we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\widetilde{V}|(R)}{\log R - \log r}.$$

Then taking the supremum of $|V(a+y)|$ inside the disk $d(0,r)$, we can derive

$$(1)\qquad |V|(r) \leq c_1 \frac{R|P|(R)|\widetilde{V}|(R)}{\log R - \log r}.$$

Let us apply Corollary B.13.30 in [7], by taking $R = r + \frac{1}{r^q}$, after noticing that the number of zeros of $\widetilde{V}(R)$ is bounded by $\beta(R,V)$. So, we have

$$(2)\qquad |\widetilde{V}|(R) \leq \Big(1 + \frac{1}{r^{q+1}}\Big)^{\beta((r+\frac{1}{r^q}),V)} |\widetilde{V}|(r).$$

Now, due to the hypothesis: $\beta(r,V) = \gamma(r,f) \leq cr^q$ in $[1,+\infty[$, we have

$$(3)\qquad \Big(1 + \frac{1}{r^{q+1}}\Big)^{\beta((r+\frac{1}{r^q}),V)} \leq \Big(1 + \frac{1}{r^{q+1}}\Big)^{[c(r+\frac{1}{r^q})^m]} =$$

$$\mathrm{Exp}\Big[c(r + \frac{1}{r^q})^q \log(1 + \frac{1}{r^{q+1}})\Big].$$

The function $h(r) = c(r + \frac{1}{r^m})^m \log(1 + \frac{1}{r^{m+1}})$ is continuous on $]0,+\infty[$ and equivalent to $\frac{c}{r}$ when r tends to $+\infty$. Consequently, it is bounded on $[l,+\infty[$. Therefore, by (2) and (3) there exists a constant $M > 0$ such that, for all $r \in [l,+\infty[$ by (3) we obtain

$$(4)\qquad |\widetilde{V}|(r + \frac{1}{r^q}) \leq M|\widetilde{V}|(r).$$

On the other hand, $\log\Big(r + \frac{1}{r^q}\Big) - \log r = \log\Big(1 + \frac{1}{r^{q+1}}\Big)$ clearly satisfies an inequality of the form $\log\Big(1 + \frac{1}{r^{q+1}}\Big) \geq \frac{c_2}{r^{q+1}}$ in $[l,+\infty[$ with $c_2 > 0$. Moreover, we can find positive constants $c_3,\ c_4$ such that $(r + \frac{1}{r^q})|P|\Big(r + \frac{1}{r^q}\Big) \leq c_3 r^{c_4}$. Consequently, by (1) and (4) we can find positive constants $c_5,\ c_6$ such that $|V|(r) \leq c_5 r^{c_6}|\widetilde{V}|(r)\ \forall r \in [l,+\infty[$. Thus, writing again $V = \overline{V}\widetilde{V}$, we have $|\overline{V}|(r)|\widetilde{V}|(r) \leq c_5 r^{c_6}|\widetilde{V}|(r)$ and hence $|\overline{V}|(r) \leq c_5 r^{c_6}\ \forall r \in [l,+\infty[$. Consequently, by Corollary B.13.31 in [7], $\overline{V}$ is a polynomial of degree $\leq c_6$ and hence it has finitely many zeros and so does V. But then, by Theorem 4.5, f must be a rational function.

Corollary 4.B: *Let f be a meromorphic function on $\mathbb{K}$ such that, for some $c,q \in]0,+\infty[$, $\gamma(r,f)$ satisfies $\gamma(r,f) \leq cr^q$ in $[1,+\infty[$. If for some $b \in \mathbb{K}$ $f' - b$ has finitely many zeros, then f is a rational function.* **Proof.** Suppose $f' - b$ has finitely many zeros. Then $f - bx$ satisfies the same hypothesis as f, hence it is a rational function and so is f.

Corollary 4.C: *Let $f \in \mathcal{M}(\mathbb{K})\setminus\mathbb{K}(x)$ be such that $\xi(r,f) \leq cr^q$ in $[1,+\infty[$ for some $c, q \in]0,+\infty[$. Then for each $k \in \mathbb{N}^*$, $f^{(k)}$ has no quasi-exceptional value.*
Proof. Indeed, if $k = 1$, the statement just comes from Corollary 4.B Now suppose $k \geq 2$. Each pole a of order n of f is a pole of order $n+k$ of $f^{(k)}$ and $f^{(k)}$ has no other pole. Consequently, we have $\gamma(r, f^{k-1}) = \xi(r, f^{(k-1)}) \leq kcr^q$. So, we can apply Corollary 4.B to $f^{(k-1)}$ to show the claim.

Theorem 4.6 suggests us the following conjecture:

Conjecture: *Let $f \in \mathcal{M}(\mathbb{K})$ be such that f' admits finitely many zeros. Then $f \in \mathbb{K}(x)$.*

In other words, the conjecture suggests that the derivative of a meromorphic function in $\mathbb{K}$ has no quasi-exceptional value, except if it is a rational function.

Remark: Of course, there exist meromorphic functions in $\mathbb{K}$ having no zero but not satisfying the hypotheses of Theorem 4.6, hence such a function cannot have primitives. For example, consider an entire function f having an infinity of zeros $(a_n)_{n\in\mathbb{N}}$ of order 2 such that $|a_n| < |a_{n+1}|$, $\lim_{n\to+\infty} |a_n| = +\infty$ and $2n \leq |a_n|$. Then the meromorphic function $g = \dfrac{1}{f}$ has no zeros but does not satisfy the hypotheses of Theorem 4.6 hence it has no primitives.

5 Small functions

Small functions with respect to a meromorphic function are well known in the general theory of complex functions. Particularly, one knows the Nevanlinna theorem on 3 small functions. Here we will recall the construction of a similar theory.

Definitions and notation: Throughout the chapter we set $a \in \mathbb{K}$ and $R \in]0,+\infty[$. For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a,R^-))$ we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a,R^-))$) the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(d(a,R^-))$) such that $T(r,h) = o(T(r,f))$ when r tends to $+\infty$ (resp. when r tends to R). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a,R^-))$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a,R^-))$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a,R^-)) \cap \mathcal{A}(d(a,R^-))$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(a,R^-))$) are called *small meromorphic functions with respect to f*, (*small functions* in brief). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a,R^-))$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a,R^-))$) are called *small analytic functions with respect to f, (small functions in brief)*.

Theorems 5.1 and Theorem 5.2 are immediate consequences of Theorems C.9.1 and C.9.2 in [7]:

Theorem 5.1: *Let $a \in \mathbb{K}$ and $r > 0$. Then $\mathcal{A}_f(\mathbb{K})$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$, $\mathcal{A}_f(d(a,R^-))$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a,R^-))$ $\mathcal{M}_f(\mathbb{K})$ is a subfield field of $\mathcal{M}(\mathbb{K})$, $\mathcal{M}_f(d(a,R^-))$ is a subfield of field of $\mathcal{M}(a,R^-))$. Moreover, $\mathcal{A}_b(d(a,R^-)$ is a sub-algebra of $\mathcal{A}_f(d(a,R^-)$ and $\mathcal{M}_b(d(a,R^-)$ is a subfield of $\mathcal{M}_f(d(a,R^-)$.*

Theorem 5.2 : *Let $f \in \mathcal{M}(\mathbb{K})$, (resp.$f \in \mathcal{M}(d(0,R^-)))$ and let $g \in \mathcal{M}_f(\mathbb{K})$, (resp.$g \in \mathcal{M}_f(d(0,R^-)))$. Then $T(r,fg) = T(r,f) + o(T(r,f))$ and $T(r,\frac{f}{g}) = T(r,f) + o(T(r,f))$, (resp. $T(r,fg) = T(r,f) + o(T(r,f))$ and $T(r,\frac{f}{g}) = T(r,f) + o(T(r,f)))$.*

Theorem 5.3 is known as Second Main Theorem on Three Small Functions in p-adic analysis [7] and [10]. It holds as well as in complex analysis, where it was showed first and it is proven in the same way.

Theorem 5.3: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-)))$ and let $w_1,\ w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2, w_3 \in \mathcal{M}_f(d(0,R^-)))$ be pairwaise distinct. Then $T(r,f) \le \sum_{j=1}^{3} \overline{Z}(r,f-w_j) + o(T(r,f))$, resp $T(r,f) \le \sum_{j=1}^{3} \overline{Z}(r,f-w_j) + o(T(r,f))$, resp. $T_R(r,f) \le \sum_{j=1}^{3} \overline{Z}_R(r,f-w_j) + o(T(r,f))$.*

Theorem 5.4: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-)))$ and let $w_1,\ w_2 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{M}_f(d(0,R^-)))$ be distinct. Then $T(r,f) \le \overline{Z}(r,f-w_1)+\overline{Z}(r,f-w_2)+\overline{N}(r,f)+o(T(r,f))$, (resp. $T(r,f) \le \overline{Z}(r,f-w_1)+\overline{Z}(r,f-w_2) + \overline{N}(r,f) + o(T(r,f)))$.*

Proof. Suppose first $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}_u(d(0,R^-))$. Let $g = \frac{1}{f},\ h_j = \frac{1}{w_j}$, $j = 1,\ 2,\ h_3 = 0$. Clearly,

$$T(r,g) = T(r,f) + O(1),\ T(r,h) = T(r,w_j),\ j = 1,2,$$

so we can apply Theorem 5.3 to $g,\ h_1,\ h_2,\ h_3$. Thus we have: $T(r,g) \le \overline{Z}(r,g-h_1) + \overline{Z}(r,g-h_2) + \overline{Z}(r,g) + o(T(r,g))$.

But we notice that $\overline{Z}(r,g-h_j) = \overline{Z}(r,f-w_j)$ for $j = 1,2$ and $\overline{Z}(r,g) = \overline{N}(r,f)$. Moreover, we know that $o(T(r,g)) = o(T(r,f))$. Consequently, the claim is proved when w_1w_2 is not identically zero.

Now, suppose that $w_1 = 0$. Let $\lambda \in \mathbb{K}^*$, let $l = f + \lambda$ and $\tau_j = u_j + \lambda,\ (j = 1,\ 2,\ 3)$. Thus, we have $T(r,l) = T(r,f) + O(1),\ T(r,\tau_j) = T(r,w_j) +$

$O(1)$, $(j = 1,\ 2)$, $\overline{N}(r,l) = \overline{N}(r,f)$. By the claim already proven whenever $w_1 w_2 \neq 0$ we may write $T(r,l) \leq \overline{Z}(r,l-\tau_1)+\overline{Z}(r,l-\tau_2)+\overline{N}(r,l)+o(T(r,l)))$ hence
$T(r,f) \leq \overline{Z}(r,f-w_1) + \overline{Z}(r,f-w_2) + \overline{N}(r,l) + o(T(r,f)))$.

Next, by setting $g = f - w_1$ and $w = w_1 + w_2$, we can write Corollary 5.A:

Corollary 5.A: *Let* $g \in \mathcal{M}(\mathbb{K})$ *(resp.* $g \in \mathcal{M}_u(d(0,R^-))$*) and let* $w \in \mathcal{M}_g(\mathbb{K})$*. Then* $T(r,g) \leq \overline{Z}(r,g) + \overline{Z}(r,g-w) + \overline{N}(r,g) + o(T(r,g))$*, (resp.* $T(r,g) \leq \overline{Z}(r,g) + \overline{Z}(r,g-w) + \overline{N}(r,g) + o(T(r,g)))$*.*

Corollary 5.B: *Let* $f \in \mathcal{A}(\mathbb{K})$ *(resp.* $f \in \mathcal{A}_u(d(0,R^-))$*) and let* $w_1,\ w_2 \in \mathcal{A}_f(\mathbb{K})$ *(resp.* $w_1, w_2 \in \mathcal{A}_f(d(0,R^-))$*) be distinct. Then* $T(r,f) \leq \overline{Z}(r,f-w_1) + \overline{Z}(r,f-w_2) + o(T(r,f))$ *(*$r \to +\infty$*), resp.(*$r \to R^-$*).*

And similarly to Corollary 5.A, we can get Corollary 5.C:

Corollary 5.C: *Let* $f \in \mathcal{A}(\mathbb{K})$ *(resp.* $f \in \mathcal{A}_u(d(0,R^-))$*, resp.* $f \in \mathcal{A}^c(D)$ *) and let* $w \in \mathcal{A}_f(\mathbb{K})$*). Then* $T(r,f) \leq \overline{Z}(r,f) + \overline{Z}(r,f-w) + o(T(r,f))$*, (resp.* $T(r,f) \leq \overline{Z}(r,f) + \overline{Z}(r,f-w) + o(T(r,f)))$*.*

We are now able to state a theorem on q small functions that is not as good as Yamanoi's Theorem [17] in complex analysis, but seems the best possible in ultrametric analysis;

Theorem 5.5 [8] (A. Escassut, C.C. Yang): *Let* $f \in \mathcal{M}(\mathbb{K})$*be transcendental (resp.* $f \in \mathcal{M}_u(d(0,R^-))$*) and let* $w_j \in \mathcal{M}_f(\mathbb{K})$ $(j = 1, ..., q)$
(resp. $w_j \in \mathcal{M}_f(d(a,R^-))$ *) be* q *distinct small functions other than the constant* ∞*. Then*

$$qT(r,f) \leq 3\sum_{j=1}^{q} \overline{Z}(r,f-w_j) + o(T(r,f)),$$

(resp.

$$qT(r,f) \leq 3\sum_{j=1}^{q} \overline{Z}(r,f-w_j) + o(T(r,f))),$$

Moreover, if f *has finitely many poles in* $\mathbb{K}$ *(resp. in* $d(0,R^-)$*), then*

$$qT(r,f) \leq 2\sum_{j=1}^{q} \overline{Z}(r,f-w_j) + o(T(r,f)),$$

(resp.

$$qT(r,f) \le 2\sum_{j=1}^{q} \overline{Z}(r, f-w_j) + o(T(r,f)).),$$

Proof. By Theorem 5.3, for every triplet (i,j,k) such that $1 \le i \le j \le k \le q$, we can write

$$T(r,f) \le \overline{Z}(r, f-w_i) + \overline{Z}(r, f-w_j) + \overline{Z}(r, f-w_k) + o(T(r,f)).$$

The number of such inequalities is C_q^3. Summing up, we obtain

(1)

$$C_q^3 T(r,f) \le \sum_{(i,j,k),\ 1\le i\le j\le k\le q} \overline{Z}(r, f-w_i)+\overline{Z}(r, f-w_j)+\overline{Z}(r, f-w_k)+o(T(r,f)).$$

In this sum, for each index i, the number of terms $\overline{Z}(r, f-w_i)$ is clearly C_{q-1}^2. Consequently, by (1) we obtain

$$C_q^3 T(r,f) \le C_{q-1}^2 \sum_{i=1}^{q} \overline{Z}(r, f-w_i) + o(T(r,f))$$

and hence

$$\frac{q}{3} T(r,f) \le \sum_{i=1}^{q} \overline{Z}(r, f-w_i) + o(T(r,f)).$$

Suppose now that f has finitely many poles. By Theorem 5.4, for every pair (i,j) such that $1 \le i \le j \le q$, we have

$$T(r,f) \le \overline{Z}(r, f-w_i) + \overline{Z}(r, f-w_j) + o(T(r,f)).$$

The number of such inequalities is then C_q^2. Summing up we now obtain

$$(2) \qquad C_q^2 T(r,f) \le \sum_{(i,j,\ 1\le i\le j\le q} \overline{Z}(r, f-w_i) + \overline{Z}(r, f-w_j) + o(T(r,f)).$$

In this sum, for each index i, the number of terms $\overline{Z}(r, f-w_i)$ is clearly $C_{q-1}^1 = q-1$. Consequently, by (1) we obtain

$$C_q^2 T(r,f) \le (q-1) \sum_{i=1}^{q} \overline{Z}(r, f-w_i) + o(T(r,f))$$

and hence

$$\frac{q}{2} T(r,f) \le \sum_{i=1}^{q} \overline{Z}(r, f-w_i) + o(T(r,f)).$$

Definition: Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}_u(d(a, R^-))$). Then f and g will be *to share a small function, I.M.* $w \in \mathcal{M}(\mathbb{K})$ *(resp.* $w \in \mathcal{M}(d(a, R^-))$*)* if $f(x) = w(x)$ implies $g(x) = w(x)$ and if $g(x) = w(x)$ implies $f(x) = w(x)$.

Theorem 5.6: *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental (resp.* $f,\ g \in \mathcal{M}_u(d(a, R^-))$*) be distinct and share* q *distinct small functions I.M.* $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ $(j = 1, ..., q)$ *(resp.* $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ $(j = 1, ..., q)$*) other than the constant* ∞*. Then*

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Proof. Suppose that f and g belong to $\mathcal{M}(\mathbb{K})$, are distinct and share q distinct small functions I.M. $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ $(j = 1, ..., q)$.

Lat b be a zero of $f - w_i$ for a certain index i. Then it is also a zero of $g - w_i$. Suppose that b is counted several times in the sum $\sum_{j=1}^{q} \overline{Z}(r, f - w_j)$, which means that it is a zero of another function $f - w_h$ for a certain index $h \neq i$. Then we have $w_i(b) = w_h(b)$ and hence b is a zero of the function $w_i - w_h$ which belongs to $\mathcal{M}_f(\mathbb{K})$. Now, put $\widetilde{Z}(r, f - w_1) = \overline{Z}(r, f - w_1)$ and for each $j > 1$, let $\widetilde{Z}(r, f - w_j)$ be the counting function of zeros of $f - w_j$ in the disk $d(0, r^-)$ ignoring multiplicity and avoiding the zeros already counted as zeros of $f - w_h$ for some $h < j$. Consider now the sum $\sum_{j=1}^{q} \widetilde{Z}(r, f - w_j)$. Since the functions $w_i - w_j$ belong to $\mathcal{M}_f(\mathbb{K})$, clearly, we have

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) = \sum_{j=1}^{q} \widetilde{Z}(r, f w_j) = o(T(r, f))$$

.

It is clear, from the assumption, that $f(x) - w_j(x) = 0$ implies $g(x) - w_j(x) = 0$ and hence $f(x) - g(x) = 0$. Since $f - g$ is not the identically zero function, it follows that

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^{q} \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Now, if f and g belong to $\mathcal{M}(d(0, R^-))$ the proof is exactly the same.

Theorem 5.7 [8] (A. Escassut, C.C. Yang): *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(a, R^-))$) be distinct and share 7 distinct small functions (other than the constant ∞) I.M. $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ $(j = 1, ..., 7)$ (resp. $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$, resp. $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$ $(j = 1, ..., 7)$,). Then $f = g$.*

Moreover, if f and g have finitely many poles and share 3 distinct small functions (other than the constant ∞) I.M. then $f = g$.

Proof. We put $M(r) = \max(T(r, f), T(r, g))$. Suppose that f and g are distinct and share q small function I.M. w_j, $(1 \leq j \leq q)$. By Theorem 5.5, we have

$$qT(r, f) \leq 3\sum_{j=1}^{q} \overline{Z}(r, f - w_j) + o(T(r, f)).$$

But thanks to Theorem 5.6, we can derive

$$qT(r, f) \leq 3T(r, f - g) + o(T(r, f))$$

and similarly

$$qT(r, g) \leq 3T(r, f - g) + o(T(r, g))$$

hence

$$qM(r) \leq 3T(r, f - g) + o(M(r)). \tag{1}$$

By Theorem C.4.8 in [7], we can derive that

$$qM(r) \leq 3(T(r, f) + T(r, g)) + o(M(r)))$$

and hence $qM(r) \leq 6M(r) + o(M(r))$. That applies to the situation when f and g belong to $\mathcal{M}(\mathbb{K})$ as well as when when f and g belong to $\mathcal{M}_u(d(0, R^-))$. Consequently, it is impossible if $q \geq 7$ and hence the first statement of Theorem 5.7 is proved.

Suppose now that f and g have finitely many poles. By Theorems C.4.8 in [7], Relation (1) gives us

$$qM(r) \leq 2M(r) + o(M(r))$$

which is obviously absurd whenever $q \geq 3$ and proves that $f = g$ when f and g belong to $\mathcal{M}(\mathbb{K})$ as well as when f and g belong to $\mathcal{M}_u(d(0, R^-))$.

Corollary 5.D: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{A}_u(d(a, R^-))$) be distinct and share 3 distinct small functions (other than the constant ∞) I.M. $w_j \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ $(j = 1, 2, 3)$ (resp. $w_j \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$, $(j = 1, 2, 3)$). Then $f = g$.*

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Alain Escassut
Laboratoire de Mathématiques Blaise Pascal
UMR CNRS 6620
Université Clermont Auvergne
F 63000 Clermont-Ferrandt

France
e-mail adress: alain.escassut@uca.fr