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A NEW CLASS OF UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS IGNORING MULTIPLICITY WITH 15 ELEMENTS

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Abstract. In this paper, we give a new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements.

1. Introduction. Main results

In this paper, by a meromorphic function we mean a meromorphic function on the complex plane $\mathbb{C}$.

Let f be a non-constant meromorphic function on $\mathbb{C}$. For every $a \in \mathbb{C}$, we define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity } d, \end{cases}$$

and set $\nu_f^\infty = \nu_{\frac{1}{f}}^0$, and define the function $\overline{\nu}_f^a : \mathbb{C} \to \mathbb{N}$ by $\overline{\nu}_f^a(z) = \min \ \{\nu_f^a(z), 1\}$, and set $\overline{\nu}_f^\infty = \overline{\nu}_{\frac{1}{f}}^0$. For $f \in \mathcal{M}(\mathbb{C})$ and a non-empty set $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{C}\}, \quad \overline{E}_f(S) = \bigcup_{a \in S} \{(z, \overline{\nu}_f^a(z)) : z \in \mathbb{C}\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{C})$. Two functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity* (share S CM) if $E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity* (share S IM) if $\overline{E}_f(S) = \overline{E}_g(S)$.

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If the condition $E_f(S) = E_g(S)$ implies $f = g$ for any two non-constant meromorphic (entire) functions f, g, then S is called a unique range set for meromorphic (entire) functions counting multiplicity, or in brief, URSM (URSE). A set $S \subset \mathbb{C} \cup \{\infty\}$ is called a unique range set for meromorphic (entire) functions ignoring multiplicity, or in brief, URSM-IM (URSE-IM), if the condition $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f = g$ for any pair of non-constant meromorphic (entire) functions.

In 1976 Gross ([8]) proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$, $j = 1, 2, 3$ must be identical. In the same paper Gross posed the following question:

Question A. *Can one find two (or possible even one) finite set S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ must be identical.*

Yi [16]-[18], [20] first gave an affirmative answer to Question A. Since then, many results have been obtained for this and related topics (see [1]-[13], [15]-[21]).

Concerning to Question A, a natural question is the following.

Question B. *What is the smallest cardinality for such a finite set S such that any two meromorphic functions f and g satisfying either $E_f(S) = E_g(S)$ or $\overline{E}_f(S) = \overline{E}_g(S)$ must be identical.*

So far, the best answer to Question B for case URSM was obtained by Frank and Reinders [5]. They proved the following result.

Theorem C. *The set $\{z \in \mathbb{C}|\ P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2} z^{n-2} - c = 0\}$, where $n \geq 11$ and $c \neq 0, 1$, is a unique range set for meromorphic functions counting multiplicity.*

In 1997, H. X. Yi [19] first gave an answer to question B for the case URSM-IM with 19 elements. Since then, many results have been obtained for this topic (see [2], [3]). So far, the best answer to Question B for the case URSM-IM was obtained by B. Chakraborty [3]. He proved the following result.

Theorem D. *Let $S = \{z \in \mathbb{C}|\ P_{FR}(z) = 0\}$. If $n \geq 15$, then S is a URSM-IM.*

In this paper, we give a new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements. Note that this class is different from B. Chakraborty's in[3].

Now let us describe main results of the paper.

Let $n \in \mathbb{N}^*, n \geq 3$.

Consider polynomial $P(z)$ as follows:

$$P(z) = z^n - \frac{2na}{n-1} z^{n-1} + \frac{na^2}{n-2} z^{n-2} + 1 = Q(z) + 1, \tag{1.1}$$

where $a \in \mathbb{C}$ $a \neq 0$. Suppose that

$$Q(1) \neq -1, \tag{1.2}$$

$$Q(1) \neq -2. \tag{1.3}$$

Note that $P(z)$, defined by (1.1) with condition (1.2), has no multiple zeros. Clearly, $P^{'}(z) = nz^{n-3}(z-a)^2$, and $P(z)$ is different from P_{FR}. Moreover $P^{'}(z)$ has a zero at 0 of order $n-3$, and a zero at a of order 2.

The polynomials of the form (1.1) were investigated in [1] and [11].

We shall prove the following theorem.

Theorem 1. *Let $P(z)$ be defined by (1.1) with conditions (1.2) and (1.3), and let $S = \{z \in \mathbb{C} | \ P(z) = 0\}$. If $n \geq 15$, then S is a URSM-IM.*

2. Lemmas, Definitions

We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example, [4], [14]).

We need some lemmas.

Lemma 2.1. **[14]** *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) + S(r,f),$$

where $S(r,f) = o(T(r,f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.2. **[4]**

For any non-constant meromorphic function f,

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r,f) + S(r,f).$$

Definition. Let f be a non-constant meromorphic function, and k be a positive integer. We denote by $\overline{N}_{(k}(r,f)$ the counting function of the poles of order $\geq k$ of f, where each pole is counted only once, and by $N_{1)}(r,f)$ the counting function of the simple poles of f.

Lemma 2.3. *Let f, g be two non-constant meromorphic functions and let $f^{-1}(0) = g^{-1}(0)$. Set*

$$F = \frac{1}{f},\ G = \frac{1}{g},\ L = \frac{F^{''}}{F'} - \frac{G^{''}}{G'}.$$

Suppose that $L \not\equiv 0$. Then

1)[**11 − 13**] $N(r, L) \leq \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) +$

$$\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1) + \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0).$$

Moreover, if a is a common simple zero of f and g, then $L(a) = 0$.

2) $\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$

$$\leq N(r, L) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}); \nu_2 \geq 2).$$

Proof. 2) By using properties of the Stieltjes integral (see [4, p. 5, p. 14]), we get:

$$N(r, \frac{1}{f}) - n(0, \frac{1}{f}) = \sum_{0<|a_m|<r} \ln \frac{r}{|a_m|},$$

where a_m are zeros of f, counting multiplicity; and

$$\overline{N}(r, \frac{1}{f}) - \overline{n}(0, \frac{1}{f})$$

is the same sum, where each zero a_m is counted only once.

Similarly, we obtain equalities for $N(r, \frac{1}{f}; \nu_1 \geq 2)$, $\overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1)$, $N(r, \frac{1}{g}); \nu_2 \geq 2)$, $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$.

We are going to prove 2) by using these inequalities and the arguments in (Lemma 2.2 [12]), (Lemma 2.4 [13]) and (Lemma 2.6 [3])).

Set

$$M = \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1),$$

$$T = N(r, L) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}); \nu_2 \geq 2).$$

Let a be a zero of f with multiplicity p. From $f^{-1}(0) = g^{-1}(0)$ it follows that a is a zero of f with multiplicity q. We consider the following cases:

Case 1. Assume that $p = q$.

If $p = q = 1$, then a is counted with $1+1+0+0 = 2$ times in M. From this and the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that a is a zero of L. Then it is counted with $1 + \frac{1}{2}(1+1) = 2$ times in T.

If $p = q \geq 2$, then a is counted with $1+1+0+0 = 2$ times on M. By the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that a is not a pole of L. Then it is counted with $0 + \frac{1}{2}(p+p) + p + p = 3p$ times in T.

Case 2. Assume that $p > q$.

If $p > q$ and $q = 1$, then $p \geq 2$ and a is counted with $1+1+1+0 = 3$ times in M. By the proof of Part 1) (Lemma 2.2[12]), (Lemma 2.4 [13]) it follows that a is a pole of L, and by $p \geq 2$ we see that a is counted with $1 + \frac{1}{2}(p+1) + p + 0 = p + 1 + \frac{p+1}{2} > 3$ times in T.

If $p > q$ and $q \geq 2$, then $p \geq 2$ and a is counted with $1+1+1+0 = 3$ times in M. From this and the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that a is a pole of L, and by $p \geq 2$, $q \geq 2$ we see that a is counted with $1 + \frac{1}{2}(p+q) + p + q = 1 + \frac{3(p+q)}{2} > 3$ times in T.

Case 3. Assume that $q > p$.

The proof is completed by using the arguments similar to ones in Case 2.

A polynomial $R(z)$ is called a *strong uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, and a nonzero constant c, the condition $R(f) = cR(g)$ implies $f = g$ (see [1], [7], [11]). In this case we say that, $R(z)$ is a SUPM (SUPE). A polynomial $R(z)$ is called a *uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, the condition $R(f) = R(g)$ implies $f = g$ (see[1], [7], [11]). In this case we say $R(z)$ is a UPM (UPE). Let $R(z)$ be a polynomial of the degree q. Assume that the derivative of $R(z)$ has mutually distinct k zeros $d_1, d_2, ..., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively. We often consider polynomials satisfying the following condition introduced by Fujimoto ([6]):

$$R(d_i) \neq R(d_j), 1 \leq i < j \leq q. \tag{2.1}$$

The number k is called the *derivative index* of R.

Lemma 2.4. *(Fujimoto [7]).*

Let $R(z)$ be a polynomial of the degree q satisfying the condition (2.1). Then $R(z)$ is a uniqueness polynomial if and only if

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{i=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2, min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

H. Fujimoto [6] proved the following:

Lemma 2.5. *Let $R(z)$ be a polynomial of the degree q satisfying the condition (2.1), we assume furthermore that $q \geq 5$ and there are two non-constant meromorphic functions f and g such that*

$$\frac{1}{R(f)} = \frac{c_0}{R(g)} + c_1$$

for two constants c_0 $(\neq 0)$ and c_1. If $k \geq 3$ or if $k = 2, min\{q_1, q_2\} \geq 2$, then $c_1 = 0$.

Lemma 2.6. [1] $\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i}$ *is not an integer, where $n, m \geq 1$ are integers.*

In [1], Banerjee proved the Lemma for $n, m \geq 3$, but it is clear that the Lemma is valid for $n, m \geq 1$.

Lemma 2.7. *Let $P(z)$ be dened by (1.1) with conditions (1.2) and (1.3), and let $n \geq 6$. Then $P(z)$ is a strong uniqueness polynomial for meromorphic functions*

Proof.

By Lemma 2.6, we see that $\dfrac{1}{n} - \dfrac{2}{n-1} + \dfrac{1}{n-2}$ is not an integer. Set $A = \dfrac{1}{n} - \dfrac{2}{n-1} + \dfrac{1}{n-2}$. Then $A \neq 0$. We have $P(0) = Q(0) + 1 = 1$, $P(a) = Q(a) + 1 = nAa^n + 1$. From this and $a \neq 0$, we get $P(a) \neq P(0)$. Set $F = P(f), G = P(g)$. From $P(f) = cP(g), c \neq 0$, it implies

$$F = cG,\ T(r, f) + S(r, f) = T(r, g) + S(r, g),\ S(r, f) = S(r, g). \tag{2.2}$$

Now we consider the following possible cases:

Case 1. $c \neq 1$.

If $c = P(a)$, from (2.2) and $P(0) = 1$ we have

$$F - 1 = P(a)(G - \frac{1}{P(a)}). \tag{2.3}$$

We consider $P(z) - \frac{1}{P(a)}$. By $P(0) = 1$ and $P(a) = c \neq 1$ we obtain $P(0) - \frac{1}{P(a)} \neq 0$. Moreover, since $P(a) = nAa^n + 1 \neq -1$ and $P(a) = c \neq 1$ we obtain

$P(a) - \frac{1}{P(a)} \neq 0$. Therefore, $P(z) - \frac{1}{P(a)}$ has only simple zeros, let they be given by $b_i^{'}, i = 1, 2, ..., n$. Note that $P(z) - 1$ has a zero at 0 of order $n-2$, and two distinct simple zeros. Let $c_i^{'}, i = 1, 2$, be distinct simple zeros of $P(z) - 1$. Applying Lemma 2.1 to the function g and the values $b_1^{'}, b_2^{'}, ..., b_n^{'}$, ∞, and by (2.2), (2.3) we get

$$\begin{aligned}
(n-1)T(r,g) &\leq \overline{N}(r,g) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g - b_i^{'}}) + S(r,g) \\
&\leq T(r,g) + \overline{N}(r, \frac{1}{f}) + \sum_{i=1}^{2} \overline{N}(r, \frac{1}{f - c_i^{'}}) + S(r,g) \\
&\leq T(r,g) + T(r,f) + 2T(r,f) + S(r,g) \\
&= 4T(r,g) + S(r,g) \\
(n-5)T(r,g) &\leq S(r,g).
\end{aligned}$$

This is a contradiction to the assumption that $n \geq 6$.

If $c \neq P(a)$, then from (2.2) we have

$$F - c = c(G - 1). \tag{2.4}$$

We consider $P(z) - c$. By $P(0) = 1$ and $c \neq 1$ we have $P(0) - c = 1 - c \neq 0$. Moreover $c \neq P(a)$. So $P(a) - c \neq 0$, $P(0) - c \neq 0$. Therefore $P(z) - c$ has only simple zeros, let they be given by $e_i, i = 1, 2, ..., n$. Now we consider $P(z) - 1$. We see that $P(0) = 1, P(z) - P(0) = P(z) - 1$ has a zero at 0 of order $n - 2$, and 2 distinct simple zeros. Let $t_i, i = 1, 2$, be distinct simple zeros of $P(z) - 1$. Applying Lemma 2.1 to the function f and the values $e_1, e_2, ..., e_n$, ∞, and by (2.4) we get

$$\begin{aligned}
(n-1)T(r,f) &\leq \overline{N}(r,f) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g - e_i}) + S(r,f) \\
&\leq T(r,f) + \overline{N}(r, \frac{1}{g}) + \sum_{i=1}^{2} \overline{N}(r, \frac{1}{f - t_i}) + S(r,f) \\
&\leq T(r,f) + T(r,g) + 2T(r,g) + S(r,f) \\
&= 4T(r,f) + S(r,f) \\
(n-5)T(r,f) &\leq S(r,f).
\end{aligned}$$

This is a contradiction to the assumption that $n \geq 6$.

Case 2. $c = 1$. Then

$$P(f) = P(g) \tag{2.5}$$

Applying Lemma 2.4 to (2.5) we obtain $f = g$.

3. Proof of Theorem 1

Now we use the above Lemmas to prove the main result of the paper.

Recall that $P(z) = (z - a_1)...(z - a_n)$, $P^{'}(z) = nz^{n-3}(z-a)^2$.

Suppose $n \geq 15$ and $\overline{E}_f(S) = \overline{E}_g(S)$, where $S = \{z \in \mathbb{C}|\ P(z) = 0\}$.

Set

$$F = \frac{1}{P(f)},\ G = \frac{1}{P(g)}, L = \frac{F^{''}}{F'} - \frac{G^{''}}{G'},$$

$$T(r) = T(r,f) + T(r,g), S(r) = S(r,f) + S(r,g).$$

Then $T(r, P(f)) = nT(r,f) + S(r,f)$ and $T(r, P(g)) = nT(r,g) + S(r,g)$, and hence $S(r, P(f)) = S(r,f)$ and $S(r,P(g)) = S(r,g)$, since $P(f)$ and f, and $P(g)$ and g have the same growth estimates, respectively.

We consider two following cases:

Case 1 $L \equiv 0$. Then, we have $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$ for some constants $c\ (\neq 0)$ and c_1. By Lemma 2.5 we obtain $c_1 = 0$.

Therefore, there is a constant $C \neq 0$ such that $P(f) = CP(g)$. Then, applying Lemma 2.7 we obtain $f = g$.

Case 2 $L \not\equiv 0$.

Claim 1 We have

$$(n-2)T(r) \leq \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r), \quad (3.1),$$

where $N_0(r, \frac{1}{f'})$ $(N_0(r, \frac{1}{g'}))$ is the counting function of those zeros of f', which are not zeros of function $(f - a_1)...(f - a_n)f(f - a)((g - a_1)...(g - a_n)g(g - a))$.

Then, applying the Lemma 2.1 to the functions f, g and the values $a_1, a_2, ...,$ a_n, $0, a, \infty$, and noting that

$$\sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) = \overline{N}(r, \frac{1}{P(f)}), \sum_{i=1}^{q} \overline{N}(r, \frac{1}{g - a_i}) = \overline{N}(r, \frac{1}{P(g)}),$$

we obtain

$$(n+1)T(r) \leq \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) +$$

$$\overline{N}(r,\frac{1}{f-a})+\overline{N}(r,\frac{1}{g-a})-N_0(r,\frac{1}{f'})-N_0(r,\frac{1}{g'})+S(r). \qquad (3.2)$$

On the other hand,

$$\overline{N}(r,f)+\overline{N}(r,g)\leq (T(r,f)+T(r,g))+S(r)=T(r)+S(r),$$

$$\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})\leq (T(r,f)+T(r,g))+S(r)=T(r)+S(r),$$

$$\overline{N}(r,\frac{1}{f-a})+\overline{N}(r,\frac{1}{g-a})\leq (T(r,f)+T(r,g))+S(r)=T(r)+S(r).$$

From this and (3.2) we obtain (3.1)

Claim 2 We have

$$\overline{N}(r,\frac{1}{P(f)})+\overline{N}(r,\frac{1}{P(g)})\leq$$

$$(\frac{n}{2}+3)T(r)+\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0)+\overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0)+S(r).$$

By $\overline{E}_f(S)=\overline{E}_g(S)$ we get $(P(f))^{-1}(0)=(P(g))^{-1}(0)$, and note that

$$\overline{N}_{(2}(r,P(f))=\overline{N}(r,f),\ \overline{N}_{(2}(r,P(g))=\overline{N}(r,g).$$

Then applying the Lemma 2.3 to the functions $P(f),P(g)$ we obtain

$$\begin{aligned} N(r,L)\leq{}& \overline{N}(r,f)+\overline{N}(r,g)+\\ & \overline{N}(r,\frac{1}{P(f)};\nu_1>\nu_2\geq 1)+\overline{N}(r,\frac{1}{P(g)};\nu_2>\nu_1\geq 1)+\\ & \overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0)+\overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0); \end{aligned} \qquad ((3.3))$$

$$\overline{N}(r,\frac{1}{P(f)})+\overline{N}(r,\frac{1}{P(g)})+\overline{N}(r,\frac{1}{P(f)};\nu_1>\nu_2\geq 1)+\overline{N}(r,\frac{1}{P(g)};\nu_2>\nu_1\geq 1)\leq$$

$$N(r,L)+\frac{1}{2}(N(r,\frac{1}{P(f)})+N(r,\frac{1}{P(g)}))+N(r,\frac{1}{P(f)};\nu_1\geq 2)+N(r,\frac{1}{P(g)};\nu_2\geq 2)). \qquad (3.4)$$

Moreover,

$$\overline{N}(r,f)+\overline{N}(r,g)\leq T(r)+S(r). \qquad (3.5)$$

Obviously,

$$N(r,\frac{1}{P(f)})\leq nT(r,f)+S(r,f);N(r,\frac{1}{P(g)})\leq nT(r,g)+S(r,g),$$

$$N(r, \frac{1}{P(f)}) + N(r, \frac{1}{P(g)}) \leq nT(r) + S(r). \tag{3.6}$$

On the other hand, from $P(f) = (f - a_1)...(f - a_n)$ it follows that every zero with multiplicity ≥ 2 of $P(f)$ is a zero of $f - a_i$ with multiplicity ≥ 2, $i = 1, 2, ..., n$, and therefore, it is a zero of $f^{'}$, so we have

$$N_{(2}(r, \frac{1}{P(f)}) \leq N(r, f^{'}).$$

From this and Lemma 2.2 we obtain

$$N_{(2}(r, \frac{1}{P(f)}) \leq N(r, \frac{1}{f}) + \overline{N}(r, f) \leq 2T(r, f) + S(r, f).$$

Similarly, we have

$$\overline{N}_{(2}(r, \frac{1}{P(g)}) \leq N(r, \frac{1}{g}) + \overline{N}(r, g) \leq 2T(r, g) + S(r, g).$$

Therefore,

$$N_{(2}(r, \frac{1}{P(f)}) + N_{(2}(r, \frac{1}{P(g)}) \leq 2T(r) + S(r). \tag{3.7}$$

Combining (3.1)-(3.7) we get

$$\overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \leq$$

$$(\frac{n}{2} + 3)T(r) + \overline{N}(r, \frac{1}{[P(f)]^{'}}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]^{'}}; P(g) \neq 0) + S(r).$$

Claim 2 is proved.

Claim 3 We have

$$\overline{N}(r, \frac{1}{[P(f)]^{'}}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]^{'}}; P(g) \neq 0) \leq$$
$$2T(r) + N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r).$$

We have

$$\begin{aligned}\overline{N}(r, \frac{1}{[P(f)]^{'}}; &P(f) \neq 0)\\ &= \overline{N}(r, \frac{1}{f^{n-3}(f-a)^2 f'}; P(f) \neq 0)\\ &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}_o(r, \frac{1}{f'})\\ &\leq 2T(r, f) + \overline{N}_o(r, \frac{1}{f'}) + S(r, f).\end{aligned} \tag{3.8}$$

Similarly,

$$\overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq 2T(r,g) + \overline{N}_o(r, \frac{1}{g'}) + S(r,g). \tag{3.9}$$

Inequalities (3.8) and (3.9) give us

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq$$
$$2T(r) + \overline{N}_o(r, \frac{1}{f'}) + \overline{N}_o(r, \frac{1}{g'}) + S(r). \tag{3.10}$$

Claim 3 is proved.

Claim 1, 2, 3 give us:

$$(n-2)T(r) \leq (\frac{n}{2} + 5)T(r) + S(r),\ (n-14)T(r) \leq S(r)$$

This is a contradiction to the assumption that $n \geq 15$. So $L \equiv 0$. Therefore $f = g$. Theorem 1 is proved.

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