J. Math. Math. Sci., 2022 (**1**), 67–85

OPTIMALITY CONDITIONS FOR EFFICIENCY OF CONSTRAINED VECTOR EQUILIBRIUM PROBLEMS

Do Van Luu (Hanoi, Vietnam)
Tran Thi Mai (Thai Nguyen, Vietnam)

(Received 20 December 2021; accepted 25 March 2022)

Abstract. Fritz John necessary conditions for local Henig and global efficient solutions of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions are established via convexificators. Under suitable constraint qualifications, Kuhn–Tucker necessary conditions for local Henig and gobally efficient solutions are derived. Note that Henig and global efficient solutions of (VEP) are studied with respect to a closed convex cone. Sufficient condition for Henig and globally efficient solutions are derived under some assumptions on asymptotic semiinvexity-infine of the problem. Some illustrative examples are also given.

1. Introduction

In recent years, vector equilibrium problems have been extensively studied with many applications. Vector equilibrium problems include a lot of other problems as special cases such as vector variational inequalities, vector optimization problems, vector saddle point problems, vector complementarity problems, vector Nash equilibrium problems. Optimality conditions for weakly efficient solutions, efficient solutions, Henig efficient solutions, globally efficient solutions and superefficient solutions of vector equilibrium problems have been

Key words and phrases: .Local Henig efficient solution, local global solution, vector equilibrium problems, Fritz John and Kuhn–Tucker necessary conditions, convexificators
2010 Mathematics Subject Classification: *90C46, 91B50, 49J52*
The Project is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No 101.01-2021.06.

studied by many authors (see, e.g., [2–5], [8–14], and references therein). There are lot of works to dealt with optimality conditions for Henig and global efficient solutions of vector equilibrium problems. Gong [5] derived optimality conditions for Henig and global efficient solutions of vector equilibrium problems with a set constraint. Long et al. [8] established optimality conditions for Henig efficient solutions of vector equilibrium problems involving a cone-constraint and a set constraint with subconvexlike functions. Recently, Luu–Hang [11] derived optimality conditions for efficient solutions of vector equilibrium problems involving equality and inequality constraints with locally Lipschitz functions in terms of the Clarke subdifferentials on using the notion of quasirelative interior of a convex set in infinite dimensional spaces, but not for Henig and global efficient solutions. Necessary and sufficient conditions for efficiency of nonsmooth constrained vector optimization problems via convexificators are established by Luu [12–14].

Motivated by the works [8, 14], in this paper we establish Fritz John and Kuhn–Tucker necessary conditions for local Henig and global efficient solutions of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions via convexificators. Under suitable constraint qualifications, Kuhn–Tucker necessary conditions for local Henig efficient solutions are derived. Sufficient conditions for Henig and global efficient solutions are derived under some assumptions on asymptotic semiinvexity-infine of the considering problem.

The remainder of the paper is organized as follows. After some preliminaries, in Section 3, based on a Fritz John necessary condition by Luu [14], we derive Fritz John necessary conditions for local Henig and global efficient solutions with respect to a closed convex cone of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions. In Section 4, Kuhn–Tucker necessary conditions for local Henig and global efficient solutions are derived under some suitable constraint qualifications. Note that Henig and global efficient solutions of (VEP) are studied with respect to a closed convex cone. Kuhn–Tucker necessary conditions via convexificators can be sharper than those expressed in terms of the Clarke subfifferentials and the Michel–Penot subdifferentials. Observe that the results obtained in this paper are more general than those obtained by Gong [5] for vector equilibrium problems with only a set constraint, and those obtained by Long et al. [8] for vector equilibrium problems with subconvexlike functions. Section 5 presents sufficient conditions for Henig and global efficient solutions under some assumptions on asymptotic semiinvexity-infine of the problem.

2. Preliminaries

Let X be a Banach space, X^* topological dual of X, $\overline{x} \in X$. We recall some notions on convexificators in [7]. The lower (upper) Dini directional derivatives of $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ at $\overline{x} \in X$ in a direction $v \in X$ is defined as

$$f^-(\overline{x}; v) := \liminf_{t \downarrow 0} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}$$

$$\Big(\text{resp. } f^+(\overline{x}; v) := \limsup_{t \downarrow 0} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}\Big).$$

In case $f^+(\overline{x}; v) = f^-(\overline{x}; v)$, their common value is denoted by $f'(\overline{x}; v)$, which is called Dini derivative of f at $\overline{x}$ in the direction v. The function f is called Dini differentiable at $\overline{x}$ iff its Dini derivative at $\overline{x}$ exists in all directions.

Recall [7] that the function f is said to have an upper (lower) convexificator $\partial^* f(\overline{x})$ (resp. $\partial_* f(\overline{x})$) at $\overline{x}$ iff $\partial^* f(\overline{x}) \subseteq X^*$ (resp. $\partial_* f(\overline{x}) \subseteq X^*$) is weakly* closed, and for all $v \in X$,

$$f^-(\overline{x}; v) \leqslant \sup_{\xi \in \partial^* f(\overline{x})} \langle \xi, v \rangle$$

$$\Big(\text{resp. } f^+(\overline{x}; v) \geqslant \inf_{\xi \in \partial_* f(\overline{x})} \langle \xi, v \rangle\Big).$$

A weakly* closed set $\partial^* f(\overline{x}) \subseteq X^*$ is said to be a convexificator of f at $\overline{x}$ iff it is both upper and lower convexificators of f at $\overline{x}$.

The function f is said to have an upper (lower) semi–regular convexificator $\partial^* f(\overline{x})$ (resp. $\partial_* f(\overline{x})$) at $\overline{x}$ iff $\partial^* f(\overline{x})$ (resp. $\partial_* f(\overline{x})$) is weakly* closed and for all $v \in X$,

$$f^+(\overline{x}; v) \leqslant \sup_{\xi \in \partial^* f(\overline{x})} \langle \xi, v \rangle$$

$$\Big(\text{resp. } f^-(\overline{x}; v) \geqslant \inf_{\xi \in \partial_* f(\overline{x})} \langle \xi, v \rangle\Big).$$

If equality holds in these inequalities, then $\partial^* f(\overline{x})$ (resp. $\partial_* f(\overline{x})$) is called an upper (resp. lower) regular convexificator of f at $\overline{x}$.

Following [1], the Clarke generalized directional derivative of f at $\overline{x}$, with respect to a direction v, is defined as

$$f^0(\overline{x}; v) = \limsup_{x \to \overline{x},\ t \downarrow 0} \frac{f(x + tv) - f(x)}{t}\,.$$

The Clarke subdifferential of f at $\overline{x}$ is

$$\partial f(\overline{x}) = \Big\{\xi \in X^* : \ \langle \xi, v\rangle \leqslant f^0(\overline{x}; v), \ \forall\, v \in X\Big\}.$$

For a locally Lipschitz function f at $\overline{x}$, $\partial f(\overline{x})$ is a convexificator of f at $\overline{x}$ (see [7]).

The Michel–Penot directional derivative of f at $\overline{x}$ in a direction $v \in X$ is defined as follows

$$f^{\diamondsuit}(\overline{x}; v) := \sup_{w\in X} \limsup_{t\downarrow 0} \frac{f(\overline{x} + t(v + w)) - f(\overline{x} + tw)}{t}.$$

The Michel–Penot subdifferential of f at $\overline{x}$ is

$$\partial^{\diamondsuit} f(\overline{x}) := \Big\{\xi \in X^* \ : \ \langle \xi, v\rangle \leqslant f^{\diamondsuit}(\overline{x}; v), \forall\, v \in X\Big\}.$$

If f is locally Lipschitz at $\overline{x}$, then $\partial^{\diamondsuit} f(\overline{x})$ is also a convexificator of f at $\overline{x}$. The convex hull of a convexificator of a locally Lipschitz function may be strictly contained in both the Clarke and Michel-Penot subdifferentials (see [7], Example 2.1). It is obvious that for a function f which is locally Lipschitz at $\overline{x}$,

$$f^{\diamondsuit}(\overline{x}; v) \leqslant f^0(\overline{x}; v) \quad (\forall v \in X),$$
$$\partial^{\diamondsuit} f(\overline{x}) \subseteq \partial f(\overline{x}).$$

The following example shows that the subdifferentials $\partial^{\diamondsuit} f(\overline{x})$ and $\partial f(\overline{x})$ may be greatly different, but they are convexificators of f at $\overline{x}$.

Example 2.1. The function f be defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} x^2|\cos\frac{\pi}{x}|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, $\partial^{\diamondsuit} f(0) = \{0\}$, $f^{\diamondsuit}(0; v) = 0$ $(\forall v \in \mathbb{R})$, $f^0(0; v) = \pi|v|$ $(\forall v \in \mathbb{R})$, $\partial f(0) = [-\pi, \pi]$. Thus, $\partial^{\diamondsuit} f(0) \subsetneqq \partial f(0)$, but $\{0\}$ and $[-\pi, \pi]$ are convexificators of f at $\overline{x}$.

Recall [1] that the Clarke tangent cone to a set $C \subseteq X$ at a point $\overline{x} \in C$ is defined as

$$T(C; \overline{x}) := \Big\{v \in X : \ \forall\, x_n \in C, x_n \to \overline{x}, \forall\, t_n \downarrow 0, \exists\, v_n \to v \text{ such that } x_n + t_n v_n \in C, \forall\, n\Big\}.$$

The Clarke normal cone to C at $\overline{x}$ is

$$N(C;\overline{x}) := \Big\{\xi \in X^* : \langle \xi, v\rangle \leqslant 0, \forall\, v \in T(C;\overline{x})\Big\}.$$

Thus $N(C;\overline{x}) = T(C;\overline{x})^\circ = -T(C;\overline{x})^*$, where $T(C;\overline{x})^\circ$ is the polar of $T(C;\overline{x})$, and $T(C;\overline{x})^*$ is the dual cone of $T(C;\overline{x})$. Note that the cones $T(C;\overline{x})$ and $N(C;\overline{x})$ are nonempty convex, $T(C;\overline{x})$ is closed and $N(C;\overline{x})$ is weakly* closed.

3. Fritz John necessary conditions for efficiency

This section deals with Fritz John necessary conditions for local Henig and globally efficient solutions of vector equilibrium problems via convexificators. The Kuhn–Tucker necessary conditions obtained here via convexificators can be sharper than those expressed in terms of the Clarke and Michel–Penot subdifferentials.

Let K a nonempty closed subset of a Banach space X. Let F be a mapping from $K \times K$ to $\mathbb{R}^r$ and Q a pointed closed convex cone in $\mathbb{R}^r$. Let us consider the following vector equilibrium problem (VEP): Finding a point $x \in K$ such that

$$F(x, y) \notin -Q \setminus \{0\} \ (\forall y \in K). \tag{1}$$

A vector $\overline{x}$ solved (1) will be called efficient solution of (VEP).

A vector $x \in K$ is called a globally efficient solution to Problem (VEP) iff there exists a pointed convex cone $M \subset \mathbb{R}^r$ with $Q \ \setminus \{0\} \subset \mathrm{int}M$ such that

$$F(x, y) \cap ((-M) \setminus \{0\}) = \emptyset \ (\forall y \in K). \tag{2}$$

For $\overline{x} \in K$, denoting $F_{\overline{x}}(y) := F(\overline{x}, y)$, we have that

$$F_{\overline{x}}(y) = (F_{1,\overline{x}}(y), \dots, F_{r,\overline{x}}(y)).$$

Denote the dual cone of Q by $Q^* := \{y^* \in Y^* : \ \langle y^*, y\rangle \geqslant 0, \ \forall y \in Q\}$. A nonempty convex subset B of Q is called a base of Q, if $Q = \mathrm{cone}B$ and $0 \notin \mathrm{cl}B$, where cl stands for the closure, cone$B$ denotes the cone hull of B: $\mathrm{cone}B = \{tb : \ t \geqslant 0, b \in B\}$. Denote the quasi-interior of Q^* by $Q^\# := \{y^* \in \mathbb{R}^r : \ \langle y^*, y\rangle > 0, \ \forall y \in Q \setminus \{0\}\}$. We set $Q^\Delta(B) := \{y^* \in Q^\# : \exists t > 0$ such that $\langle y^*, b\rangle \geqslant t, \forall b \in B\}$. Then $Q^\Delta(B)$ is a cone in $\mathbb{R}^r$ and $Q^\Delta(B) \subseteq Q^\#$. Moreover, if $y^* \in Q^\Delta(B)$, then $y^* \neq 0$. If B is a base of the cone Q, by a separation theorem see, e.g., Theorem 3.6 [6]), there exists $y^* \in \mathbb{R}^r \setminus \{0\}$ such that

$$\alpha := \ \inf\{\langle y^*, b\rangle : \ b \in B\} > y^*(0) = 0.$$

Then the set $V_B := \{y \in \mathbb{R} : \mid \langle y^*, y\rangle \mid < \frac{\alpha}{2}\}$ is an absolutely convex open neighborhood of $0 \in Y$ (see [5]), and $\inf\{\langle y^*, y\rangle : y \in B + V_B\} \geqslant \frac{\alpha}{2}$. For each convex neighborhood U of 0, $U \subseteq V_B$, one has $0 \notin \mathrm{cl}(B+U)$. Hence, the set $Q_U(B) := \mathrm{cone}(U+B)$ is a pointed convex cone, and

$$Q \setminus \{0\} \subseteq \mathrm{int} Q_U B. \tag{3}$$

A vector $\overline{x} \in K$ is called Henig efficient solution of (VEP) if there is an absolutely convex neighborhood U of 0, $U \subseteq V_B$ such that

$$\mathrm{cone} F_{\overline{x}}(K) \cap (-\mathrm{int} Q_U(B)) = \emptyset,$$

where $F_{\overline{x}}(K) = \bigcup_{y \in K} F_{\overline{x}}(y)$. Since $Q_U(B)$ is a pointed convex cone, $\overline{x}$ is a Henig solution if and only if

$$F_{\overline{x}}(K) \cap (-\mathrm{int} Q_U(B)) = \emptyset. \tag{4}$$

Note that $\overline{x} \in K$ is a Henig efficient solution of (VEP) if and only if there is an absolutely convex neighborhood U of 0, $U \subseteq V_B$ such that (see [5])

$$\mathrm{cone} F_{\overline{x}}(K) \cap (U - B) = \emptyset. \tag{5}$$

If in the definitions of efficient solution, globally efficient solution and Henig efficient solution, K is replaced by $K \cap W$ for some neighborhood W of $\overline{x}$, we obtain the notions of local efficient solution, local global solution and local Henig efficient solution for (VEP), respectively.

Remark 3.1 It follows from (1)–(4) that a Henig efficient solution is an efficient solution, and globally efficient solution is also an efficient solution.

Let g and h be mappings from X into $\mathbb{R}^m$ and $\mathbb{R}^\ell$, respectively, and let C be nonempty closed subsets of X. Then g and h can be expressed as follows: $g = (g_1, \ldots, g_m), h = (h_1, \ldots, h_\ell)$. This paper deals with the vector equilibrium problem (VEP) in which K is described by

$$K = \Big\{x \in C : \ g(x) \leqslant 0, \ h(x) = 0\Big\}.$$

This constrained vector equilibrium problem is denoted by (CVEP). We set $I := \{1, \ldots, m\}$, $L := \{1, \ldots, \ell\}$, and

$$I(\overline{x}) := \{i \in I : g_i(\overline{x}) = 0\},$$
$$H := \{x \in C : \ h(x) = h(\overline{x})\}.$$

Recall that a point $\overline{x}$ is said to be a regular point in the sense of Ioffe for h relative to C if there exist numbers $K > 0$ and $\delta > 0$ such that for all $x \in C \cap B(\overline{x}; \delta)$,

$$d_H(x) \leqslant K \parallel h(x) - h(\overline{x}) \parallel,$$

where $d_H(x)$ denotes the distance from x to H, $B(\overline{x};\delta)$ stands for the open ball of radius δ around $\overline{x}$ (see, e.g., [14]).

The following assumptions are posed on Problem (CVEP).

Assumption 3.1

(a) $\overline{x}\in H$; C is convex; Q has a base B; $F_{\overline{x}}, h_j$ $(j\in L)$ are locally Lipschitz at $\overline{x}$, $g_i(i\in I(\overline{x}))$ are continuous in a neighborhood of $\overline{x}$.

(b) The functions $F_{k,\overline{x}}$ and h_j admit upper convexificators $\partial^* F_{k,\overline{x}}(x)$ $(k\in J)$ and $\partial^* h_j(x)$ $(j\in L)$ at x near $\overline{x}$, respectively; g_i $(i\in I(\overline{x}))$ admit upper convexificators $\partial^* g(\overline{x})$ at $\overline{x}$; the functions $\mid h_j \mid$ $(j\in L)$ are regular in the sense of Clarke at $\overline{x}$, that is for every $v\in X$ there exists $f'(\overline{x};v)$ and $f'(\overline{x};v)=f^0(\overline{x};v)$.

(c) $\partial^* F_{1,\overline{x}}(\overline{x}),\dots,\partial^* F_{r,\overline{x}}(\overline{x})$, $\partial^* h_1(\overline{x}),\dots,\partial^* h_\ell(\overline{x})$ are bounded; the convexificator maps $\partial^* F_{1,\overline{x}},\dots,\partial^* F_{r,\overline{x}}$, $\partial^* h_1,\dots,\partial^* h_\ell$ are upper semicontinuous at $\overline{x}$.

We shall begin with establishing a Fritz John necessary optimality condition for local Henig efficient solution of (CVEP).

Theorem 3.1. *Let $\overline{x}$ be a local Henig efficient solution of (CVEP). Assume that $F_{\overline{x}}(\overline{x})=0$, Assumption 3.1 hold. Then there exist $\overline{\tau}\geqslant 0, \overline{\mu}_i\geqslant 0$ $(i\in I(\overline{x}))$, $\overline{\nu}:=(\overline{\nu}_1,\dots,\overline{\nu}_\ell)\in\mathbb{R}^\ell$ with $(\overline{\tau},\overline{\mu}_1,\dots,\overline{\mu}_{|I(\overline{x})|})\neq 0$, and a continuous positively homogeneous function Λ on Y satisfying*
(i) If $y_2-y_1\in Q\setminus\{0\}$, then $\Lambda(y_1)<\Lambda(y_2)$,
(ii) $\exists\beta_0>0$ such that $\Lambda(-b)\leqslant-\beta_0$ $(\forall b\in B)$,
such that

$$0\in cl\Big(\overline{\tau}\, conv\,\partial^*(\Lambda\circ F_{\overline{x}})(\overline{x})+\sum_{i\in I(\overline{x})}\overline{\mu}_i\, conv\,\partial^* g_i(\overline{x})+\sum_{j\in L}\overline{\nu}_j\, conv\,\partial^* h_j(\overline{x})+N(C;\overline{x})\Big). \tag{6}$$

where $|I(\overline{x})|$ denotes the capacity of $|I(\overline{x})|$.

Proof Since $\overline{x}$ is a local Henig efficient solution of (CVEP), there are a neighborhood W of $\overline{x}$ and an absolutely convex neighborhood U of 0, $U\subseteq V_B$ such that

$$\mathrm{cone}F_{\overline{x}}(K\cap W)\cap(-\mathrm{int}Q_U(B))=\emptyset.$$

Applying Theorem 3.2 in [5] yields the existence a continuous positively homogeneous subadditive function Λ on Y such that (i), (ii) hold, and

$$(\Lambda\circ F_{\overline{x}})(x)\geqslant 0\ (\forall x\in K\cap W). \tag{7}$$

Since $F_{\overline{x}}(\overline{x})=0$ and Λ is positively homogeneous, we have $(\Lambda\circ F_{\overline{x}})(\overline{x})=0$. In view of (7), we deduce that $\overline{x}$ is a local minimum of the following scalar optimization problem:

$$(\mathrm{P}) \qquad \begin{aligned} &\min(\Lambda\circ F_{\overline{x}})(x),\\ \text{s.t.}\quad & g_i(x)\leqslant 0\ (i\in I),\\ & h_j(x)=0\ (j\in J),\\ & x\in C\cap W. \end{aligned}$$

Since the function Λ is continuous and convex, we can apply Proposition 2.2.6 [1] to deduce that it is locally Lipschitz. Observe that in the scalar case, a local Henig solution is a local minimum. Taking account of Theorem 3.2 [14] to the scalar problem (P) yields the existence of $\overline{\tau}\geqslant 0, \overline{\mu}_i\geqslant 0$ ($\forall i\in I(\overline{x})$) with $(\overline{\tau},\overline{\mu}_1,\ldots,\overline{\mu}_{|I(\overline{x})|})\neq 0$, $\overline{\nu}_j\in\mathbb{R}$ ($\forall j\in L$) such that

$$\begin{aligned} 0\in \mathrm{cl}\Big(&\overline{\tau}\mathrm{conv}\,\partial^*(\Lambda\circ F_{\overline{x}})(\overline{x})+\sum_{i\in I(\overline{x})}\overline{\mu}_i\mathrm{conv}\,\partial^* g_i(\overline{x})+\\ &\sum_{j\in L}\overline{\nu}_j\mathrm{conv}\ \partial^* h_j(\overline{x})+N_{C\cap W}(\overline{x})\Big). \end{aligned} \tag{8}$$

On the other hand,

$$N_{C\cap W}(\overline{x})=N_C(\overline{x}).$$

Hence, (8) implies (6) . $\square$

A Fritz John necessary optimality condition for local global efficient solution of (CVEP) can be stated as follows.

Theorem 3.2. *Let $\overline{x}$ be a local global efficient solution of (CVEP). Assume that $F_{\overline{x}}(\overline{x})=0$, Assumption 3.1 hold. Then there exist $\overline{\tau}\geqslant 0,\overline{\mu}_i\geqslant 0$ ($i\in I(\overline{x})$), $\overline{\nu}:=(\overline{\nu}_1,\ldots,\overline{\nu}_\ell)\in\mathbb{R}^\ell$ with $(\overline{\tau},\overline{\mu}_1,\ldots,\overline{\mu}_{|I(\overline{x})|})\neq 0$, and a continuous positively homogeneous function Λ on Y satisfying that if $y_2-y_1\in Q\setminus\{0\}$, then $\Lambda(y_1)<\Lambda(y_2)$, such that*

$$\begin{aligned} 0\in cl\Big(&\overline{\tau}\,conv\,\partial^*(\Lambda\circ F_{\overline{x}})(\overline{x})+\sum_{i\in I(\overline{x})}\overline{\mu}_i\,conv\,\partial^* g_i(\overline{x})+\\ &\sum_{j\in L}\overline{\nu}_j\,conv\ \partial^* h_j(\overline{x})+N(C;\overline{x})\Big). \end{aligned} \tag{9}$$

Proof Since $\overline{x}$ is a local global efficient solution of (CVEP), there are a neighborhood W of $\overline{x}$ and a pointed convex cone H satisfying $Q\setminus\{0\}\subset \mathrm{int}H$ such that

$$F_{\overline{x}}(K)\cap(-H)\setminus\{0\}=\emptyset. \tag{10}$$

Taking $a \in Q \setminus \{0\}$, it follows that $a \in \text{int}H$. We invoke Theorem 3.3 [5] to deduce that the function defined by $\Lambda(y) = \inf\{t \in \mathbb{R} : y \in ta - H\}$ satisfies the following conditions:

$$\begin{aligned}&\text{(a) if } y_2 - y_1 \in Q \setminus \{0\}, \text{then } \Lambda(y_1) < \Lambda(y_2);\\ &\text{(b) } \Lambda(F_{\overline{x}}(y)) \geq 0 \text{ (for all } y \in K).\end{aligned}$$

Since $F_{\overline{x}}(\overline{x}) = 0$ and Λ is positively homogeneous, we have $(\Lambda \circ F_{\overline{x}})(\overline{x}) = 0$. In view of (b), we deduce that $\overline{x}$ is a local minimum of the following scalar optimization problem:

$$\text{(P1)} \qquad \begin{aligned} &\min(\Lambda \circ F_{\overline{x}})(x),\\ \text{s.t.} \quad & g_i(x) \leqslant 0 \ (i \in I),\\ & h_j(x) = 0 \ (j \in J),\\ & x \in C \cap W.\end{aligned}$$

Observe that in the scalar case, a local global solution is a local minimum. Hence, we can apply Theorem 3.2 [14] to the scalar problem (P1). In the same way as in the proof of Theorem 3.1 we deduce that there exist $\overline{\tau} \geqslant 0, \overline{\mu}_i \geqslant 0$ $(\forall i \in I(\overline{x}))$ with $(\overline{\tau}, \overline{\mu}_1, \ldots, \overline{\mu}_{|I(\overline{x})|}) \neq 0$, $\overline{\nu}_j \in \mathbb{R}$ $(\forall j \in L)$ such that

$$\begin{aligned} 0 \in \text{cl}\Big(&\overline{\tau}\text{conv}\,\partial^*(\Lambda \circ F_{\overline{x}})(\overline{x}) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \text{conv}\,\partial^* g_i(\overline{x}) +\\ &\sum_{j \in L} \overline{\nu}_j \text{conv}\ \partial^* h_j(\overline{x}) + N_{C \cap W}(\overline{x})\Big), \end{aligned} \tag{11}$$

which together with the fact that $N_{C \cap W}(\overline{x}) = N_C(\overline{x})$ implies (9) . □

4. Kuhn–Tucker necessary conditions for efficiency

To derive Kuhn–Tucker necessary conditions for Henig efficiency, we introduce the following constraint qualification, which is called (CQ1): There exist $d_0 \in T(C; \overline{x})$ and numbers $b_i > 0$ $(i \in I(\overline{x}))$ such that
(i) $\langle \eta_i, d_0 \rangle \leqslant -b_i$ $(\forall \eta_i \in \partial^* g_i(\overline{x}), \forall i \in I(\overline{x}))$;
(ii) $\langle \zeta_j, d_0 \rangle = 0$ $(\forall \zeta_j \in \partial^* h_j(\overline{x}), \forall j \in L)$.

We also introduce another constraint qualification (CQ2): For every $\mu_i \geqslant 0$ $(\forall i \in I(\overline{x}))$, not all zero, and $\gamma_j \in \mathbb{R}$ $(\forall j \in L)$,

$$0 \notin \text{cl}\big(\sum_{i \in I(\overline{x})} \mu_i \text{conv}\,\partial^* g_i(\overline{x}) + \sum_{j \in L} \gamma_j \text{conv}\ \partial^* h_j(\overline{x}) + N(C; \overline{x})\big).$$

Remark 4.1 By an argument analogous to that for the proof of Proposition 4.1 [12], we deduce that (CQ1) implies (CQ2).

A Kuhn-Tucker necessary condition for Henig efficiency can be stated as follows.

Theorem 4.1. *Let $\overline{x}$ be a local Henig efficient solution of (CVEP). Assume that $F_{\overline{x}}(\overline{x}) = 0$; Assumption 3.1 is fulfilled; (CQ1) or (CQ2) holds. Then there exist $\overline{\lambda} \in Q^{\Delta}(B), \overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} \in \mathbb{R}^{\ell}$, such that*

$$0 \in cl\Big(\sum_{k=1}^{r} \overline{\lambda}_k \, conv \, \partial^* F_{k,\overline{x}}(\overline{x}) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \, conv \, \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\nu}_j \, conv \, \partial^* h_j(\overline{x}) + N(C;\overline{x}) \Big). \tag{12}$$

Proof Applying Theorem 3.1 yields the existence of $\overline{\tau} \geqslant 0, \overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} := (\overline{\nu}_1, \ldots, \overline{\nu}_\ell) \in \mathbb{R}^{\ell}$ with $(\overline{\tau}, \overline{\mu}_1, \ldots, \overline{\mu}_{|I(\overline{x})|}) \neq 0$, and a continuous positively homogeneous function Λ on Y satisfying $(\alpha)$and (β) such that (6) holds. Since (CQ1) or (CQ2) holds, one gets that $\overline{\tau} > 0$.

Due to Assumption 3.1(c), $\partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \partial^* F_{r,\overline{x}}(\overline{x})$ are bounded convexificators of $F_{1,\overline{x}}, \ldots, F_{r,\overline{x}}$ at $\overline{x}$, respectively, and the set-valued mappings $\partial^* F_{1,\overline{x}}$, $\ldots, \partial^* F_{r,\overline{x}}$ are upper semicontinuous at $\overline{x}$. Therefore, due to Proposition 5.1 on a chain rule in [7], $\partial \Lambda(F_{\overline{x}}(\overline{x}))(\partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \partial^* F_{r,\overline{x}}(\overline{x}))$ is a convexificator of $\Lambda \circ F_{\overline{x}}$ at $\overline{x}$. Observing that $F_{\overline{x}}(\overline{x}) = 0$, it follows from (6) that there exists a sequence

$$\begin{aligned} z_n \in \overline{\tau} \partial \Lambda(0) \big(&\text{conv } \partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \text{conv } \partial^* F_{r,\overline{x}}(\overline{x}) \big) \\ &+ \sum_{i \in I(\overline{x})} \overline{\mu}_i \text{conv } \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\gamma}_j \text{conv } \partial^* h_j(\overline{x}) + N(C;\overline{x}), \end{aligned} \tag{13}$$

such that $\lim_{n \to \infty} z_n = 0$. By (13), there exists a sequence $\{\chi_n\} \subset \partial \Lambda(0) \subset \mathbb{R}^r$ such that

$$\begin{aligned} z_n \in \overline{\tau} \chi_n (\text{conv } \partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \text{conv } \partial^* F_{r,\overline{x}}(\overline{x})) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \text{conv } \partial^* g_i(\overline{x}) \\ + \sum_{j \in L} \overline{\gamma}_j \text{conv } \partial^* h_j(\overline{x}) + N(C;\overline{x}). \end{aligned} \tag{14}$$

Since $\partial \Lambda(F_{\overline{x}}(\overline{x}))$ is a compact set in $\mathbb{R}^r$, without loss of generality, we can assume that $\chi_n \to \overline{\chi} \in \partial \Lambda(F_{\overline{x}}(\overline{x}))$. Putting $\bar{\lambda} = \bar{\tau} \bar{\chi}$, one has $\bar{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_r) \in \mathbb{R}^r$. By virtue of (14), it holds that

$$\begin{aligned} 0 \in \text{cl}\Big(&\overline{\lambda} (\text{conv } \partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \text{conv } \partial^* F_{r,\overline{x}}(\overline{x})) \\ &+ \sum_{i \in I(\overline{x})} \overline{\mu}_i \text{conv } \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\gamma}_j \text{conv } \partial^* h_j(\overline{x}) + N(C;\overline{x}) \Big), \end{aligned}$$

which implies (12).

Let us see that $\overline{\lambda} \in Q^{\Delta}(B)$. Indeed, observing that Λ is a convex function and $\overline{\chi} \in \partial(\Lambda(F_{\overline{x}}(\overline{x})))$, according to Theorem 3.1, there exists $\beta_0 > 0$ such that for every $y \in B$,

$$\begin{aligned}\langle \overline{\chi}, -y \rangle &\leq \Lambda(F_{\overline{x}}(\overline{x}) - y) - \Lambda(F_{\overline{x}}(\overline{x})) \\ &= \Lambda(-y) < -\beta_0.\end{aligned}$$

Consequently, $\langle \chi, y \rangle \geqslant \beta_0$ $(\forall y \in B)$. Hence, $\overline{\chi} \in Q^{\Delta}(B)$, and so $\overline{\lambda} \in Q^{\Delta}(B)$, which completes the proof. □

A Kuhn–Tucker necessary condition via the Clarke subdifferentials can be stated as follows.

Corollary 4.1. *Let $\overline{x}$ be a local Henig efficient solution of (CVEP). Assume that all the hypotheses of Theorem 3.1(a), (b) hold. Suppose, in addition, that (CQ1) or (CQ2) is fulfilled with $\partial F_{k,\overline{x}}, \partial g_i$ instead of $\partial^* F_{k,\overline{x}}, \partial^* g_i$, respectively. Then, there exist $\overline{\lambda} := (\overline{\lambda}_1, \ldots, \overline{\lambda}_r) \in Q^{\Delta}(B)$, $\overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu}_j \in \mathbb{R}$ $(j \in J)$, such that*

$$0 \in \sum_{k=1}^{r} \overline{\lambda}_k \partial F_{k,\overline{x}}(\overline{x}) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \partial g_i(\overline{x}) + \sum_{j \in L} \overline{\nu}_j \, conv \, \partial^* h_j(\overline{x}) + N_C(\overline{x}), \qquad (12)$$

where $\partial F_{k,\overline{x}}(\overline{x}), \partial g_i(\overline{x})$ indicate the Clarke subdifferential of $F_{k,\overline{x}}, g_i$ at $\overline{x}$, respectively.
Moreover, if the base B of Q is bounded and closed, then $\overline{\lambda} \in intQ^*$.

Proof Making use of Corollary 5.2 [1], it follows that $F_{1,\overline{x}}, \ldots, F_{r,\overline{x}}, g_i$ $(i \in I(\overline{x}))$ admit convexificators $\partial F_{1,\overline{x}}(\overline{x}), \ldots, \partial F_{r,\overline{x}}(\overline{x}), \partial g_i(\overline{x})$ $(i \in I(\overline{x}))$ at $\overline{x}$; $\partial F_{1,\overline{x}}(\overline{x}), \ldots,$ $\partial F_{r,\overline{x}}(\overline{x})$ are bounded; $\partial F_{1,\overline{x}}, \ldots, \partial F_{r,\overline{x}}$ are upper semicontinuous at $\overline{x}$. Hence, Assumption 3.1 is fulfilled. We invoke Theorem 4.1 to deduce that there exist $\overline{\lambda} \in Q^{\Delta}(B), \overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} \in \mathbb{R}^{\ell}$ such that

$$0 \in \mathrm{cl}\Big(\sum_{k=1}^{r} \overline{\lambda}_k \mathrm{conv}\, \partial F_{k,\overline{x}}(\overline{x}) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \mathrm{conv}\, \partial g_i(\overline{x}) + \sum_{j \in L} \overline{\nu}_j \nabla h_j(\overline{x}) + N(C; \overline{x}) \Big). \qquad (13)$$

Since $\partial F_{k,\overline{x}}(\overline{x}), \partial g_i(\overline{x})$ are compact and $N_C(\overline{x})$ is closed, it follows that the right hand side of (12) is closed, and so, the closure in (13) can be removed. Hence, (13) implies (12).

If the base B of Q is bounded and closed, by Lemma 4.5(iii) [5], we have $Q^{\Delta}(B) = \mathrm{int}Q^*$. Hence, we get the desired conclusion. □

In case the mapping $F_{\overline{x}}$ is Gâteaux differentiable at $\overline{x}$ we get the following Kuhn–Tucker necessary condition.

Corollary 4.2. *Let $\overline{x}$ be a local Henig efficient solution of (CVEP). Assume that all the hypotheses of Theorem 4.1 hold. Suppose, in addition, that $F_{\overline{x}}$ is Gâteaux differentiable with the Gâteaux derivative $\nabla_G F_{\overline{x}}(\overline{x})$. Then, there exist $\overline{\lambda} \in Q^{\Delta}(B)$, $\overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu}_j \in \mathbb{R}$ $(j \in J)$, such that*

$$0 \in [\nabla_G F_{\overline{x}}(\overline{x})]^* \overline{\lambda} + \sum_{i \in I(\overline{x})} \overline{\mu}_i \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\nu}_j \, conv \, \partial^* h_j(\overline{x}) + N_C(\overline{x}), \qquad (16)$$

where $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_r)$.

Proof Since $F_{\overline{x}}$ is Gâteaux differentiable at $\overline{x}$, the set $\{\nabla_G F_{\overline{x}}(\overline{x})\}$ is a convexificator of f at $\overline{x}$. Applying Theorem 4.1 yields the existence of $\overline{\lambda} \in Q^{\Delta}(B), \overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} \in \mathbb{R}^{\ell}$, such that (9) holds. Taking $\partial^* F_{\overline{x}}(\overline{x}) = \{\nabla_G F_{\overline{x}}(\overline{x})\}$, we obtain the desired conclusion. □

Theorem 4.1 is illustrated by the following example.

Example 4.1. Let $X = \mathbb{R}^2, Y = \mathbb{R}^2, C = [0,1] \times [0,1]$, $\overline{x} = (0,0)$, $Q = \mathbb{R}^2_+$. Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, g : \mathbb{R}^2 \to \mathbb{R}^2, h : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$F(x,y) = \begin{cases} (\sin \frac{\pi}{y_1} + y_1 + 3y_2)(1 - x_2), \frac{1}{2}|y_1| - \frac{1}{4}y_1 + 3y_2^2 + x_2^2 y_1), & \text{if} \quad y_1 \neq 0, \\ (0, -3y_2^2), & \text{if} \quad y_1 = 0, \end{cases}$$

$$g = (g_1, g_2),$$

$$g_1(y) = \begin{cases} -\frac{y_1}{1 + e^{\frac{1}{y_1}}} - y_2, & \text{if} \quad y_1 \neq 0, \\ -y_2, & \text{if} \quad y_1 = 0, \end{cases}$$

$$g_2(y) = y_2^2 - \frac{5}{2} y_2 + 1,$$

$$h(y) = y_1 - \frac{1}{2} y_2$$

$(x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2)$. Then

$$F_{\overline{x}}(y) = \begin{cases} (\sin \frac{\pi}{y_1} + y_1 + 3y_2, \frac{1}{2}|y_1| - \frac{1}{4}y_1 + 3y_2^2), & \text{if} \quad y_1 \neq 0, \\ (0, -3y_2^2), & \text{if} \quad y_1 = 0, \end{cases}$$

We have $K = \{(y_1, y_2) \in [0,1] \times [\frac{1}{2}, 1] : y_1 = \frac{1}{2} y_2\}, T(C; \overline{x}) = \mathbb{R}^2_+$, and $N(C; \overline{x}) = \mathbb{R}^2_-$, where $\mathbb{R}_- = -\mathbb{R}_+$. The pointed closed cone Q has the following bounded closed convex base $B = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$. It is easy to check that $\text{dist}(0, B) = \frac{\sqrt{2}}{2}$, where $\text{dist}(0, B)$ indicates the distance from 0 to B. Taking U being the open ball of radius $\delta = \frac{1}{2}$ around 0, then U is an absolutely convex neighborhood of 0 in $\mathbb{R}^2$, and

$$\text{cone} F_{\overline{x}}(K) \cap (U - B) = \emptyset.$$

Hence, $\overline{x} = 0$ is a Henig efficient solution of the following vector equilibrium problem: Finding $x \in K$ such that

$$F(x, y) \notin -Q \setminus \{0\} \ (\forall y \in K).$$

It can be seen that $\partial^* g_1(0) = \{(-1, -1), (0, -1)\}, \partial^* g_2(0) = \{(0, -\frac{5}{2})\}, \partial^* h(0) = \{(1, -\frac{1}{2})\}$. Hence, taking $\alpha > 0$, one has $(\alpha, 2\alpha) \in \mathbb{R}^2_+$, and (CQ1) holds with $b_1 = b_2 = 2\alpha$. Thus all the hypotheses of Theorem 4.1 are fulfilled. Since the base B of Q is bounded and closed, then $\overline{\lambda} \in \mathrm{int}Q^*$, and $Q^{\Delta}(B) = \mathrm{int}Q^* = \mathbb{R}^2_{++}$. We have $\partial^* F_{1,\overline{x}}(0) = \{(1, 3), (-1, 3)\}$, $\partial^*_{2,\overline{x}}(0) = \{(-\frac{3}{4}, 0), (\frac{1}{4}, 0)\}$. For $\overline{\lambda} = (2, 2), \overline{\mu} = (\frac{1}{2}, \frac{1}{2}), \overline{\nu} = 7$, the optimality condition (9) in Theorem 3.2 holds at $\overline{x} = (0, 0)$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in 2 \begin{pmatrix} \xi \\ 3 \end{pmatrix} + 2 \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \zeta \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ -\frac{5}{2} \end{pmatrix} + 7 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \mathbb{R}^2_-$$

for $-1 \leqslant \xi \leqslant 1, -\frac{3}{4} \leqslant \eta \leqslant \frac{1}{4}, -1 \leqslant \zeta \leqslant 0$.

In what follows we give a Kuhn-Tucker necessary condition for global efficiency of (VEP).

Theorem 4.2. *Let $\overline{x}$ be a local global efficient solution of (CVEP). Assume that $F_{\overline{x}}(\overline{x}) = 0$; Assumption 3.1 is fulfilled; (CQ1) or (CQ2) holds. Then there exist $\overline{\lambda} \in Q^{\#}$, $\overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} \in \mathbb{R}^{\ell}$, such that*

$$0 \in cl\Big(\sum_{k=1}^{r} \overline{\lambda}_k \, conv \, \partial^* F_{k,\overline{x}}(\overline{x}) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \, conv \, \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\nu}_j \, conv \ \partial^* h_j(\overline{x}) + N(C; \overline{x}) \Big).$$

Proof As also in the proof of Theorem 4.1, we invoke Theorem 3.2 to deduce that there exist $\overline{\tau} \geqslant 0, \overline{\mu}_i \geqslant 0$ $(i \in I(\overline{x}))$, $\overline{\nu} := (\overline{\nu}_1, \ldots, \overline{\nu}_\ell) \in \mathbb{R}^\ell$ with $(\overline{\tau}, \overline{\mu}_1, \ldots, \overline{\mu}_{|I(\overline{x})|}) \neq 0$, and a continuous positively homogeneous function Λ on Y satisfying that if $y_2 - y_1 \in Q \setminus \{0\}$, then $\Lambda(y_1) < \Lambda(y_2)$ and (9) holds. Since (CQ1) or (CQ2) holds, one gets that $\overline{\tau} > 0$. By the same way as in the proof of Theorem 4.1, we arrive at

$$0 \in \mathrm{cl}\Big(\overline{\tau} \bar{\chi} (\mathrm{conv} \ \partial^* F_{1,\overline{x}}(\overline{x}), \ldots, \mathrm{conv} \ \partial^* F_{r,\overline{x}}(\overline{x})) + \sum_{i \in I(\overline{x})} \overline{\mu}_i \mathrm{conv} \ \partial^* g_i(\overline{x}) + \sum_{j \in L} \overline{\gamma}_j \mathrm{conv} \ \partial^* h_j(\overline{x}) + N(C; \overline{x}) \Big),$$

where $\bar{\chi} \in \partial \Lambda(F_{\overline{x}}(\overline{x}))$.

Let us see that $\bar{\lambda} \in Q^{\#}$. Indeed, observing that Λ is a convex function and $\overline{\chi} \in \partial(\Lambda(F_{\overline{x}}(\overline{x})))$. Hence, by Theorem 3.2, for any $y \in Q \setminus \{0\}$, one has

$$\begin{aligned}\langle \bar{\chi}, -y \rangle &\leq \Lambda(F_{\bar{x}}(\bar{x}) - y) - \Lambda(F_{\bar{x}}(\bar{x})) \\ &= \Lambda(-y) < \Lambda(0) = 0,\end{aligned}$$

as it can be rewritten as $y = 0 - (-y) \in Q \setminus \{0\}$. Hence, $\bar{\chi} \in Q^{\#}$. Since $\bar{\tau} > 0$, we obtain $\bar{\lambda} := \bar{\tau}\bar{\chi} \in Q^{\#}$. □

5. Sufficient conditions for efficiency

To derive sufficient conditions for Henig and global efficient solutions of (CVEP), we introduce some notions of generalized convexity. Let f be a function defined on X which admits an upper convexificator $\partial^* f(\overline{x})$. Adapting to the definition of asymptotic pseudoconvex functions in [14] and L-invex-infine on C at $\overline{x} \in C$ in [2], we introduce the following definition.

Definition 5.1 For $\overline{\lambda} \in Q^{\triangle}(B)$, the triple functions $(\overline{\lambda} F_{\overline{x}}, g, h)$ is called asymptotic semiinvex-infine of type I at $\overline{x}$ on C iff for any $x \in C$, $\chi_k^{(n)} \in \operatorname{conv} \partial^* F_{k,\overline{x}}(\overline{x})$ $(k \in J)$, $\xi_i^{(n)} \in \operatorname{conv} \partial^* g_i(\overline{x})$ $(i \in I(\overline{x}))$, $\eta_j^{(n)} \in \operatorname{conv} \partial^* h_j(\overline{x})$ $(j \in L)$, there exists $v \in N(C;\overline{x})^\circ$ satisfying

(a) Asymptotic pseudoinvexity-infine to $\overline{\lambda} F_{\overline{x}}$:

$$\lim_{n\to\infty} \overline{\lambda}_k \langle \chi_k^{(n)}, v \rangle \geqslant 0 \Longrightarrow \overline{\lambda} F_{\overline{x}}(x) \geqslant \overline{\lambda} F_{\overline{x}}(\overline{x}),$$

(b) Asymptotic quasiinvexity-infine to g :

$$g_i(x) \leqslant g_i(\overline{x}) \Longrightarrow \lim_{n\to\infty} \langle \xi_i^{(n)}, v \rangle \leqslant 0 \ (\forall i \in I(\overline{x})),$$

(c) Asymptotic linearinvexity-infine to h :

$$h_j(x) = h_j(\overline{x}) \Longrightarrow \lim_{n\to\infty} \langle \eta_j^{(n)}, v \rangle = 0 \ (\forall j \in L).$$

Remark 5.1

(a) In case that C is convex, it can be taken $v := x - \overline{x}$, as $T(C;\overline{x}) = \mathbb{R}_+(C - \overline{x})$.

(b) If C is convex, $\overline{\lambda} F_{\overline{x}}$ is pseudoconvex at $\overline{x}$ on C, g_i $(\forall i \in I(\overline{x}))$ are quasiconvex at $\overline{x}$ on C, $\pm h_j$ $(\forall j \in L)$ are quasiconvex at $\overline{x}$ on C, then $(\overline{\lambda} F_{\overline{x}}, g, h)$ is asymptotic semiinvex-infine of type I at $\overline{x}$ on C.

(c) If C is convex, $\overline{\lambda}F_{\overline{x}}$ is asymptotic pseudoconvex at $\overline{x}$ on C, g_i ($i \in I(\overline{x})$) are asymptotic at $\overline{x}$ on C, h_j ($j \in L$) are asymptotic quasilinear at $\overline{x}$ (see [14]), then $(\overline{\lambda}F_{\overline{x}}, g, h)$ is asymptotic semiinvex-infine of type II at $\overline{x}$ on C.

Definition 5.2 Let $M \subset \mathbb{R}^r$ be a pointed convex cone such that $Q \setminus \{0\} \subset \mathrm{int}M$. For $\overline{\lambda} \in M^{\#}$, the triple functions $(\overline{\lambda}F_{\overline{x}}, g, h)$ is called asymptotic semiinvex-infine of type II at $\overline{x}$ on C iff for any $x \in C$, $\chi_k^{(n)} \in \mathrm{conv}\, \partial^* F_{k,\overline{x}}(\overline{x})$ ($k \in J$), $\xi_i^{(n)} \in \mathrm{conv}\, \partial^* g_i(\overline{x})$ ($i \in I(\overline{x})$), $\eta_j^{(n)} \in \mathrm{conv}\, \partial^* h_j(\overline{x})$ ($j \in L$), there exists $v \in N(C;\overline{x})^\circ$ satisfying (a)–(c).

In what follows we shall give a sufficient condition for Henig efficient solutions of (CVEP).

Theorem 5.1. *Let $\overline{x} \in K$. Assume that $F_{\overline{x}}(\overline{x}) = 0$, and*
(i) There exist $\overline{\lambda} := (\overline{\lambda}_1, \ldots, \lambda_r) \in Q^{\Delta}(B)$, $\overline{\mu}_i \geqslant 0$ ($\forall i \in I(\overline{x})$), $\overline{\gamma}_j \in \mathbb{R}$ ($\forall j \in L$) such that

$$0 \in cl\Big(\sum_{k\in J} \overline{\lambda}_k \, conv\, \partial^* F_{k,\overline{x}}(\overline{x}) + \sum_{i\in I(\overline{x})} \overline{\mu}_i \, conv\, \partial^* g_i(\overline{x}) + \sum_{j\in L} \overline{\gamma}_j \, conv\, \partial^* h_j(\overline{x}) + N(C;\overline{x})\Big). \tag{17}$$

(ii) All but at most one of the upper convexificators $\partial^* F_{k,\overline{x}}(\overline{x})$ ($k \in J$) are upper regular at $\overline{x}$. Assume that the functions $(\overline{\lambda}F_{\overline{x}}, g, h)$ is asymptotic semiinvex-infine of type I at $\overline{x}$ on C.
Then $\overline{x}$ is a Henig efficient solution of (CVEP).

Proof By (17), there exist $\chi_k^{(n)} \in \mathrm{conv}\, \partial^* F_{k,\overline{x}}(\overline{x})$ ($k \in J$), $\xi_i^{(n)} \in \mathrm{conv}\, \partial^* g_i(\overline{x})$ ($i \in I(\overline{x})$), $\eta_j^{(n)} \in \mathrm{conv}\, \partial^* h_j(\overline{x})$ ($j \in L$), $\zeta^{(n)} \in N(C;\overline{x})$ such that

$$0 = \lim_{n\to\infty} \Big[\sum_{k\in J} \overline{\lambda}_k \chi_k^{(n)} + \sum_{i\in I(\overline{x})} \overline{\mu}_i \xi_i^{(n)} + \sum_{j\in L} \overline{\gamma}_j \eta_j^{(n)} + \zeta^{(n)}\Big].$$

Then, by the asymptotic semiinvexity-infine of $(\overline{\lambda}F_{\overline{x}}, g, h)$ at $\overline{x}$ on C, there exists $v \in N(C;\overline{x})^\circ$ such that

$$\begin{aligned}\lim_{n\to\infty} \Big[\sum_{k\in J} \overline{\lambda}_k \langle \chi_k^{(n)}, v\rangle &+ \sum_{i\in I(\overline{x})} \overline{\mu}_i \langle \xi_i^{(n)}, v\rangle \\ &+ \sum_{j\in L} \overline{\gamma}_j \langle \eta_j^{(n)}, v\rangle + \langle \zeta^{(n)}, v\rangle\Big] = 0.\end{aligned} \tag{18}$$

Obersve that for all $x \in K, g_i(x) \leqslant 0 = g_i(\overline{x})$ ($\forall i \in I(\overline{x})$). In view of the asymptotic quasiinvexity-infine of g_i at $\overline{x}$ on C, for all $x \in K$, we have

$$\lim_{n\to\infty} \langle \xi_i^{(n)}, v\rangle \leqslant 0. \tag{19}$$

Since $h_j(x) = 0 = h_j(\overline{x})$ $(\forall x \in K)$, by virtue of the asymptotic linearinvexity-infine of h_j $(\forall j \in L)$ at $\overline{x}$ on C, it follows that for all $x \in K$,

$$\lim_{n\to\infty} \langle \eta_j^{(n)}, v\rangle = 0. \tag{20}$$

Due to $v \in N(C;\overline{x})^\circ$, one has

$$\lim_{n\to\infty} \langle \zeta^{(n)}, v\rangle \leqslant 0. \tag{21}$$

Since all but at most one of the upper convexificators $\partial^* F_{k,\overline{x}}(\overline{x})$ $(k \in J)$ are upper regular, by Rule 4.2 [7], $\sum_{k\in J} \overline{\lambda}_k \partial^* F_{k,\overline{x}}(\overline{x})$ is an upper convexificator for the function $\sum_{k\in J} \overline{\lambda}_k F_{k,\overline{x}}$ at $\overline{x}$. Combining (18)–(21) yields that for all $x \in K$,

$$\lim_{n\to\infty} \langle \sum_{k\in J} \overline{\lambda}_k \chi_k^{(n)}, x - \overline{x}\rangle \geqslant 0.$$

In view of the asymptotic pseudoconvexity-infine of $\overline{\lambda} F_{\overline{x}}(.)$ at $\overline{x}$, we claim that for all $x \in K$,

$$\overline{\lambda} F_{\overline{x}}(x) \geqslant \overline{\lambda} F_{\overline{x}}(\overline{x}) = 0. \tag{22}$$

Let us see that $\overline{x}$ is a Henig efficient solution of (CVEP). If this were not so, for every open absolutely convex neighborhood $U \subseteq V_B$ of 0, by (3), there would be

$$F_{\overline{x}}(K) \cap (-\mathrm{int} Q_U(B)) \neq \emptyset. \tag{23}$$

Lemma 4.5 [5] shows that for $\overline{\lambda} \in Q^\Delta(B)$, there exists an open absolutely convex neighborhood $U_0 \subseteq V_B$ of 0 such that $\overline{\lambda} \in (Q_{U_0}(B))^* \setminus \{0\} \subseteq Q^\Delta(B)$. Hence, there exists $y_1 \in K$ such that

$$F_{\overline{x}}(y_1) \in -\mathrm{int} Q_{U_0}(B).$$

Therefore, $\overline{\lambda} F_{\overline{x}}(y_1) < 0 = F_{\overline{x}}(\overline{x})$, which conflicts with (22). Consequently, $\overline{x}$ is a Henig efficient solution of (CVEP). □

In the sequel we give a sufficient condition for globally efficient solutions of (CVEP).

Theorem 5.2. *Let $\overline{x} \in K$. Assume that $F_{\overline{x}}(\overline{x}) = 0$, and $M \subset \mathbb{R}^r$ is a pointed convex cone such that $Q \setminus \{0\} \subset intM$, and*
(i) There exist $\overline{\lambda} := (\overline{\lambda}_1, \ldots, \lambda_r) \in M^{\#}$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I(\overline{x}))$, $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$ such that

$$0 \in cl\Big(\sum_{k\in J} \overline{\lambda}_k\, conv\ \partial^* F_{k,\overline{x}}(\overline{x}) + \sum_{i\in I(\overline{x})} \overline{\mu}_i\, conv\ \partial^* g_i(\overline{x}) + \sum_{j\in L} \overline{\gamma}_j\, conv\ \partial^* h_j(\overline{x}) + N(C;\overline{x})\Big).$$

(ii) All but at most one of the upper convexificators $\partial^* F_{k,\overline{x}}(\overline{x})$ $(k \in J)$ *are upper regular at* $\overline{x}$*. Assume that the functions* $(\overline{\lambda} F_{\overline{x}}, g, h)$ *is asymptotic semiinvex-infine of type II at* $\overline{x}$ *on* C*.*
Then $\overline{x}$ *is a globally efficient solution of (CVEP).*

Proof By an argument analogous to that use for the proof of Theorem 5.1, we get that for all $x \in K$,

$$\overline{\lambda} F_{\overline{x}}(x) \geqslant \overline{\lambda} F_{\overline{x}}(\overline{x}) = 0. \tag{24}$$

Let us see that $\overline{x}$ is a globally efficient solution of (CVEP). If $\overline{x}$ is not a global efficient solution of (CVEP), then

$$F_{\overline{x}}(K) \cap (-M \setminus \{0\}) \neq \emptyset.$$

Hence, there exists $y_1 \in K$ such that

$$F_{\overline{x}}(y_1) \in -M \setminus \{0\}.$$

Since $\overline{\lambda} \in M^{\#}$, it floows that $\overline{\lambda} F_{\overline{x}}(y_1) < 0 = F_{\overline{x}}(\overline{x})$, which conflicts with (24). Consequently, $\overline{x}$ is a globally efficient solution of (CVEP). □

6. Conclusions

We derive Fritz John and Kuhn-Tucker necessary conditions for local Henig and global efficient solutions solutions of vector equilibrium problems involving nonsmooth equality, inequality and set constraints via convexificators. The Kuhn–Tucker necessary conditions obtained here via convexificators can be sharper than those expressed in terms of the Clarke subdifferentials. Under assumptions on asymptotic semiinvexity-infine of type I or type II of the triple $(\overline{\lambda} F_{\overline{x}}, g, h)$, sufficient conditions for Henig and globally efficient solutions are established. The results obtained in this paper are more general than those obtained by Gong [5] for vector equilibrium problems with only a set constraint, and those obtained by Long et al. [8] for vector equilibrium problems with subconvexlike functions.

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, 1983.

[2] T. D. Chuong, *L-invex-infine functions and applications*, Nonlinear Anal., **75** (2012), 5044-5052.

[3] P. Daniele, *Lagrange multipliers and infinite-dimensional equilibrium problems*, J. Glob. Optim., **40** (2008), 65-70.

[4] F. Giannessi, G. Mastroeni and L. Pellegrini, *On the theory of vector optimization and variational inequalities, image space analysis and separation*, in: *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, F. Giannessi (ed.), Kluwer, Dordrecht, 2000, 153-215.

[5] X. H. Gong, *Scalarization and optimality conditions for vector equilibrium problems*, Nonlinear Anal., **73** (2010), 3598-3612.

[6] I. V. Girsanov, *Lectures on Mathematical Theory of Extremum Problems*, Berlin-Heidenberg, Springer-Verlag, 1972.

[7] V. Jeyakumar and D. T. Luc, *Nonsmooth calculus, minimality, and monotonicity of convexificators*, J. Optim. Theory Appl., **101** (1999), 599-621.

[8] X. J. Long, Y. Q. Huang and Z. Y. Peng, *Optimality conditions for the Henig efficient solution of vector equilibrium problems with constraints*, Optim. Lett., **5** (2011), 717-728.

[9] J. Morgan and M. Romaniello, *Scalarization and Kuhn-Tucker-like conditions for weak vector generalized quasivariational inequalities*, J. Optim. Theory Appl., **130** (2006), 309-316.

[10] D. V. Luu, D. D. Hang, *On optimality conditions for vector variational inequalities*, J. Math. Anal. Appl., **412** (2014), 792-804.

[11] D. V. Luu, D. D. Hang, *Efficient solutions and optimality conditions for vector equilibrium problems*, Math. Meth. Oper. Res., **79** (2014), 163-177.

[12] D. V. Luu, *Necessary and sufficient conditions for efficiency via convexificators*, J. Optim. Theory Appl., **160** (2014), 510-526.

[13] D. V. Luu, *Necessary conditions for efficiency in terms of the Michel-Penot subdifferentials*, Optimization, **61** (2012), 1099-1117.

[14] D. V. Luu, *Optimality condition for local efficient solutions of vector equilibriuam problems via convexificators and applications*, J. Optim. Theory Appl., **171** (2016), 643-665.

Do Van Luu
Thang Long Institute of Mathematics and Applied Sciences
Hanoi
Vietnam
dvluu@math.ac.vn

Tran Thi Mai
University of Economics and Business Administration
Thai Nguyen
Vietnam
tranthimai879@gmail.com