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Finiteness of holomorphic mappings from a complex disc sharing $2n+1$ hyperplanes with finite growth index

Nguyen The Hoang (Hanoi, Vietnam)

Nguyen Thi Nhung (Hanoi, Vietnam)

Hoang Thu Thuy (Hanoi, Vietnam)

Dedicated to Professor Ha Huy Khoai on the occasion of his 80 birthday.

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Abstract. In this paper, we prove a finiteness theorem for holomorphic mappings from a complex disc $\Delta(R) \subset \mathbb{C}$ with finite growth index into $\mathbb{P}^n(\mathbb{C})$ sharing $2n+1$ hyperplanes in general position regardless of multiplicity. We also give a theorem on the algebraic dependence of three such mappings. These results extend the previous finiteness theorem for meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$.

1. Introduction

In 1926, R. Nevanlinna [4] showed that two distinct nonconstant meromorphic functions on $\mathbb{C}$ cannot have the same inverse images for five distinct values, and that one of them is a linear fractional transformation of the other

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if they have the same inverse images counted with multiplicities for four distinct values. The second assertion is usually called the four values theorem. Many authors have extended this theorem to meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes in general position. In these extensions the mappings need not coincide, but their number is bounded. Results of this type are called finiteness theorems.

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly non-degenerate over $\mathbb{C}$. Let d be a positive integer and $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq q).$$

For each point z, let $\nu_{H_j(f)}(z)$ denote the intersecting multiplicity of the mapping f with the hyperplane H_j at z. We consider $\mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^q, d)$ the set of all linearly non-degenerate over $\mathbb{C}$ meromorphic mappings $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ that satisfies

- $\min\{\nu_{(f,H_j)}(z), d\} = \min\{\nu_{(g,H_j)}(z), d\} \quad (1 \leq j \leq q)$,
- $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

If $d = 1$, we will say that f and g share q hyperplanes $\{H_j\}_{j=1}^q$ regardless of multiplicity.

In 1998, H. Fujimoto [1] proved that there are at most two distinct linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ sharing $3n+1$ hyperplanes in general position with multiplicities truncated to level 2,

$$\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{3n+1}, 2) \leq 2.$$

In 2012, S. D. Quang [7] established a finiteness theorem for meromorphic mappings sharing $2n+2$ hyperplanes. This result was subsequently extended in 2019 [9], where he also showed that

$$\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{2n+2}, 1) \leq 2 \qquad (n \geq 2).$$

So far, $2n+2$ is the smallest known number of hyperplanes for which this result holds without truncating multiplicities.

If the multiplicities are truncated to level n, the number of the shared hyperplanes can be reduced further. In 2020, P. D. Thoan and N. Vangty [11] proved that $\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{2n+1}, n) \leq 2$ for $n \geq 8$, and recently H. T. Thuy, P. D. Thoan and N. T. Nhung [13] proved that $\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{2n}, n) \leq 2$ for $n \geq 17$.

The truncation in the result of [11] was removed by P. D. Thoan, D. D. Thai and H. T. Thuy [12]. In 2024, they proved that

$$\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{2n+1}, 1) \leq 2 \qquad (n \geq 12).$$

Now, we turn our consideration to a case that f is a holomorphic mapping from a complex disc $\Delta(R) := \{z; |z| < R\} \subset \mathbb{C}$ in to $\mathbb{P}^n(\mathbb{C})$. In 2020, M. Ru and N. Sibony [10] introduced the definition of growth index for f as follows.

$$c_f = \inf\left\{c > 0 \,\Big|\, \int_0^R \exp(cT_f(r))dr = +\infty\right\}.$$

For convenient, we set $c_f = +\infty$ in case $\left\{c > 0 \,\big|\, \int_0^R \exp(cT_f(r))dr = +\infty\right\} = \varnothing$. They also studied a new class of holomorphic mappings from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$, which has finite growth index and they showed the second main theorem for these ones involving hyperplanes in general position (see 2.2).

Similarly, let f be a linearly nondegenerate holomorphic mapping of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \ldots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $1 \le i < j \le q$. We denote by $\mathcal{F}_\Delta\big(f, \{H_i\}_{i=1}^q, d\big)$ the set of all linearly nondegenerate holomorphic mappings g of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ such that $\min\{\nu_{(g,H_i)}, d\} = \min\{\nu_{(f,H_i)}, d\}$ for all $1 \le i \le q$ and $g = f$ on $\bigcup_{i=1}^q f^{-1}(H_i)$.

Our main results are stated as follows. In these theorems, $g(n)$ and $g^*(n)$ are functions of n defined in Section 4, where their values are given explicitly. In the proofs of Theorems 1.1 and 1.2, the corresponding functions are positive for every $n \ge 12$.

Theorem 1.1. *Let f be a linearly nondegenerate holomorphic mapping of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \ldots, H_{2n+1}$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $1 \le i < j \le 2n+1$. Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$. If $n \ge 12$ and*

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} < g(n),$$

then $f_1 \wedge f_2 \wedge f_3 \equiv 0$.

Theorem 1.2. *Let f and $H_1, \ldots, H_{2n+1}$ be as in Theorem 1.1. Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$. If $n \ge 12$ and $\min\{c_{f_1}, c_{f_1}, c_{f_1}\} < g^*(n)$ then $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_1$.*

If $R = +\infty$, then $c_g = 0$ for every holomorphic mapping g of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Hence Theorem 1.2 gives the following.

Corollary 1.3. *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \ldots, H_{2n+1}$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $1 \le i < j \le 2n+1$. If $n \ge 12$, then $\sharp\, \mathcal{F}_{\mathbb{C}}(f, \{H_i\}_{i=1}^{2n+1}, 1) \le 2$.*

This is the result of [12] for the case $m = 1$. In this sense, Theorem 1.2 is an extension of that result to holomorphic mappings from a complex disc with finite growth index.

We now give an outline of the proof. Theorem 1.1 is proved along the lines of the one of Theorem 13 in [12]. That proof consists of two parts. The first part is a chain of inequalities between divisors, together with a case analysis on the number l_1 of the indices i with $V_i \cong V_1$ and on the vanishing of the Cartan auxiliary functions Φ_{ij}. This part uses no property of the domain of the mappings, and we quote it from [12] with the corresponding numbers. The second part consists in integrating these inequalities and in applying the Second Main Theorem. Here the disc differs from $\mathbb{C}^m$, and this part is carried out again.

Three points need to be handled differently. First, we use the Second Main Theorem of Ru and Sibony (Theorem 2.2 below), whose error term contains a function γ; we take $\gamma(r) = \exp\big((c+\varepsilon)T(r)\big)$, where $c = \min\{c_{f_1}, c_{f_2}, c_{f_3}\}$, $T(r) = T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r)$, and $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon\log r$. Second, the ordering of the counting functions used in [12] holds here only on a subset of $(0, R)$, and we choose this subset so that the integral of γ over it is infinite. Third, each case of [12] ends with a numerical inequality which is impossible for $n \geq 12$. On the disc the same case gives an inequality of the form $\|_E\, A\,T(r) \geq B\,T(r) - \kappa S(r) + o(T(r))$, and Lemma 3.11 below turns it into the lower bound $c \geq \frac{B-A}{\kappa}$. The function $g(n)$ is the minimum of the bounds so obtained, and $g^*(n)$ is defined in the same way from the proof of Theorem 1.2.

2. Basic notions and auxiliary results from Nevanlinna theory

We denote by $\Delta(R)$ a disc in $\mathbb{C}$, $\Delta(R) := \{z \in \mathbb{C}; |z| < R\}$, $(0 < R \leq +\infty)$. Let ν be a divisor on $\Delta(R)$. We consider ν as a function on $\Delta(R)$ with values in $\mathbb{Z}$ such that $\mathrm{Supp}\,(\nu) := \{z; \nu(z) \neq 0\}$ is a discrete subset of $\Delta(R)$. Let k be a positive integer or $+\infty$. The truncated counting function of ν is defined by:

$$n^{[k]}(t) = \sum_{|z|\leq t} \min\{k, \nu(z)\}\ (0 \leq t \leq R) \text{ and } N^{[k]}(r,\nu) = \int_{r_0}^{r} \frac{n^{[k]}(t) - n^{[k]}(0)}{t} dt.$$

We will omit the character $^{[k]}$ if $k = +\infty$.

Let $\varphi : \Delta(R) \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function. We denote by ν_φ^0 (resp. ν_φ^∞) the divisor of zeros (resp. divisor of poles) of φ and

set $\nu_\varphi = \nu_\varphi^0 - \nu_\varphi^\infty$. As usual, we will write $N_\varphi^{[k]}(r)$ and $N_{1/\varphi}^{[k]}(r)$ for $N^{[k]}(r, \nu_\varphi^0)$ and $N^{[k]}(r, \nu_\varphi^\infty)$ respectively. The proximity function of φ with respect to the point ∞ is defined by

$$m(r, \varphi) = \int_0^{2\pi} \log^+ |\varphi(re^{t\theta})| d\theta.$$

We consider φ as a holomorphic map into $\mathbb{P}^1(\mathbb{C})$ and denote by Ω_1 the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. The characteristic function of φ is defined by

$$T_\varphi(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} \varphi^* \Omega_1.$$

By Jensen's formula, we have

$$T_\varphi(r) = m(r, \varphi) + N_{1/\varphi}(r) + O(1).$$

Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with a reduced representation $(f_0 : \cdots : f_n)$, where $f_0, \ldots, f_n$ are holomorphic functions on $\Delta(R)$ without common zeros. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ defined by $H := \{(\omega_0 : \cdots : \omega_n); \sum_{i=0}^n a_i \omega_i = 0\}$, where a_i $(0 \le i \le n)$ are constants, not all zero. We define

$$(f, H) = \sum_{i=0}^n a_i f_i.$$

The function (f, H) depends on the choices of the reduced representation of f and the presentation of H, but the divisor $\nu_{(f,H)}$ is well-defined, not depending on these choices. The proximity function of f with respect to H is defined by

$$m_f(r, H) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f\|(re^{t\theta}) \cdot \|H\|}{|(f, H)(re^{t\theta})|} d\theta,$$

where $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$ and $\|H\| = (|a_0|^2 + \cdots + |a_n|^2)^{1/2}$. The characteristic function of f (with respect to the Fubini-Study form Ω_n on $\mathbb{P}^n(\mathbb{C})$) is defined by

$$T_f(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^* \Omega_n.$$

The first main theorem states that

$$T_f(r) = m_f(r, H) + N_{(f,H)}(r) + O(1).$$

Theorem 2.1. *(Lemma on logarithmic derivatives [10, Theorem 5.1]). Let $0 < R \leq +\infty$ and let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $f(z)$ be a meromorphic function on $\Delta(R)$. Then, for every $\varepsilon > 0$, we have the following inequality*

$$\|_E \; m\left(r, \frac{f'}{f}\right) \leq (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + O\left(\log T_f(r)\right).$$

Here and throughout this paper, we use the notation $\|_E P$ to say that the proposition P holds for all $r \in (0; R)$ outside a subset E of $(0; R)$ with $\int_E \gamma(r)dr < +\infty$.

Theorem 2.2. *(see [10, Theorem 1.7]). Let f be a linearly non-degenerate holomorphic map from $\Delta(R)$ $(0 < R \leq +\infty)$ into $\mathbb{P}^n(\mathbb{C})$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$ and let $H_1, \ldots, H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then, for every $\varepsilon > 0$, we have*

$$\begin{aligned}\|_E (q-n-1)T_f(r) \leq \sum_{i=1}^{q} N^{[n]}_{(f,H_i)}(r) + \frac{n(n+1)}{2}(1+\varepsilon)\log\gamma(r)\\ + O(\log T_f(r)) + \frac{n(n+1)}{2}\varepsilon \log r.\end{aligned}$$

In case f has finite growth index (i.e., $c_f < +\infty$), then in Theorem 2.1 and Theorem 2.2, we may take $\gamma(r) = \exp\left((c_f + \varepsilon)\, T_f(r)\right)$.

We recall the definition of the Cartan's auxiliary function due to H.Fujimoto [1].

For meromorphic functions F, G, H on $\Delta(R)$, the Cartan's function is defined by

$$\Phi(F,G,H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ \left(\frac{1}{F}\right)' & \left(\frac{1}{G}\right)' & \left(\frac{1}{H}\right)' \end{vmatrix}.$$

Lemma 2.3 (see [6], Lemma 3.2)**.** *If $\Phi(F, G, H) \equiv 0$ then there exist constants α_0, β_0, not all zeros, such that*

$$\alpha_0\left(\frac{1}{F} - \frac{1}{G}\right) = \beta_0\left(\frac{1}{F} - \frac{1}{H}\right).$$

3. Auxiliary lemmas

Throughout this section, for three mappings $f_1, f_2, f_3 \in \mathcal{F}_\Delta(f, \{H_i\}_{i=1}^q, 1)$, we set

- $F_u^{ij} = \dfrac{(f_u, H_i)}{(f_u, H_j)} \quad (1 \le u \le 3,\ 1 \le i \ne j \le q)$;
- $T(r) = T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r)$;
- $\gamma(r) = \exp\big((c+\varepsilon)T(r)\big)$, where $c := \min\{c_{f_1}, c_{f_2}, c_{f_3}\}$;
- $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon\log r = (1+\varepsilon)(c+\varepsilon)T(r) + \varepsilon\log r$;
- $T_i = \{z \in \Delta(R)\ :\ \nu_{(f,H_i)}(z) > 0\} \quad (1 \le i \le q)$.

If $c_{f_1}, c_{f_2}, c_{f_3}$ are all finte, then $c < +\infty$. Let u_0 be an index with $c_{f_{u_0}} = c$. Since $T(r) \ge T_{f_{u_0}}(r)$, we have

$$\int_0^R \gamma(r)\,dr \ge \int_0^R \exp\big((c_{f_{u_0}} + \varepsilon)T_{f_{u_0}}(r)\big)dr = +\infty$$

by the definition of the growth index. Hence Theorem 2.1 and Theorem 2.2 may be applied to each of the mappings f_1, f_2, f_3 with this function γ, and all the estimates of this section hold outside a set $E \subset (0,R)$ with $\int_E \gamma(r)dr < +\infty$.

A measurable subset E_0 of $(0,R)$ is said to be of infinite γ-measure if $\int_{E_0}\gamma(r)dr = +\infty$. Thus $(0,R)$ itself is of infinite γ-measure, while the exceptional set E in the notation $\|_E$ is not.

Lemma 3.1. *Let $g \in \mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$. Assume that c_f and c_g are both finite. Then $\|_E T_g(r) = O(T_f(r))$ and $\|_E T_f(r) = O(T_g(r))$.*

Proof. We first prove that $\|_E T_g(r) = O(T_f(r))$. By the Second Main Theorem (Theorem 2.2), applied to g and the $2n+1$ hyperplanes $H_1, \dots, H_{2n+1}$, we have

$$\begin{aligned}
\|_E nT_g(r) &\le \sum_{i=1}^{2n+1} N^{[n]}_{(g,H_i)}(r) + \frac{n(n+1)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + o(T_g(r)) \\
&\le \sum_{i=1}^{2n+1} n\ N^{[1]}_{(g,H_i)}(r) + \frac{n(n+1)}{2}S(r) + o(T_g(r)) \\
&= \sum_{i=1}^{2n+1} n\ N^{[1]}_{(f,H_i)}(r) + \frac{n(n+1)}{2}S(r) + o(T_g(r)) \\
&\le n(2n+1)T_f(r) + \frac{n(n+1)}{2}S(r) + o(T_g(r)).
\end{aligned}$$

By taking $\gamma(r) = \exp((c_f + \varepsilon)T_f(r))$, which satisfies $\int_0^R \gamma(r)dr = +\infty$ since $c_f < +\infty$, and letting $\varepsilon \to 0$, we get

$$\|_E T_g(r) \leq \Big(2n + 1 + \frac{n+1}{2}c_f + C\Big)T_f(r) + o(T_g(r)),$$

where C is a positive constant. This implies that $\|_E T_g(r) = O(T_f(r))$. Similarly, applying Theorem 2.2 to f with $\gamma(r) = \exp((c_g + \varepsilon)T_g(r))$, we get $\|_E T_f(r) = O(T_g(r))$. ■

The proof of the following lemma is a pointwise inequality between divisors, and it does not depend on the domain of the mappings; we therefore state it without proof, in the form given in [12].

Lemma 3.2 (see [12], Lemma 5). *Let $f^1, f^2, f^3 \in \mathcal{F}_\Delta(f, \{H_j\}_{j=1}^q, 1)$. Suppose that there exist $s, t, l \in \{1, 2, \ldots, q\}$ such that*

$$P_{(s,t,l)} := \det \begin{pmatrix} (f^1, H_s) & (f^1, H_t) & (f^1, H_l) \\ (f^2, H_s) & (f^2, H_t) & (f^2, H_l) \\ (f^3, H_s) & (f^3, H_t) & (f^3, H_l) \end{pmatrix} \neq 0.$$

Then

$$\nu_{P_{(s,t,l)}}(z) \geq \sum_{i=s,t,l} \Big(\min_{1\leq u\leq 3} \{\nu_{(f^u,H_i)}(z)\} - \nu^{[1]}_{(f,H_i)}(z) \Big) + 2\sum_{i=1}^{q} \nu^{[1]}_{(f,H_i)}(z).$$

If $V_s \simeq V_k$ for some k, then

$$\nu_{P_{(s,t,l)}}(z) \geq \min_{1\leq u\leq 3} \{\nu_{(f^u,H_s)}(z)\} + \sum_{i=t,l} \Big(\min_{1\leq u\leq 3} \{\nu_{(f^u,H_i)}(z)\} - \nu^{[1]}_{(f,H_i)}(z) \Big) + 2\sum_{i=1}^{q} \nu^{[1]}_{(f,H_i)}(z).$$

By taking $q = 2n + 1$ in Lemma 3.7 of [8], we get the following lemma.

Lemma 3.3 (see [8], Lemma 3.7). *With the assumption of Theorem 1.2, let h and g be two elements of the family $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$ such that $\min\{c_h, c_g\} < \frac{2(n-2)}{n(n+1)}$. If there exists a constant λ and two indices i, j such that $\frac{(h,H_i)}{(h,H_j)} = \lambda\frac{(g,H_i)}{(g,H_j)}$, then $\lambda = 1$.*

The following lemma is the version of Lemma 7 in [12] for holomorphic mappings from a complex disc. We prove it by the method of Lemma 3.9 in [8].

Lemma 3.4. *Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$ such that*

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} < \frac{2(n-2)}{n(n+1)}.$$

Let $i, j \in \{1, \dots, 2n+1\}$ $(i \neq j)$. Assume that $F_1^{ji}, F_2^{ji}, F_3^{ji}$ are distinct and $\Phi_{ij} := \Phi(F_1^{ij}, F_2^{ij}, F_3^{ij}) \equiv 0$. Then we have

$$\|_E\, N^{[1]}_{(f,H_i)}(r) + N^{[1]}_{(f,H_j)}(r) \geq \sum_{s\neq i,j} N^{[1]}_{(f,H_s)}(r) - S(r) + O\Big(\sum_{u=1}^{3} \log T_{f_u}(r)\Big).$$

Proof. The assumptions of the lemma do not depend on the order of f_1, f_2, f_3. Hence, we may assume that $c_{f_1} = \min\{c_{f_1}, c_{f_2}, c_{f_3}\}$.

Since $\Phi_{ij} \equiv 0$, by Lemma 2.3 there exist constants α_0, β_0, not all zeros, such that

$$\alpha_0\Big(\frac{1}{F_1^{ij}} - \frac{1}{F_2^{ij}}\Big) = \beta_0\Big(\frac{1}{F_1^{ij}} - \frac{1}{F_3^{ij}}\Big),$$

that is,

$$\alpha_0\big(F_1^{ji} - F_2^{ji}\big) = \beta_0\big(F_1^{ji} - F_3^{ji}\big).$$

If $\alpha_0 = 0$ then $\beta_0 \neq 0$ and $F_1^{ji} = F_3^{ji}$, and if $\beta_0 = 0$ then $F_1^{ji} = F_2^{ji}$. Both cases are impossible, since $F_1^{ji}, F_2^{ji}, F_3^{ji}$ are distinct. Setting $c = \alpha_0/\beta_0$, we have

$$F_3^{ji} = (1-c)F_1^{ji} + cF_2^{ji}, \tag{3.5}$$

where $c \neq 0$ and $c \neq 1$.

Let h be a meromorphic function on $\Delta(R)$ such that $F := (F_1^{ji}h : F_2^{ji}h)$ is a reduced representation of a holomorphic map F of $\Delta(R)$ into $\mathbb{P}^1(\mathbb{C})$.

By Lemma 3.3, we can show that F is not constant.

Since $T(r, F_u^{ji}) \leq T_{f_u}(r) + O(1)$ for $u = 1, 2$, we have

$$T_F(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1).$$

We consider the three hyperplanes

$$\tilde{H}_1 = \{\omega_0 = 0\}, \quad \tilde{H}_2 = \{\omega_1 = 0\}, \quad \tilde{H}_3 = \{(1-c)\omega_0 + c\omega_1 = 0\}$$

of $\mathbb{P}^1(\mathbb{C})$. Since $c \neq 0, 1$, they are in general position, and by (3.5) we have $(F, \tilde{H}_u) = F_u^{ji}h$ for $u = 1, 2, 3$.

By (3.5) and since $(F_1^{ji}h : F_2^{ji}h)$ is a reduced representation, no two of the functions $F_1^{ji}h, F_2^{ji}h, F_3^{ji}h$ have a common zero. Moreover, if z is a zero of neither (f, H_i) nor (f, H_j), then $F_1^{ji}, F_2^{ji}, F_3^{ji}$ are holomorphic and nonzero at

z, and hence $h(z) \neq 0$, i.e., z is a zero of none of these functions. Therefore, for every $z \in \Delta(R)$,

$$\sum_{u=1}^{3} \nu^{[1]}_{F_u^{ji}h}(z) \leq \nu^{[1]}_{(f,H_i)}(z) + \nu^{[1]}_{(f,H_j)}(z),$$

and hence

$$\sum_{u=1}^{3} N^{[1]}_{F_u^{ji}h}(r) \leq N^{[1]}_{(f,H_i)}(r) + N^{[1]}_{(f,H_j)}(r).$$

By Theorem 2.2, applied to the map F and the hyperplanes $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$, we get

$$\begin{aligned} \|_E T_F(r) &\leq \sum_{u=1}^{3} N^{[1]}_{F_u^{ji}h}(r) + S(r) + O(\log T_F(r)) \\ &\leq N^{[1]}_{(f,H_i)}(r) + N^{[1]}_{(f,H_j)}(r) + S(r) + O\Big(\sum_{u=1}^{3} \log T_{f_u}(r)\Big). \end{aligned}$$

On the other hand, we apply the First Main Theorem to the map F and the hyperplane $\{\omega_0 - \omega_1 = 0\}$ of $\mathbb{P}^1(\mathbb{C})$, we have

$$T_F(r) \geq N_{(F_1^{ji} - F_2^{ji})h}(r) + O(1) \geq \sum_{s \neq i,j} N^{[1]}_{(f,H_s)}(r) + O(1).$$

Combining the two above inequalities, we obtain

$$\|_E N^{[1]}_{(f,H_i)}(r) + N^{[1]}_{(f,H_j)}(r) \geq \sum_{s \neq i,j} N^{[1]}_{(f,H_s)}(r) - S(r) + O\Big(\sum_{u=1}^{3} \log T_{f_u}(r)\Big).$$

The lemma is proved. ■

The following lemma is the version of Lemma 8 in [12] for holomorphic mappings from a complex disc. This is Lemma 3.2 in [5] for the case $k_i = +\infty$ $(1 \leq i \leq 2n+1)$.

Lemma 3.6 (see [5], Lemma 3.2)**.** *Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$. Assume that there exist $i, j \in \{1, \ldots, 2n+1\}$ $(i \neq j)$ such that $\Phi_{ij} := \Phi(F_1^{ij}, F_2^{ij}, F_3^{ij}) \not\equiv 0$. Then for every $\varepsilon > 0$, the following assertions hold.*

(i) We have

$$\|_E T(r) + 6S(r) + o(T(r)) \geqslant \sum_{u=1}^{3} N^{[n]}_{(f_u,H_i)}(r) + \sum_{u=1}^{3} N^{[n]}_{(f_u,H_j)}(r) + 2\sum_{\substack{t=1\\ t\neq i,j}}^{2n+1} N^{[1]}_{(f,H_t)}(r) - (2n+1)N^{[1]}_{(f,H_i)}(r) - (n+1)N^{[1]}_{(f,H_j)}(r).$$

(ii) Furthermore, if $\Phi(F_1^{ji}, F_2^{ji}, F_3^{ji}) \not\equiv 0$, *then*

$$\|_E T(r) + 6S(r) + o(T(r)) \geqslant \sum_{t=i,j} \Big(\sum_{u=1}^{3} N^{[n]}_{(f_u,H_t)}(r) - \frac{3n+6}{2} N^{[1]}_{(f,H_t)}(r)\Big) + 2\sum_{t=1}^{2n+1} N^{[1]}_{(f,H_t)}(r).$$

The following lemma is an algebraic consequence of the identity theorem for holomorphic functions, applied at the points where $V_i \sim V_j$; it does not depend on the domain of the mappings, and we quote it from [9].

Lemma 3.7 (see [9], Lemma 3.4)**.** *Let* f^1, f^2, f^3 *be three elements of* $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^q, 1)$. *Suppose that* $f^1 \wedge f^2 \wedge f^3 \equiv 0$ *and* $V_i \sim V_j$ *for some distinct indices* i *and* j. *Then* f^1, f^2, f^3 *are not distinct.*

By taking $q = 2n+1$ in Lemma 3.9 of [8], we get the following lemma.

Lemma 3.8 (see [8], Lemma 3.9)**.** *Let* f_1, f_2, f_3 *be three mappings in* $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$ *satisfying* $f_1 \wedge f_2 \wedge f_3 \equiv 0$. *Suppose that* f_1, f_2, f_3 *are distinct and there exist two indices* $i, j \in \{1, \ldots, 2n+1\}$ $(i \neq j)$ *such that* $V_i \ncong V_j$ *and* $\Phi_{ij} \equiv 0$. *Then, for every* $t \in \{1, \ldots, 2n+1\} \setminus \{i\}$, *the following assertions hold.*

(i) $\Phi_{it} \equiv 0$.

(ii) If $V_i \ncong V_t$, *then* $F_1^{ti}, F_2^{ti}, F_3^{ti}$ *are distinct and*

$$\|_E N^{[1]}_{(f,H_i)}(r) \geq \sum_{s\neq i,t} N^{[1]}_{(f,H_s)}(r) - N^{[1]}_{(f,H_t)}(r) - S(r) + O\Big(\sum_{u=1}^{3} \log T_{f_u}(r)\Big).$$

Lemma 3.9 (see [12], Lemma 11)**.** *If* a, b, c, p, n, k *are positive integers satisfying*

$$\min\{a,p\} = \min\{b,p\} = \min\{c,p\}, \qquad 1 \leq p \leq n$$

and $k \leq 6n$, *then*

$$3\min\{a,b,c\} + k \geq \frac{k+3p}{2n+p}\left(\min\{a,n\} + \min\{b,n\} + \min\{c,n\}\right).$$

Lemma 3.10 ([11], Proof of Theorem 1)**.** *Let n be a positive integer and let $V_1, \dots, V_{2n+1}$ be a family of vectors in 3-dimensional vector space satisfying $\operatorname{rank}\{V_j\}_{j\in R} = 3$ for all subset $R \subset \{1, \dots, 2n+1\}$ with $\#R = n+1$. For each $1 \le j \le 2n+1$, we define $l_j := \#\{V_{j_1}, \dots, V_{j_{l_j}}\}$ if $\operatorname{rank}\{V_{j_1}, \dots, V_{j_{l_j}}\} = 1$ and $V_{j_1} = V_j$, and l_j is maximal. Then, for $s \notin \{j_1, \dots, j_{l_1}\}$, there exists a partition $J_1^s, \dots, J_n^s$ of $Q_s := \{1, \dots, s-1, s+1, \dots, 2n+1\}$ with $\#J_t^s = 2$ $(1 \le t \le n)$ such that $\operatorname{rank}\{V_v, V_{i_1}\}_{v\in J_1^s} = \cdots = \operatorname{rank}\{V_v, V_{i_n}\}_{v\in J_n^s} = 3$, where*

$$i_t = \begin{cases} 1, & \text{if } j_l \notin J_t^s \text{ for all } 1 \le l \le l_1, \\ s, & \text{if } j_l \in J_t^s \text{ for some } 1 \le l \le l_1, \end{cases}$$

and for all $1 \le t \le n$.

Lemma 3.11. *Let f_1, f_2, f_3 be three mappings in $\mathcal{F}_\Delta(f, \{H_i\}_{i=1}^{2n+1}, 1)$ and let E_0 be a subset of $(0, R)$ of infinite γ-measure. Let A, B and κ be nonnegative constants which do not depend on ε. Assume that, for every $\varepsilon > 0$,*

$$\|_E\, A\,T(r) \ge B\,T(r) - \kappa\,S(r) + o(T(r)) \qquad \text{for all } r \in E_0.$$

Then we have

$$A \ge B - \kappa \min\{c_{f_1}, c_{f_2}, c_{f_3}\}.$$

Proof. We first note that $\int_E \gamma(r)dr < +\infty$, and hence $\int_{E_0\setminus E} \gamma(r)dr = +\infty$. By the definitions of γ and S, we have

$$S(r) = (1+\varepsilon)\big(\min\{c_{f_1}, c_{f_2}, c_{f_3}\} + \varepsilon\big)T(r) + \varepsilon \log r.$$

Dividing both sides of the inequality of the assumption by $T(r)$, we get

$$A \ge B - \kappa(1+\varepsilon)\big(\min\{c_{f_1}, c_{f_2}, c_{f_3}\} + \varepsilon\big) - \kappa\varepsilon\frac{\log r}{T(r)} + \frac{o(T(r))}{T(r)}$$

for all $r \in E_0 \setminus E$. Letting $r \to R$ with $r \in E_0 \setminus E$, letting $\varepsilon \to 0$, we obtain

$$A \ge B - \kappa \min\{c_{f_1}, c_{f_2}, c_{f_3}\}.$$

The lemma is proved. ∎

4. Proof of the main theorems

Proof of Theorem 1.1

In the proof below, the phrase *as in* [12], as well as *as in* an earlier case, refers to a computation involving inequalities between divisors on $\Delta(R)$ and

using only the inequalities and lemmas stated immediately before it. This computation does not depend on the domain of the mappings and is therefore the same as the corresponding computation in the proof of Theorem 13 of [12]. All steps after integration are carried out here for the disc.

Proof. For each $i \in \{1, \dots, 2n+1\}$, we put

$$N_i(r) := \sum_{u=1}^{3} \Big(N^{[n]}_{(f_u, H_i)}(r) - \frac{n+2}{2} N^{[1]}_{(f_u, H_i)}(r) \Big).$$

We denote by $\mathcal{I}$ the set of all permutations $I = (i_1, \dots, i_{2n+1})$ of $(1, \dots, 2n+1)$. For each $I \in \mathcal{I}$, we put

$$E_I := \{ r \in (0, R) \ : \ N_{i_1}(r) \geq \cdots \geq N_{i_{2n+1}}(r) \}.$$

It is clear that $\bigcup_{I \in \mathcal{I}} E_I = (0, R)$. Since $\mathcal{I}$ is a finite set and $\int_0^R \gamma(r) dr = +\infty$, there exists $I_0 \in \mathcal{I}$ such that E_{I_0} is of infinite γ-measure. Without loss of generality, we may assume that $I_0 = (1, \dots, 2n+1)$, and hence

$$N_1(r) \geq N_2(r) \geq \cdots \geq N_{2n+1}(r) \qquad \text{for all } r \in E_{I_0}. \tag{4.1}$$

We will use the following two estimates. Firstly, if φ is a nonzero holomorphic function on $\Delta(R)$ of the form $\varphi = \det(V_s, V_t, V_l)$ with $s, t, l \in \{1, \dots, 2n+1\}$, then $|\varphi| \leq C \cdot \|f_1\| \cdot \|f_2\| \cdot \|f_3\|$ for some positive constant C, and hence, by Jensen's formula,

$$N_\varphi(r) \leq T(r) + O(1). \tag{4.2}$$

Secondly, applying Theorem 2.2 to each mapping f_u and the hyperplanes $H_1, \dots, H_{2n+1}$, and then summing up the three inequalities obtained, we get

$$\|_E \, nT(r) \leq \sum_{u=1}^{3} \sum_{i=1}^{2n+1} N^{[n]}_{(f_u, H_i)}(r) + \frac{3n(n+1)}{2} S(r) + O\Big(\sum_{u=1}^{3} \log T_{f_u}(r) \Big). \tag{4.3}$$

Now we suppose that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. Since $H_1, \dots, H_{2n+1}$ are in general position, for each subset $\mathcal{R} \subset \{1, \dots, 2n+1\}$ with $\sharp \mathcal{R} = n+1$ there exist three indices $s, t, l \in \mathcal{R}$ such that $\operatorname{rank}\{V_s, V_t, V_l\} = 3$, and hence $P_I := \det(V_s, V_t, V_l) \not\equiv 0$, where $I = \{s, t, l\}$.

For each $j \in \{1, \dots, 2n+1\}$, we denote by l_j the number of the indices i such that $V_i \cong V_j$ and by $j_1, \dots, j_{l_j}$ these indices, where $V_{j_1} = V_j$. We see that $1 \leq l_j \leq n-1$. Indeed, if $l_j \geq n$ then we may choose a subset $\mathcal{R}$ of $\{1, \dots, 2n+1\}$ with $\sharp \mathcal{R} = n+1$ containing $j_1, \dots, j_n$, and then $\operatorname{rank}\{V_v\}_{v \in \mathcal{R}} \leq 2$, which is impossible. Without loss of generality, we may assume that $l_1 = \max\{l_j : 1 \leq j \leq 2n+1\}$.

For each $i \in \{1, \dots, 2n+1\}$, we put

$$\nu_i(z) := 3\Big(\min_{1\le u\le 3}\{\nu_{(f_u,H_i)}(z)\} - \nu^{[1]}_{(f,H_i)}(z)\Big),$$

or

$$\nu_i(z) := 3\min_{1\le u\le 3}\{\nu_{(f_u,H_i)}(z)\} \quad \text{if } V_i \cong V_k \text{ for some } k.$$

Then, by Lemma 3.2, we have

$$\nu_{P_I}(z) \ge \frac{1}{3}\sum_{i\in I}\nu_i(z) + 2\sum_{i=1}^{2n+1}\nu^{[1]}_{(f,H_i)}(z). \tag{4.4}$$

We write $q = 2n+1$ and denote $Q := \{1, \dots, 2n+1\} \setminus \{1_1, \dots, 1_{l_1}\}$. For each $s \in Q$, we choose a partition $J^s_1, \dots, J^s_n$ of $Q_s := \{1, \dots, 2n+1\} \setminus \{s\}$ as in Lemma 3.10. For each $1 \le t \le n$ and $1 \le l \le l_1$, we define

$$I_{lts} := \begin{cases} J^s_t \cup \{1_l\} & \text{if } 1_v \notin J^s_t \text{ for all } 1 \le v \le l_1, \\ J^s_t \cup \{s\} & \text{if } 1_v \in J^s_t \text{ for some } 1 \le v \le l_1. \end{cases}$$

Then $P_{I_{lts}} \not\equiv 0$ for all $1 \le t \le n$. We note that each J^s_t contains at most one index of $\{1_1, \dots, 1_{l_1}\}$, and hence the second case occurs for exactly l_1 indices t. For each $1 \le l \le l_1$, we put

$$P^s_l := \prod_{t=1}^{n} P_{I_{lts}}, \qquad P^s := \prod_{l=1}^{l_1} P^s_l \qquad \text{and} \qquad P_Q := \prod_{s\in Q} P^s,$$

and then P_Q is the product of $\gamma := nl_1(q-l_1)$ functions $P_{I_{lts}}$. By (4.4), we have

$$\nu_{P^s_l}(z) \ge \frac{1}{3}\Big(\sum_{i=1}^{q}\nu_i(z) + (n-l_1)\nu_{1_l}(z) + (l_1-1)\nu_s(z)\Big) + 2n\sum_{i=1}^{q}\nu^{[1]}_{(f,H_i)}(z),$$

and therefore

$$\nu_{P^s}(z) \ge \frac{1}{3}\Big(l_1\sum_{i=1}^{q}\nu_i(z) + (n-l_1)\sum_{l=1}^{l_1}\nu_{1_l}(z) + l_1(l_1-1)\nu_s(z)\Big) + 2nl_1\sum_{i=1}^{q}\nu^{[1]}_{(f,H_i)}(z).$$

This implies that

$$\begin{aligned}\nu_{P_Q}(z) &\geq \frac{1}{3}\Big(l_1(q-l_1)\sum_{i=1}^{q}\nu_i(z)+(n-l_1)(q-l_1)\sum_{l=1}^{l_1}\nu_{1_l}(z)\\ &\quad + l_1(l_1-1)\sum_{s\in Q}\nu_s(z)\Big)+2nl_1(q-l_1)\sum_{i=1}^{q}\nu^{[1]}_{(f,H_i)}(z)\\ &= \frac{1}{3}\Big(n(q-l_1)\sum_{l=1}^{l_1}\nu_{1_l}(z)+(q-1)l_1\sum_{s\in Q}\nu_s(z)\Big)\\ &\quad + 2nl_1(q-l_1)\sum_{i=1}^{q}\nu^{[1]}_{(f,H_i)}(z).\end{aligned} \tag{4.5}$$

Each of the cases below ends with an inequality of the form

$$\big\|_E\, A\,T(r)\geq B\,T(r)-\kappa\,S(r)+o(T(r))\qquad (r\in E_0),$$

where A, B and κ are nonnegative constants which do not depend on ε, and where E_0 is E_{I_0} or a subset of it with $\int_{E_0}\gamma(r)dr=+\infty$. By Lemma 3.11, such an inequality gives $\min\{c_{f_1},c_{f_2},c_{f_3}\}\geq\frac{B-A}{\kappa}$, which contradicts the assumption of the theorem. We denote by g_k the constant $\frac{B-A}{\kappa}$ obtained in the k-th case and we set

$$g(n):=\min\{g_k(n,l_1)\ :\ 1\leq k\leq 13\},$$

the minimum being taken over all admissible values of l_1 in the corresponding case. In each case, we will show that $g_k(n,l_1)>0$ and these ensures the positivity of $g(n)$.

We now consider the following cases.

Case 1. $2\leq l_1\leq n-6$. Since $l_1\geq 2$, we have $V_{1_l}\cong V_1$ for all $1\leq l\leq l_1$, and hence $\nu_{1_l}=3\min_{1\leq u\leq 3}\{\nu_{(f_u,H_{1_l})}\}$. Then, by (4.5), we have

$$\begin{aligned}3\nu_{P_Q}(z)&\geq n(q-l_1)\sum_{l=1}^{l_1}\Big(3\min_{1\leq u\leq 3}\{\nu_{(f_u,H_{1_l})}(z)\}+6l_1\nu^{[1]}_{(f,H_{1_l})}(z)\Big)\\ &\quad + 2nl_1\sum_{s\in Q}\Big(3\min_{1\leq u\leq 3}\{\nu_{(f_u,H_s)}(z)\}+\big(3(q-l_1)-3\big)\nu^{[1]}_{(f,H_s)}(z)\Big).\end{aligned} \tag{4.6}$$

We put $a:=\dfrac{n+2}{2}$ and choose two numbers b and x satisfying

$$(q-l_1-b)\frac{6l_1-x+3n}{3n}=(q-l_1)\frac{6l_1-x}{2n+1},\ x=\frac{3al_1\big(n(2n+1)-2l_1(n-1)\big)}{(2n+1)^2(n-a)-l_1a(n-1)}.$$

Transforming the right hand side of (4.6) by these numbers and applying Lemma 3.9 to each of the brackets, we get

(4.7)
$$\frac{3}{n}\nu_{P_Q}(z) \geq (q-l_1)l_1\frac{3n(2n+1)-(6l_1-x)(n-1)}{3n(2n+1)}\sum_{u=1}^{3}\nu^{[n]}_{(f_u,H_1)}(z) + (q-l_1)\frac{6l_1-x}{2n+1}\sum_{u=1}^{3}\sum_{i=1}^{q}\nu^{[n]}_{(f_u,H_i)}(z) + (q-l_1)x\sum_{i=1}^{q}\nu^{[1]}_{(f,H_i)}(z).$$

These transformations are the same as those in the proof of Theorem 13 in [12]; they use only (4.5) and Lemma 3.9, and hence they do not depend on the domain of the mappings.

We now integrate both sides of (4.7). Since P_Q is the product of $\gamma = nl_1(q-l_1)$ functions $P_{I_{lts}}$, the estimate (4.2) gives $\frac{3}{n}N_{P_Q}(r) \leq 3l_1(q-l_1)T(r)+O(1)$. On the other hand, since $N^{[1]}_{(f_u,H_1)}(r) \geq \frac{1}{n}N^{[n]}_{(f_u,H_1)}(r)$, we have

$$N_1(r) \leq \Big(1-\frac{a}{n}\Big)\sum_{u=1}^{3}N^{[n]}_{(f_u,H_1)}(r)$$

and hence (4.1) gives

$$\sum_{u=1}^{3}N^{[n]}_{(f_u,H_1)}(r) \geq \frac{n}{n-a}N_1(r) \geq \frac{n}{n-a}\cdot\frac{1}{2n+1}\sum_{i=1}^{q}N_i(r) \qquad (r\in E_{I_0}).$$

Moreover, $\sum_{u=1}^{3}N^{[1]}_{(f_u,H_i)}(r) = 3N^{[1]}_{(f,H_i)}(r)$ for all i. Dividing by $q-l_1$ and using the above choice of x, all the terms containing $N^{[1]}_{(f,H_i)}(r)$ cancel, and we obtain

$$\|_E\, 3l_1T(r) \geq \lambda\sum_{u=1}^{3}\sum_{i=1}^{q}N^{[n]}_{(f_u,H_i)}(r)+o(T(r)) \qquad (r\in E_{I_0}), \tag{4.8}$$

where $\lambda := \dfrac{x}{3a}+\dfrac{6l_1-x}{2n+1}$.

Combining (4.8) with (4.3), we get

$$\|_E\, 3l_1T(r) \geq \lambda nT(r) - \frac{3n(n+1)}{2}\lambda S(r)+o(T(r)) \qquad (r\in E_{I_0}).$$

Applying Lemma 3.11 with $A=3l_1$, $B=\lambda n$ and $\kappa = \dfrac{3n(n+1)}{2}\lambda$, we obtain

$$\min\{c_{f_1},c_{f_2},c_{f_3}\} \geq \frac{2}{3n(n+1)}\Big(n-\frac{3l_1}{\lambda}\Big) =: g_1(n,l_1).$$

Now we need to show that $g_1(n,l_1)>0$. As in [12], $g_1(n,l_1)>0$ is equivalent to

$$\frac{(n-6)(n-1)l_1-(n^3-10n^2+9n+6)}{(2n+1)^2(n-2)-l_1(n-1)(n+2)}<0.$$

This inequality is true for $2\le l_1\le n-6$ and $n\ge 12$. Since $g(n)\le g_1(n,l_1)$, this contradicts the assumption of Theorem 1.1.

Case 2. $l_1=1$. We first suppose that $\nu_{(f_1,H_1)}(z)=\nu_{(f_2,H_1)}(z)=\nu_{(f_3,H_1)}(z)$ for all z outside the set S defined in the proof of Lemma 3.2. Then, from (4.5), we have

$$\begin{aligned}\frac{3}{n}\nu_{P_Q}(z)\ge 2n\Big(3\min_{1\le u\le 3}\{\nu_{(f_u,H_1)}(z)\}+3\nu^{[1]}_{(f,H_1)}(z)\Big)\\ +2\sum_{i=2}^{q}\Big(3\min_{1\le u\le 3}\{\nu_{(f_u,H_i)}(z)\}+(6n-3)\nu^{[1]}_{(f,H_i)}(z)\Big).\end{aligned}$$

We put

$$x^*:=\frac{a\big((2n+1)(3n+3)-18n\big)}{(2n+1)^2(n-a)-a(n-1)},\qquad \lambda^*:=\frac{x^*}{3a}+\frac{6-x^*}{2n+1}.$$

Transforming the right hand side of the above inequality by x^*, applying Lemma 3.9 to the two brackets, integrating and using (4.1) and (4.2) as in Case 1, we obtain

$$\|_E\, 3T(r)\ge\lambda^*\sum_{u=1}^{3}\sum_{i=1}^{q}N^{[n]}_{(f_u,H_i)}(r)+o(T(r))\qquad (r\in E_{I_0}).$$

Combining this with (4.3), we get

$$\|_E\, 3T(r)\ge\lambda^* n\,T(r)-\frac{3n(n+1)}{2}\lambda^*\,S(r)+o(T(r))\qquad (r\in E_{I_0}).$$

Applying Lemma 3.11 with $A=3$, $B=\lambda^*n$ and $\kappa=\frac{3n(n+1)}{2}\lambda^*$, we get

$$\min\{c_{f_1},c_{f_2},c_{f_3}\}\ge\frac{2}{3n(n+1)}\Big(n-\frac{3}{\lambda^*}\Big)=:g_2(n).$$

Similarly, we can prove that (as in [12]) $g_2(n)>0$ is equivalent to

$$\frac{6(2n+1)(n-2)(n-10)}{n(n-4)(4n^2-5n-8)}>0.$$

It is true for $n\ge 11$. This contradicts the assumption of Theorem 1.1, since $g(n)\le g_2(n,l_1)$.

We now suppose that the above situation does not happen. Then, as in the proof of Theorem 13 in [12], we may assume that $V_1 \not\sim V_j$ for all $j \geq n+1$, and hence $F_1^{1j}, F_2^{1j}, F_3^{1j}$ are distinct for all $j \geq n+1$. We distinguish three subcases.

Subcase 2.1. $\Phi_{1j} \not\equiv 0$ and $\Phi_{j1} \not\equiv 0$ for all $j \geq n+1$. Applying Lemma 3.6(ii) to the indices 1 and $n+1$, we get

$$\|_E\, T(r) + 6S(r) \geq N_1(r) + N_{n+1}(r) + 2\sum_{i=1}^{q} N^{[1]}_{(f,H_i)}(r) + o(T(r)).$$

Multiplying both sides by $n+1$ and using (4.1) together with $nN_1(r) \geq \sum_{i=1}^{n} N_i(r)$, $N_1(r) \geq \frac{1}{2n+1}\sum_{i=1}^{q} N_i(r)$ and $(n+1)N_{n+1}(r) \geq \sum_{j=n+1}^{q} N_j(r)$, we obtain

$$\|_E\, (n+1)T(r) + 6(n+1)S(r) \geq \mu \sum_{u=1}^{3}\sum_{i=1}^{q} N^{[n]}_{(f_u,H_i)}(r) + o(T(r)),$$

where $\mu := \dfrac{2n+2}{2n+1} + \dfrac{(n+1)(n-4)}{3n(2n+1)}$. Here we have used $\sum_{u=1}^{3} N^{[1]}_{(f_u,H_i)}(r) = 3N^{[1]}_{(f,H_i)}(r)$ and $N^{[1]}_{(f_u,H_i)}(r) \geq \frac{1}{n}N^{[n]}_{(f_u,H_i)}(r)$. Combining this with (4.3), we get

$$\|_E\, (n+1)T(r) \geq \mu n\, T(r) - \Big(\frac{3n(n+1)}{2}\mu + 6(n+1)\Big)S(r) + o(T(r)).$$

Applying Lemma 3.11 with $A = n+1$, $B = \mu n$ and $\kappa = \frac{3n(n+1)}{2}\mu + 6(n+1)$, we get

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{(n+1)(n-7)}{3(2n+1)\kappa} =: g_3(n).$$

We can easily see that $g_3(n) > 0$ for $n \geq 8$. These lead to contradiction to the assumption of Theorem 1.1.

Subcase 2.2. There exist two distinct indices i, j such that $\Phi_{1i} \equiv 0$ or $\Phi_{i1} \equiv 0$, and $\Phi_{1j} \equiv 0$ or $\Phi_{j1} \equiv 0$. Applying Lemma 3.4 to the pairs $(1,i)$ and $(1,j)$ and summing up the two inequalities obtained, we get

$$\|_E\, N^{[1]}_{(f,H_1)}(r) \geq \sum_{t\neq 1,i,j} N^{[1]}_{(f,H_t)}(r) - S(r) + o(T(r)).$$

Applying Theorem 2.2 to each f_u and the $2n-2$ hyperplanes $\{H_t\}_{t\neq 1,i,j}$, and using $N^{[1]}(r,\nu) \geq \frac{1}{n}N^{[n]}(r,\nu)$ for every positive divisor ν, we obtain

$$\|_E\, N^{[1]}_{(f,H_1)}(r) \geq \frac{n-3}{3n}T(r) - \frac{n+3}{2}S(r) + o(T(r)).$$

On the other hand, from the first inequality of this case we get

$$\frac{3}{n}\nu_{P_Q}(z) \geq 12n\,\nu^{[1]}_{(f,H_1)}(z) + \frac{12n}{2n+1}\sum_{u=1}^{3}\sum_{i=2}^{q}\nu^{[n]}_{(f_u,H_i)}(z).$$

Integrating both sides, using (4.2) and applying Theorem 2.2 to each f_u and the $2n$ hyperplanes $H_2,\ldots,H_q$, we obtain

$$\begin{aligned}\|_E\, 6nT(r) \geq{}& \Big(4(n-3)+\frac{12n(n-1)}{2n+1}\Big)T(r)\\ &-\Big(6n(n+3)+\frac{18n^2(n+1)}{2n+1}\Big)S(r)+o(T(r)).\end{aligned}$$

By Lemma 3.11, we get

$$\begin{aligned}min\{c_{f_1},c_{f_2},c_{f_3}\} &\geq \frac{4(n-3)+\frac{12n(n-1)}{2n+1}-6n}{6n(n+3)+\frac{18n^2(n+1)}{2n+1}}\\ &= \frac{4n^2-19n-6}{3n(n+3)(2n+1)+9n^2(n+1)} =: g_4(n).\end{aligned}$$

If $n\geq 6$, then $g_4(n)>0$. Because $g(n)\leq g_4(n)$, this is a contradiction.

Subcase 2.3. There exists an index i_0 such that $\Phi_{1i_0}\equiv 0$ or $\Phi_{i_01}\equiv 0$, and $\Phi_{1j}\not\equiv 0$ for all $j\neq i_0$ with $j\geq n+1$. Applying Lemma 3.4 to the pair $(1,i_0)$ and Theorem 2.2 to the $2n-1$ hyperplanes $\{H_t\}_{t\neq 1,i_0}$, we get

$$\|_E\, N^{[1]}_{(f,H_1)}(r)+N^{[1]}_{(f,H_{i_0})}(r)\geq \frac{n-2}{3n}T(r)-\frac{n+3}{2}S(r)+o(T(r)).$$

Moreover, applying Lemma 3.6(ii) to the indices 1 and j for each $j\geq n+1$ with $j\neq i_0$, and summing up the n inequalities obtained, we get

$$\|_E\, nT(r)+6nS(r)\geq nN_1(r)+\sum_{j\geq n+1,\,j\neq i_0}N_j(r)+2n\sum_{i=1}^{q}N^{[1]}_{(f,H_i)}(r)+o(T(r)).$$

Transforming the right hand side by (4.1) as in [12] and using the two above estimates, we obtain

$$\|_E\, nT(r)\geq \frac{(n-2)(11n-10)}{3(2n-1)}T(r)-\kappa\, S(r)+o(T(r)),$$

where $\kappa := n(n+9)+\Big(\dfrac{n-2}{2n-1}+\dfrac{2}{3}\Big)\dfrac{3n(n+1)}{2}$. By Lemma 3.11, we get

$$\min\{c_{f_1},c_{f_2},c_{f_3}\}\geq \frac{1}{\kappa}\Big(\frac{(n-2)(11n-10)}{3(2n-1)}-n\Big)=\frac{5n^2-29n+20}{3(2n-1)}=:g_5(n).$$

The positivity of $g_5(n)$ can be proved for $n \geq 6$. As $g(n) \leq g_5(n)$, we obtain a contradiction with the assumption of Theorem 1.1.

Case 3. $n-5 \leq l_1 \leq n-1$. We may assume that $\operatorname{rank}\{V_1, \ldots, V_{l_1}\} = 1$. We put $t := l_1 + 3(n-l_1) + 1$, hence $t - 1 = 3n - 2l_1$, and

$$E^*_{I_0} := \{r \in E_{I_0} \ : \ N_t(r) < 0\},$$

$$x := \frac{3l_1 a\big(n(2n+1) - 2l_1(n-1)\big)}{(2n+1)(3n-2l_1)(n-a) - l_1 a(n-1)}, \qquad \lambda := \frac{x}{3a} + \frac{6l_1 - x}{2n+1}.$$

Assume first that $E^*_{I_0}$ is of infinite γ-measure. Then $N_j(r) < 0$ for all $r \in E^*_{I_0}$ and all $j \geq t$. Transforming (4.7) by means of (4.1) and the above choice of x, as in [12], we get

$$\|_E\, 3l_1 T(r) \geq \lambda \sum_{u=1}^{3} \sum_{i=1}^{q} N^{[n]}_{(f_u, H_i)}(r) + o(T(r)) \qquad (r \in E^*_{I_0}).$$

Combining this with (4.3), we get

$$\|_E\, 3l_1 T(r) \geq \lambda n\, T(r) - \frac{3n(n+1)}{2} \lambda\, S(r) + o(T(r)) \qquad (r \in E^*_{I_0}).$$

Applying Lemma 3.11 with $E_0 = E^*_{I_0}$, $A = 3l_1$, $B = \lambda n$ and $\kappa = \frac{3n(n+1)}{2}\lambda$, we obtain

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{2}{3n(n+1)}\Big(n - \frac{3l_1}{\lambda}\Big) =: g_6(n, l_1).$$

As in [12], we also see that $g_6(n, l_1) > 0$ is equivalent to $(n-l_1)(2n+1)(n^2 - 13n + 18) > 0$. This inequality is true for $n \geq 12$. Since $g_6(n, l_1) \geq g(n)$, this is impossible under the assumption of Theorem 1.1. Hence $\int_{E^*_{I_0}} \gamma(r)dr < +\infty$, and therefore, replacing E_{I_0} by $E_{I_0} \setminus E^*_{I_0}$, we may assume that $N_t(r) \geq 0$ for all $r \in E_{I_0}$.

Suppose now that there exists an index $j_0 \in \{l_1 + 1, \ldots, n\}$ such that $\Phi_{1j_0} \not\equiv 0$ and $\Phi_{j_0 1} \not\equiv 0$. Applying Lemma 3.6(ii) to the indices 1 and j_0 and using (4.1), we get the same inequality as in Subcase 2.1, and hence the same contradiction. Therefore $\Phi_{1j} \equiv 0$ or $\Phi_{j1} \equiv 0$ for all $j \in \{l_1 + 1, \ldots, n\}$. We also note that $\Phi_{1k} \equiv 0$ if and only if $\Phi_{ik} \equiv 0$ for all $i \in \{1, \ldots, l_1\}$, since $\Phi_{ik} = h_i \Phi_{1k}$ for some nonzero meromorphic function h_i.

Suppose next that there exist two indices $k, t' \in \{l_1 + 1, \ldots, q\}$ such that $\Phi_{ik} \equiv 0$ or $\Phi_{ki} \equiv 0$, and $\Phi_{it'} \equiv 0$ or $\Phi_{t'i} \equiv 0$ for all $1 \leq i \leq l_1$, and such that $F_1^{ik}, F_2^{ik}, F_3^{ik}$ as well as $F_1^{t'i}, F_2^{t'i}, F_3^{t'i}$ are distinct. Applying Lemma 3.4 to the pairs (i, k) and (i, t') and summing up the two inequalities obtained, we get

$$\|_E\, N^{[1]}_{(f, H_i)}(r) \geq \sum_{s \neq i,k,t'} N^{[1]}_{(f, H_s)}(r) - S(r) + o(T(r))$$

for all $1 \le i \le l_1$. Summing up over i and using $\{1,\dots,l_1\} \subset \{s : s \ne k, t'\}$, we obtain

$$\|_E\, (l_1-2) \sum_{s\ne k,t'} N^{[1]}_{(f,H_s)}(r) \le l_1 S(r) + o(T(r)).$$

Applying Theorem 2.2 to each f_u and the $2n-1$ hyperplanes $\{H_s\}_{s\ne k,t'}$, and using $N^{[1]} \ge \frac{1}{n} N^{[n]}$, we get

$$\|_E\, 0 \ge \frac{n-2}{3n} T(r) - \Big(\frac{n+1}{2} + \frac{l_1}{l_1-2}\Big) S(r) + o(T(r)).$$

By Lemma 3.11, this gives

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \ge \frac{n-2}{3n}\Big(\frac{n+1}{2} + \frac{l_1}{l_1-2}\Big)^{-1} =: g_7(n,l_1).$$

Here, $g_7(n,l_1) > 0$ for $n \ge 8$. Because $g(n) \le g_7(n,l_1)$, this is a contradiction. Hence this situation does not happen.

We put $\mathcal{Q} := \{l_1+1 \le k \le q \ : \ \Phi_{1k} \equiv 0 \text{ or } \Phi_{k1} \equiv 0\}$ and we may assume that $\mathcal{Q} = \{l_1+1,\dots,l_1+\sharp\mathcal{Q}\}$. We distinguish four subcases.

Subcase 3.1. $\sharp\mathcal{Q} \ge 3(n-l_1)+2$. As in [12], there exist two indices k, t' satisfying the conditions of the last situation above, which does not happen.

Subcase 3.2. $\sharp\mathcal{Q} = 3(n-l_1)+1$. Then there exists an index, for instance t, such that $\Phi_{it} \equiv 0$ or $\Phi_{ti} \equiv 0$ and $F_1^{ti}, F_2^{ti}, F_3^{ti}$ are distinct, and moreover $\Phi_{ij} \not\equiv 0$ and $\Phi_{ji} \not\equiv 0$ for all $j \ge t+1$ and $1 \le i \le l_1$. Applying Lemma 3.4 to the pair (i,t), and Lemma 3.6(ii) to the indices i and j, we get

$$\|_E\, 2N^{[1]}_{(f,H_i)}(r) + 2N^{[1]}_{(f,H_t)}(r) \ge \sum_{s=1}^{q} N^{[1]}_{(f,H_s)}(r) - S(r) + o(T(r))$$

and

$$\|_E\, T(r) + 6S(r) \ge N_i(r) + N_j(r) + 2\sum_{s=1}^{q} N^{[1]}_{(f,H_s)}(r) + o(T(r)).$$

Summing up the first inequality over $1 \le i \le l_1$ and applying Theorem 2.2 to the $2n$ hyperplanes $\{H_s\}_{s\ne t}$, we obtain

$$\|_E\, l_1 N^{[1]}_{(f,H_t)}(r) \ge \frac{(l_1-2)(n-1)}{3n} T(r) - \kappa_1 S(r) + o(T(r)), \tag{4.9}$$

where $\kappa_1 := \dfrac{(l_1-2)(n+1)}{2} + l_1$.

If $N_{t+1}(r) < 0$ on a subset of E_{I_0} of infinite γ-measure, then, transforming (4.7) as above and using (4.9), we get for $r \in E^*_{I_0}$,

$$\|_E\, 3l_1 T(r) \geq \Big(\frac{(6l_1-x)n}{2n+1}+\frac{x(n-1)}{3a}+\frac{x(l_1-2)(n-1)}{3nl_1}\Big)T(r)-\kappa_2 S(r)+o(T(r)),$$

where $\kappa_2 := \Big(\dfrac{6l_1-x}{2n+1}+\dfrac{x}{3a}\Big)\dfrac{3n(n+1)}{2}+\dfrac{x}{l_1}\kappa_1$. By Lemma 3.11 with $E_0 = E^*_{I_0}$, we obtain

$$\begin{aligned}\min\{c_{f_1}, c_{f_2}, c_{f_3}\} &\geq g_8(n,l_1)\\ &:= \frac{1}{\kappa_2}\Big(\frac{(6l_1-x)n}{2n+1}+\frac{x(n-1)}{3a}+\frac{x(l_1-2)(n-1)}{3nl_1}-3l_1\Big).\end{aligned}$$

As in [12], $g_8(n,l_1) > 0$ for $n \geq 8$. We thus obtain a contradiction with the assumption of Theorem 1.1, as $g(n) \leq g_8(n,l_1)$. Hence we may assume that $N_{t+1}(r) \geq 0$ for all $r \in E_{I_0}$.

Summing up the second inequality above over $1 \leq i \leq q-t$, with $j = t+1$, and using (4.1), (4.3) and (4.9), we obtain

$$\|_E\, T(r) \geq \Big(\frac{(n-1)(n-2)}{2n(3n-2l_1)}+\frac{2(n-1)}{3n}+\frac{2(l_1-2)(n-1)}{3l_1 n}\Big)T(r)-\kappa_3 S(r)+o(T(r)),$$

where

$$\kappa_3 := 6+\Big(\frac{1}{3n-2l_1}+\frac{2}{3n}-\frac{a}{n(3n-2l_1)}\Big)\frac{3n(n+1)}{2}+\frac{2}{l_1}\kappa_1.$$

By Lemma 3.11, we get $\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq g_9(n,l_1)$, where

$$g_9(n,l_1) := \frac{1}{\kappa_3}\Big(\frac{(n-1)(n-2)}{2n(3n-2l_1)}+\frac{2(n-1)}{3n}+\frac{2(l_1-2)(n-1)}{3l_1 n}-1\Big),$$

which is a contradiction. The positivity of $g_9(n,l_1)$ was shown in [12].

Subcase 3.3. $\sharp\mathcal{Q} \leq (n-l_1)+2$. Then $\Phi_{ij} \not\equiv 0$ and $\Phi_{ji} \not\equiv 0$ for all $j \geq n+3$ and $1 \leq i \leq l_1$, and $n+3 \leq t$. As above, we may assume that $N_{n+3}(r) \geq 0$ for all $r \in E_{I_0}$. Applying Lemma 3.6(ii) to the indices i and $n+3$ for $1 \leq i \leq l_1$, summing up the l_1 inequalities obtained and using (4.1) and (4.3), we get

$$\|_E\, T(r) \geq \Big(\frac{n}{n+2}+\frac{2}{3}-\frac{a}{n+2}\Big)T(r)-\kappa_4 S(r)+o(T(r)),$$

where $\kappa_4 := 6+\Big(\dfrac{1}{n+2}+\dfrac{2}{3n}-\dfrac{a}{n(n+2)}\Big)\dfrac{3n(n+1)}{2}$. By Lemma 3.11, we obtain

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{1}{\kappa_4}\Big(\frac{n}{n+2}+\frac{2}{3}-\frac{a}{n+2}-1\Big) =: g_{10}(n).$$

We can see that $g_{10}(n) > 10$ if and only if $3n^2 - 8n - 4 > 0$. This is true for $n \geq 4$. It leads to a contradiction.

Subcase 3.4. $(n - l_1) + 3 \leq \sharp\mathcal{Q} \leq 3(n - l_1)$. This implies that $l_1 \leq n - 2$. If there exists an index $j \in \{l_1 + 1, \ldots, 3(n - l_1)\}$ such that $\Phi_{1j} \equiv 0$ or $\Phi_{j1} \equiv 0$ and $F_1^{1j}, F_2^{1j}, F_3^{1j}$ are distinct, then we are in the same situation as in Subcase 3.2, and hence we get a contradiction. Therefore, for every such index j, the functions $F_1^{1j}, F_2^{1j}, F_3^{1j}$ are not distinct, and hence, writing $V_1 = (a_1, b_1, c_1)$, the vector V_j is of one of the three forms

$$V_j = h_j(a_j, b_1, c_1), \qquad V_j = h_j(a_1, b_1, c_j), \qquad V_j = h_j(a_1, b_j, c_1),$$

where h_j are meromorphic functions on $\Delta(R)$. As in [12], we choose two indices $k, h \in \{l_1 + 2, \ldots, n + 2\}$ such that V_{l_1+1} and V_k, as well as V_{l_1+1} and V_h, are not of the same form, and we put $\mathcal{R} := \{l_1 + 1, k, h\}$.

Assume first that $\nu_{(f_1,H_j)}(z) = \nu_{(f_2,H_j)}(z) = \nu_{(f_3,H_j)}(z)$ for all $j \in \mathcal{R}$. Splitting the sum over $\mathcal{R}$ and over $\{l_1 + 1, \ldots, q\} \setminus \mathcal{R}$, applying Lemma 3.9 to each of the brackets and choosing b and x as in Case 1, we obtain, after integrating and using (4.2),

$$\|_E\, 3l_1 T(r) \geq \lambda \sum_{u=1}^{3} \sum_{i=1}^{q} N^{[n]}_{(f_u,H_i)}(r) + \frac{D}{q - l_1} \sum_{u=1}^{3} \sum_{i \in \mathcal{R}} N^{[n]}_{(f_u,H_i)}(r) + o(T(r)),$$

where λ is as in Case 1 and $D := \dfrac{(n-1)\big(6l_1(2n+1) - (q-l_1)(6l_1 - x)\big)}{3n(2n+1)}$.

On the other hand, since $N^{[1]}_{(f_u,H_i)} \geq \frac{1}{n} N^{[n]}_{(f_u,H_i)}$ and by (4.1), we have

$$\sum_{u=1}^{3} \sum_{i \in \mathcal{R}} N^{[n]}_{(f_u,H_i)}(r) \geq \frac{n}{n - a} \sum_{i \in \mathcal{R}} N_i(r) \geq \frac{3n}{(n-2)(n-l_1)} \sum_{i=n+1}^{t-1} N_i(r).$$

Moreover, since $\Phi_{ij} \not\equiv 0$ and $\Phi_{ji} \not\equiv 0$ for all $1 \leq i \leq l_1$ and $t \leq j \leq q$, Lemma 3.6(ii) applied to the indices 1 and t, together with (4.1) and (4.3), gives

$$\|_E \sum_{i=n+1}^{t-1} N_i(r) \geq \frac{n-6}{6} T(r) - \kappa_5 S(r) + o(T(r)), \qquad \kappa_5 := 6n + \Big(\frac{5}{3} - \frac{a}{n}\Big) \frac{3n(n+1)}{2}.$$

Combining the three above inequalities with (4.3) and applying Lemma 3.11 with $A = 3l_1$,

$$B := \lambda n + \theta\, \frac{n-6}{6}, \qquad \kappa := \frac{3n(n+1)}{2} \lambda + \theta\, \kappa_5, \qquad \theta := \frac{D}{q - l_1} \cdot \frac{3n}{(n-2)(n-l_1)},$$

we obtain $\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{B - 3l_1}{\kappa} =: g_{11}(n, l_1)$. The positivity of $g_{11}(n, l_1), n \geq 11$ was proved in [12]. We then have a contradiction.

Therefore, there exists an index $j_0 \in \mathcal{R}$ such that $\nu_{(f_1,H_{j_0})}, \nu_{(f_2,H_{j_0})}, \nu_{(f_3,H_{j_0})}$ do not all coincide. As in [12], this implies that there exists $i_0 \in \mathcal{R} \setminus \{l_1+1\}$ such that $F_1^{l_1+1\,i_0}, F_2^{l_1+1\,i_0}, F_3^{l_1+1\,i_0}$ are distinct. We consider two possibilities.

If $\Phi_{l_1+1\,i_0} \equiv 0$ or $\Phi_{i_0\,l_1+1} \equiv 0$, then Lemma 3.4 applied to the pair (l_1+1, i_0) and Theorem 2.2 applied to the $2n-1$ hyperplanes $\{H_s\}_{s\neq l_1+1,i_0}$ give

$$\|_E\, N^{[1]}_{(f,H_{l_1+1})}(r) + N^{[1]}_{(f,H_{i_0})}(r) \geq \frac{n-2}{3n}T(r) - \frac{n+3}{2}S(r) + o(T(r)).$$

On the other hand, by Lemma 3.6(ii) applied to the indices 1 and t, we have

$$\|_E\, T(r)+6S(r) \geq 2\sum_{j\neq l_1+1,i_0} N^{[1]}_{(f,H_j)}(r)+2\big(N^{[1]}_{(f,H_{l_1+1})}(r)+N^{[1]}_{(f,H_{i_0})}(r)\big)+o(T(r)).$$

Estimating the first sum by Theorem 2.2 and the second one by the above inequality, we get

$$\|_E\, T(r) \geq \frac{4(n-2)}{3n}T(r) - (2n+10)S(r) + o(T(r)).$$

By Lemma 3.11, this gives

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{1}{2n+10}\Big(\frac{4(n-2)}{3n} - 1\Big) =: g_{12}(n).$$

It is seen that $g_{12}(n) > 0$ for $n \geq 9$. As $g(n) \leq g_{12}(n)$, we obtain a contradiction with the assumption of Theorem 1.1.

If $\Phi_{l_1+1\,i_0} \not\equiv 0$ and $\Phi_{i_0\,l_1+1} \not\equiv 0$, then Lemma 3.6(ii) applied to the indices l_1+1 and i_0, together with (4.1), gives

$$\|_E\, \frac{3(n-l_1)}{2l_1}\big(T(r)+6S(r)\big) \geq \frac{1}{l_1}\sum_{j=l_1+1}^{t-1} N_j(r)+\frac{3(n-l_1)}{l_1}\sum_{j=1}^{q} N^{[1]}_{(f,H_j)}(r)+o(T(r)),$$

while Lemma 3.6(ii) applied to the indices 1 and t gives

$$\|_E\, T(r) + 6S(r) \geq \frac{1}{l_1}\sum_{i=1}^{l_1} N_i(r) + \frac{1}{l_1}\sum_{j=t}^{q} N_j(r) + 2\sum_{j=1}^{q} N^{[1]}_{(f,H_j)}(r) + o(T(r)).$$

Summing up both sides of these two inequalities and using (4.3), we obtain

$$\|_E\, \frac{3n-l_1}{2l_1}T(r) \geq \Big(\frac{n}{l_1} + \frac{3n-2l_1-6}{6l_1}\Big)T(r) - \kappa_6 S(r) + o(T(r)),$$

where

$$\kappa_6 := 6 + \frac{9(n-l_1)}{l_1} + \frac{3n(n+1)}{2l_1} + \frac{n+1}{2}\Big(2 + \frac{3(n-l_1-a)}{l_1}\Big).$$

By Lemma 3.11, we get

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{1}{\kappa_6}\Big(\frac{n}{l_1} + \frac{3n-2l_1-6}{6l_1} - \frac{3n-l_1}{2l_1}\Big) =: g_{13}(n, l_1).$$

We have $g_{13}(n, l_1) > 0$ as in [12] for $n \geq 12$. This is impossible under the assumption of Theorem 1.1.

From all the above cases, we conclude that $f_1 \wedge f_2 \wedge f_3 \equiv 0$. The theorem is proved. ■

Proof of Theorem 1.2

Proof. As in the proof of Theorem 1.1, each of the cases below ends with an inequality $\|_E\, AT(r) \geq BT(r) - \kappa S(r) + o(T(r))$ on some set E_0 of infinite γ-measure, and Lemma 3.11 turns it into a lower bound $\frac{B-A}{\kappa}$ for $\min\{c_{f_1}, c_{f_2}, c_{f_3}\}$. We denote by $h_1, \ldots, h_4$ the four constants obtained in this way and we set

$$g^*(n) := \min\{g(n), h_1(n), h_2(n, l_1), h_3(n), h_4(n)\},$$

where $g(n)$ is defined in Theorem 1.1 and the minimum over h_2 is taken over all admissible values of l_1. As before, we will show that $h_k > 0$ for every k; since $g(n) > 0$ for $n \geq 12$, this proves that $g^*(n) > 0$ for every $n \geq 12$, as used in the statement of Theorem 1.2.
Suppose that f_1, f_2, f_3 are distinct. Since $g^*(n) \leq g(n)$, Theorem 1.1 gives $f_1 \wedge f_2 \wedge f_3 \equiv 0$. We set $q := 2n+1$ and $\Sigma(r) := \sum_{s=1}^{q} N^{[1]}_{(f,H_s)}(r)$. By Theorem 2.2 and by $N^{[1]}_{(f,H_s)}(r) \geq \frac{1}{n} N^{[n]}_{(f,H_s)}(r)$, we have

$$\|_E\, \Sigma(r) \geq \frac{1}{3} T(r) - \frac{n+1}{2} S(r) + o(T(r)). \tag{4.10}$$

We consider two cases.

Case 1. There exists an index $i_0 \neq 1$ such that $V_1 \cong V_{i_0}$ and $\Phi_{1i_0} \equiv 0$. We may assume that $\operatorname{rank}\{V_1, \ldots, V_{l_1}\} = 1$, and then $\Phi_{ii_0} \equiv 0$ for all $1 \leq i \leq l_1$. By Lemma 3.8, we have $\Phi_{ij} \equiv 0$ for all $j \geq l_1 + 1$ and

$$\|_E\, 2N^{[1]}_{(f,H_i)}(r) + 2N^{[1]}_{(f,H_j)}(r) \geq \Sigma(r) - S(r) + o(T(r))$$

for all $1 \leq i \leq l_1$ and $l_1 + 1 \leq j \leq q$. Summing up both sides of these $l_1(q-l_1)$ inequalities, we obtain

$$\|_E\, 2(q-2l_1) \sum_{i=1}^{l_1} N^{[1]}_{(f,H_i)}(r) \geq l_1(q-l_1-2)\Sigma(r) - l_1(q-l_1)S(r) + o(T(r)). \tag{4.11}$$

Suppose that $l_1 = 2$. Then (4.11) gives

$$\|_E\, 0 \geq \sum_{s=3}^{q} N^{[1]}_{(f,H_s)}(r) - \frac{2n-1}{2n-3} S(r) + o(T(r)),$$

and applying Theorem 2.2 to the $2n-1$ hyperplanes $H_3, \ldots, H_q$ we get

$$\|_E\, 0 \geq \frac{n-2}{3n} T(r) - \Big(\frac{n+1}{2} + \frac{2n-1}{2n-3}\Big) S(r) + o(T(r)).$$

By Lemma 3.11, this gives

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{2(n-2)(2n-3)}{3n(n-1)(2n+5)} =: h_1(n).$$

It is easy to see that $h_1(n) > 0$ for $n \geq 3$. Since $g^*(n) < h_1(n)$, this contradicts the assumption of Theorem 1.2.

Suppose that $l_1 \geq 3$. We can see that $2(q-2l_1) < l_1(q-l_1-2)$. Since $\sum_{i=1}^{l_1} N^{[1]}_{(f,H_i)}(r) \leq \Sigma(r)$, (4.11) gives

$$\|_E\, c_{l_1} \Sigma(r) \leq \frac{q-l_1}{q-l_1-2} S(r) + o(T(r)), \qquad c_{l_1} := 1 - \frac{2(q-2l_1)}{l_1(q-l_1-2)} > 0.$$

Combining with (4.10), we obtain

$$\|_E\, 0 \geq \frac{c_{l_1}}{3} T(r) - \Big(\frac{c_{l_1}(n+1)}{2} + \frac{2n+1-l_1}{2n-1-l_1}\Big) S(r) + o(T(r)),$$

and therefore, by Lemma 3.11,

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{2\big(-l_1^2 + (2n+3)l_1 - 4n - 2\big)}{3\big[(n+1)p(l_1) + 2l_1(2n+1-l_1)\big]} =: h_2(n, l_1).$$

For $3 \leq l_1 \leq n-1$, we can prove that $-l_1^2 + (2n+3)l_1 - 4n - 2 > 0$. Hence $h_2(n, l_1) > 0$. This leads to a contradiction.

Therefore $l_1 = 1$, and (4.11) becomes

$$\|_E\, (q-1) N^{[1]}_{(f,H_1)}(r) \geq (q-3) \sum_{i=2}^{q} N^{[1]}_{(f,H_i)}(r) - (q-1) S(r) + o(T(r)). \tag{4.12}$$

Assume that there exist two distinct indices $k, t \neq 1$ having the same property as the index 1. Then (4.12) also holds with 1 replaced by k and by t. Summing up both sides of these three inequalities and using $N^{[1]}_{(f,H_1)}(r) + N^{[1]}_{(f,H_k)}(r) + N^{[1]}_{(f,H_t)}(r) \leq \Sigma(r)$, we get

$$\|_E\, (q-5)\Sigma(r) \leq 3(q-1) S(r) + o(T(r)).$$

Together with (4.10), we obtain

$$\|_E\, 0 \geq \frac{q-5}{3}T(r) - \Big(\frac{(q-5)(n+1)}{2} + 3(q-1)\Big)S(r) + o(T(r)).$$

Lemma 3.11 gives

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{2(n-2)}{3(n^2+5n-2)} =: h_3(n).$$

We have $h_3(n) > 0$ if $n \geq 3$. Because $g^*(n) \leq h_3(n)$, this is a contradiction.

Hence there exists at most one index k having this property. This means that $\Phi_{ij} \not\equiv 0$ and $\Phi_{ji} \not\equiv 0$ for all $i, j \in \mathcal{R}$, where $\mathcal{R} := \{1, \ldots, q\} \setminus \{1, k, j_k\}$ and $\sharp\mathcal{R} = q - 3 = 2n - 2$. Let $i < j$ be the two smallest indices of $\mathcal{R}$. By (4.1), we have $N_i(r) \geq N_j(r) \geq N_s(r)$ for all $s \in \mathcal{R}$ and all $r \in E_{I_0}$, and hence

$$N_i(r) + N_j(r) \geq \frac{2}{\sharp\mathcal{R}} \sum_{s\in\mathcal{R}} N_s(r) = \frac{1}{n-1} \sum_{s\in\mathcal{R}} N_s(r) \qquad (r \in E_{I_0}).$$

Then, by Lemma 3.6(ii), by (4.12) and by $N^{[1]}_{(f_u,H_s)}(r) \geq \frac{1}{n} N^{[n]}_{(f_u,H_s)}(r)$, we obtain

$$\begin{aligned}
\|_E\, T(r) &\geq N_i(r) + N_j(r) + 2\sum_{s=1}^{q} N^{[1]}_{(f,H_s)}(r) - 6S(r) + o(T(r)) \\
&\geq \frac{1}{n-1}\Big(1 - \frac{n+2}{2n}\Big) \sum_{u=1}^{3} \sum_{s\in\mathcal{R}} N^{[n]}_{(f_u,H_s)}(r) \\
&\quad + 2\Big(\frac{q-3}{q-1} + 1\Big) \sum_{s=2}^{q} N^{[1]}_{(f,H_s)}(r) - 8S(r) + o(T(r)) \\
&\geq \frac{n-3}{n-1}\Big(1 - \frac{n+2}{2n}\Big)T(r) + \frac{4n-2}{3n} \cdot \frac{n-1}{n} T(r) - \kappa S(r) + o(T(r)) \\
&= BT(r) - \kappa S(r) + o(T(r)),
\end{aligned}$$

where

$$\kappa := 8 + \frac{3(n-2)(n+1)}{4(n-1)} + \frac{(2n-1)(n+1)}{n} = \frac{11n^3 + 25n^2 - 46n + 4}{4n(n-1)},$$

$$B := \frac{(n-3)(n-2)}{2n(n-1)} + \frac{(4n-2)(n-1)}{3n^2} = \frac{11n^3 - 35n^2 + 34n - 4}{6n^2(n-1)},$$

and therefore Lemma 3.11 gives

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq \frac{B-1}{\kappa} = \frac{2\big(5n^3 - 29n^2 + 34n - 4\big)}{3n\big(11n^3 + 25n^2 - 46n + 4\big)} =: h_4(n).$$

We can show that $h_4(n) > 0$ for $n \geq 12$. This leads to a contradiction.

Case 2. $\Phi_{1i} \not\equiv 0$ for all $i \neq 1$ with $V_1 \cong V_i$. Since $H_1, \ldots, H_q$ are in general position, in the set $\{1, \ldots, n+1\}$ there exists an index $i_0 \neq 1$ such that $V_{i_0} \cong V_1$. Then $\Phi_{1i_0} \not\equiv 0$ and $\Phi_{i_0 1} \not\equiv 0$, and Lemma 3.6(ii), applied to the pair of indices $(1, i_0)$, gives

$$\|_E\, T(r) + 6S(r) \geq N_1(r) + N_{i_0}(r) + 2\Sigma(r) + o(T(r)).$$

Since $i_0 \leq n+1$, (4.1) gives $N_{i_0}(r) \geq N_{n+1}(r)$ for all $r \in E_{I_0}$, and hence

$$\|_E\, T(r) + 6S(r) \geq N_1(r) + N_{n+1}(r) + 2\Sigma(r) + o(T(r)) \qquad (r \in E_{I_0}).$$

This is exactly the inequality obtained in Subcase 2.1 of the proof of Theorem 1.1. Multiplying both sides by $n+1$ and using (4.1) together with $nN_1(r) \geq \sum_{i=1}^{n} N_i(r)$, $N_1(r) \geq \frac{1}{2n+1}\sum_{i=1}^{q} N_i(r)$ and $(n+1)N_{n+1}(r) \geq \sum_{j=n+1}^{q} N_j(r)$ as there, we obtain

$$\|_E\, (n+1)T(r) + 6(n+1)S(r) \geq \mu \sum_{u=1}^{3}\sum_{i=1}^{q} N^{[n]}_{(f_u, H_i)}(r) + o(T(r)),$$

where $\mu = \frac{2n+2}{2n+1} + \frac{(n+1)(n-4)}{3n(2n+1)}$. Combining this with (4.3) and applying Lemma 3.11 with $A = n+1$, $B = \mu n$ and $\kappa = \frac{3n(n+1)}{2}\mu + 6(n+1)$, we get

$$\min\{c_{f_1}, c_{f_2}, c_{f_3}\} \geq g_3(n),$$

which is a contradiction.

From all the above cases, we conclude that f_1, f_2, f_3 are not distinct. The theorem is proved. ■

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Nguyen The Hoang
Faculty of General Studies, CMC University
Hanoi
Vietnam
nthoang@cmc.edu.vn

Nguyen Thi Nhung
Department of Mathematics, Thang Long University
Hanoi
Vietnam
nhungnt@thanglong.edu.vn

Hoang Thu Thuy
Department of Mathematics, Hanoi University of Civil Engineering
Hanoi
Vietnam
thuyht@huce.edu.vn