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NONAUTONOMOUS ATTRACTORS FOR YOUNG DIFFERENTIAL EQUATIONS DRIVEN BY UNBOUNDED VARIATION PATHS

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Abstract. We prove the existence of the pullback attractor of the generated flow by a dissipative nonautonomous differential equations driven by unbounded variation paths under the condition of smallness of nonlinear term. In case perturbed term is linear we prove that the attractor is singleton and also is forward one.

1. Introduction

This work extends the study on the long term behavior of the solution of the dissipative Young equations driven by Hölder paths in [10], [11] to the general case where coefficient functions now depend on time. Namely, we consider system

$$dx_t = [A(t)x + f(t, x_t)]dt + g(t, x_t)d\omega_t, \tag{1.1}$$

in which A, f, g are continuous functions, the driving path ω is of bounded $p-$variation for some $p \in (1, 2)$. This equation is understood in the form

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\omega_s, \tag{1.2}$$

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where the first integral is of Riemann type, meanwhile the second one is defined in the Young sense.

The existence and uniqueness of the solution of (1.1) is proved in e.g. [17], [2] and the references there in. It is proved in [2] that (1.1) generates a two-parameter flow of homeomorphism on $\mathbb{R}^d$. This allow to study the asymptotic behaviour of the solution of system in the frame work of dynamical system theory. One interesting topic is pull back and forward attractors. They are invariant sets those attract all the trajectories of the system. In this paper we develop techniques from [11] which deal with autonomous equation, to prove the existence of a nonautonomous attractor for the generated flow from (1.1). In case g is linear the attractor is singleton and is also forward one. Since the notation of nonautonomous attractor is understood as $\omega-$wise we keep the presentation simple by deal with the problem for a deterministic system. The results in this paper can be applied to a stochastic equation with Hölder noises where a random attractor is established.

2. Preliminaries and main results

Young integral

Let us first briefly make a survey on Young integrals. Let $\mathcal{C}([a,b],\mathbb{R}^r)$, $r\geq 1$, denote the space of all continuous paths $x:\ [a,b]\to\mathbb{R}^r$ equipped with supremum norm $\|\cdot\|_{\infty,[a,b]}$ given by $\|x\|_{\infty,[a,b]}=\sup_{t\in[a,b]}|x_t|$, where $|\cdot|$ is the Euclidean norm of a vector in $\mathbb{R}^r$. For $p\geq 1$ and $[a,b]\subset\mathbb{R}$, $\mathcal{C}^p([a,b],\mathbb{R}^r)\subset C([a,b],\mathbb{R}^r)$ denotes the space of all continuous paths $x:[a,b]\to\mathbb{R}^r$ which is of finite $p-$variation, i.e.

$$\left|\!\left|\!\left| x \right|\!\right|\!\right|_{p,[a,b]} := \left(\sup_{\Pi(a,b)}\sum_{i=1}^{n}\|x_{t_{i+1}}-x_{t_i}\|^p\right)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taken over the whole class of finite partitions of $[a,b]$ (see e.g. [12]). $\mathcal{C}^p([a,b],\mathbb{R}^r)$ with the $p-$var norm

$$\|x\|_{p,[a,b]} \quad := \quad |x_a| + \left|\!\left|\!\left| x \right|\!\right|\!\right|_{p,[a,b]},$$

is a nonseparable Banach space [12, Theorem 5.25, p. 92]. Also for each $0<\alpha<1$, we denote by $\mathcal{C}^{\alpha-\mathrm{Hol}}([a,b],\mathbb{R}^r)$ the space of Hölder continuous functions with exponent α on $[a,b]$ equipped with the norm

$$\|x\|_{\alpha-\mathrm{Hol},[a,b]} := \|x_a\| + \sup_{a\leq s<t\leq b}\frac{|x_t-x_s|}{(t-s)^\alpha}.$$

It is known that $\mathcal{C}^{\alpha-\mathrm{Hol}}([a,b],\mathbb{R}^r) \subset \mathcal{C}^{\frac{1}{p}}([a,b],\mathbb{R}^r)$.

Now, consider $y \in \mathcal{C}^q([a,b],\mathbb{R}^{d\times m})$ and $x \in \mathcal{C}^p([a,b],\mathbb{R}^m)$ with $\frac{1}{p}+\frac{1}{q}>1$, the Young integral $\int_a^b y_t dx_t$ can be defined as

$$\int_a^b y_s dx_s := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} y_u(x_v - x_u),$$

where the limit is taken over all the finite partitions Π of $[a,b]$ with $|\Pi| := \max_{[u,v]\in\Pi} |v-u|$ (see [19]). This integral satisfies the additive property by construction, and the so-called *Young-Loeve estimate* [12, Theorem 6.8, p. 116]

$$\Big| \int_s^t y_u dx_u - y_s[x_t - x_s] \Big| \leq K \, \| y \|_{q,[s,t]} \, \| x \|_{p,[s,t]} \, , \ \forall [s,t] \subset [a,b],$$

where

$$K := (1 - 2^{1-\frac{1}{p}-\frac{1}{q}})^{-1}. \tag{2.2}$$

This implies

$$\Big| \int_s^t y_u dx_u \Big| \leq \| x \|_{p,[s,t]} \Big(|y_s| + (K+1) \, \| y \|_{q,[s,t]} \Big) .$$

Assumptions

Now we introduce conditions on driving path ω and coefficient functions A, f, g.

$(\mathbf{H}_0)$ For $p \in (1,2)$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=-n}^{n-1} \| \omega \|^p_{p-\mathrm{var},[k,k+1]} < +\infty$$

$(\mathbf{H}_1)$ A is continuous and bounded on $\mathbb{R}$ by $\|A\|$. Moreover A satisfies the *uniform contraction condition*, i.e. there exists $C_A \geq 1, \lambda_A > 0$ such that for all $s < t$

$$\|\Phi(t,s)\| \leq C_A e^{-\lambda_A(t-s)}, \tag{2.3}$$

where $\Phi(t,s)$ is the Cauchy matrix of the equation $dz_t = A(t) z_t dt$.

$(\mathbf{H}_2)$ $f(t,x)$ is continuous and locally Lipchitz continuity w.r.t. x uniformly on t and there exists $C_f > 0$ and and $b \in L^1(\Delta,\mathbb{R}^d)$, for all closed interval $\Delta \subset \mathbb{R}$ such that the following properties hold:

$$\begin{cases} (i) \ \ |f(t,x)| \leq C_f|x| + b(t), \quad \forall x \in \mathbb{R}^d, \ \forall t \in \mathbb{R}, \\ (ii) \ \ \sup_{k\in\mathbb{Z}} \|b\|_{L^1(k,k+1)} < \infty. \end{cases}$$

($\mathbf{H}_3$) $g(t,x)$ is differentiable in x with $\partial_x g$ is locally δ - Holder continuous w.r.t. x uniformly in t for some $\delta > p-1$, and there exist some constants $0 < C_g$, $1 - \dfrac{1}{p} < \beta \leq 1$, an increasing convex function $k : \mathbb{R}_+ \to \mathbb{R}_+$ vanish at 0

$$\begin{cases} (i)\ \ |g(t,x) - g(t,y)| \leq C_g|x-y|, \quad \forall x,y \in \mathbb{R}^d, \quad \forall t \in \mathbb{R}, \\ (ii)\ \ |g(t,x) - g(s,x)| + \|\partial_x g(t,x) - \partial_x g(s,x)\| \leq k(|t-s|)^\beta =: h^*(|t-s|) \\ \quad \forall x \in \mathbb{R}^d, \quad \forall s,t \in \mathbb{R}. \\ (iii) \lim_{t\to\infty} \frac{\log h^*(t)}{|t|} = 0 \end{cases}$$

Remark 2.1. (i) *($\boldsymbol{H}_0$) is satisfied for almost all realizations of fractional Brownian motion with Hurst index $H > 1/2$ ([15]). We introduce the notation*

$$\Gamma_p := \max\left\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|^p_{p-\mathrm{var},[k,k+1]}, \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=-n}^{-1} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|^p_{p-\mathrm{var},[k,k+1]} \right\} \tag{2.4}$$

which is finite under assumption ($\boldsymbol{H}_0$).

(ii) *Assumption* ($\mathbf{H}_1$) *ensures that the semigroup $\Phi(t) = e^{At}, t \in \mathbb{R}$ generated by A satisfies the following properties: for all $a < b \leq t$*

$$\|\Phi(t,\cdot)\|_{\infty,[a,b]} \leq C_A e^{-\lambda_A(t-b)}, \tag{2.5}$$

$$\left|\!\left|\!\left|\Phi(t,\cdot)\right|\!\right|\!\right|_{p,[a,b]} \leq C_A^2\|A\| e^{-\lambda_A(t-b)}(b-a). \tag{2.6}$$

We recall here the theorem on existence and unqueness of solution from [2]. Under these conditions, system (1.1) possesses a unique solution on whole $\mathbb{R}$ which starts at an arbitrary time t_0 and generates a stochastic two-parameter flow of homeomorphism$\Psi(t,s)$, $t \geq s$ in which $\Psi(t,s)x_0$ is the solution to (1.1) at time t with initial value x_0 at time s. Moreover, we have the following estimate for the growth of the solution.

Proposition 2.1. *The solution x of* (1.1) *is of bounded $p-$variation on each $[u,v] \subset \mathbb{R}$ and satisfies*

$$\begin{aligned} \|x\|_{p,[u,v]} &\leq \Big[|x_u| + D[1 + h^*(|u| \vee |v| \vee |u-v|)](1 + \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{p,[u,v]})N_{[u,v]}\Big] \times \\ &\quad \times e^{2L(v-u)+\kappa N_{[u,v]}} N_{[u,v]}^{\frac{p-1}{p}}, \end{aligned}$$

where $\kappa = \log\frac{K+2}{K+1}$, $L = \|A\| + C_f$, and D is a generic constant and $N_{[u,v]}$ is estimated as

$$N_{[u,v]} \leq 1 + [2(K+1)C_g]^p \left|\!\left|\!\left|\omega\right|\!\right|\!\right|^p_{p,[u,v]}. \tag{2.7}$$

Proof. By computation we have for $s,t \in [u,v]$, $s<t$

$$|x_t - x_s| \leq \hat{A}^{1/p}_{s,t} + C_f \int_s^t |x_r| dr + C_g \left\|\omega\right\|_{p,[s,t]} \Big(|x_s| + K \left\|x\right\|_{p,[s,t]} \Big),$$

where

$$\hat{A}_{s,t} := \left[\int_s^t b(r)dr + \left\|\omega\right\|_{p,[s,t]} \Big(|g(0,0)| + h^*(|u| \vee |v|) + K h^*(|u-v|) \Big) \right]^p.$$

The rest follows step by step in [11, Theorem 2.4].

From now on, we always denote by D a generic constant.

3. Nonautonomous attractors

In what follows we recall the notion of the (global) pullback attractor of a two-parameter flow (see more for instance in [8], [14], [9]).

Definition 3.1. *([7]) For a given two-parameter flow $\Psi(t,s)$, a family of sets $\mathcal{A}_t$ of $\mathbb{R}^d$, $t \in \mathbb{R}$ is called the pullback (forward) attractor of Ψ if*

(i) is compact set for $t \in \mathbb{R}$,

(ii) is invariant, i.e $\Psi(t,s)\mathcal{A}_s = \mathcal{A}_t$ for all $s \leq t$ in $\mathbb{R}$,

(iii) globally pullback (forward) attracting, i.e for every $t \in \mathbb{R}$ and every $\tilde{D}$ bounded

$$\lim_{s\to -\infty} d(\Psi(t,s)\tilde{D}|\mathcal{A}_t) = 0, \quad (\lim_{t\to +\infty} d(\Psi(t,s)\tilde{D}|\mathcal{A}_t) = 0),$$

in which d is Hausdorff semi-distance between nonempty closed subsets E,F of $\mathbb{R}^d$ is defined as $d(E|F) = \sup\{\inf\{d(x,y)|y\in F\}|x \in E\}$.

In general, one may consider the attracting on a family of nonempty sets $(\tilde{D}_t)$ instead of a single set as in Definition 3.1. Below, we consider the family $\tilde{\mathcal{D}}$ of tempered set $\tilde{D}_t$, i.e. $\tilde{D}_t$ is a subset of the closed ball $\bar{B}(0,r_t)$ where the radius r_t is tempered, i.e.

$$\lim_{t\to\pm\infty} \frac{1}{t} \max\{\log r_t,\ 0\} = 0.$$

The pullback attracting property now can be written as

$$\lim_{s\to -\infty} d(\Psi(t,s)\tilde{D}_s|\mathcal{A}_t) = 0,$$

It is known that the existence of a nonautonomous pullback attractor is ensured by the existence of the pullback absorbing set. A family of set $\mathcal{B}_t$ is said to be nonautonomous pullback absorbing if for almost all ω, for each t there exists $T = T(t, \tilde{\mathcal{D}})$ such that

$$\Psi(t,s)\tilde{D}_s \subset \mathcal{B}_t$$

fot all $s < t - T(t, \tilde{\mathcal{D}})$. Assume that there exists a family of compact pullback absorbing sets $\mathcal{B}_t$. Then there is a pullback attractor $\mathcal{A}_t$ given by ([7])

$$\mathcal{A}_t = \bigcap_{\tau \le t} \overline{\bigcup_{s\le \tau} \Psi(t,s)\mathcal{B}_s}. \tag{3.1}$$

3.1. Existence

We consider

$$x_t = x_0 + \int_0^t f(s,x_s)ds + \int_0^t g(s,x_s)d\omega_s,$$

with $(\mathbf{H_0}) - (\mathbf{H_4})$. Thanks to the "variation of constants" formula for Young differential equations (see e.g. [20] or [10]), x_t satisfies

$$x_t = \Phi(t,t_0)x_{t_0} + \int_{t_0}^t \Phi(t,s)f(s,x_s)ds + \int_{t_0}^t \Phi(t,s)g(s,x_s)d\omega_s,\ t \ge t_0. \tag{3.2}$$

In the Lemma below we are going to estimate the solution base on (3.2).

Lemma 3.1. *The following estimate holds for any $a < b \le t$*

$$\begin{aligned}\Big|\int_a^b \Phi(t,s)g(s,x_s)d\omega_s\Big| \quad &\le \quad KC_A\Big(1 + C_A\|A\|(b-a)\Big)\, \|\!|\omega|\!\|_{p,[a,b]}\, e^{-\lambda_A(t-b)} \times \\ &\times \Big[C_g\|x\|_{p,[a,b]} + (h^*(|a|\vee|b|) + h^*(|b-a|)) + |g(0,0)|\Big].\end{aligned}$$

Proof. Firstly by assumption, we choose $2 > q \le p$ such that $q\beta \le 1$, then

$$\begin{aligned}|g(t,x)| \quad &\le \quad C_g|x| + h^*(|t|) + |g(0,0)|, \ \ \forall t,\\ \|\!|g(\cdot,x.)|\!\|_{q,[s,t]} \quad &\le \quad C_g\,\|\!|x|\!\|_{p,[s,t]} + h^*(|t-s|), \ \ \forall\ s \le t.\end{aligned}$$

Using (2.5) and (2.6) we have

$$
\begin{aligned}
&\Big|\int_a^b \Phi(t,s)g(s,x_s)d\omega_s\Big| \\
\leq\ & \left\|\!\left|\omega\right|\!\right\|_{p-\mathrm{var},[\mathrm{a,b}]}\Big(|\Phi(t,a)g(a,x_a)| + K\left\|\!\left|\Phi(t,\cdot)g(\cdot,x_\cdot)\right|\!\right\|_{q,[a,b]}\Big)\\
\leq\ & \left\|\!\left|\omega\right|\!\right\|_{p,[a,b]}\Big[\|\Phi(t,a)\|.|g(a,x_a)| + K\left\|\!\left|\Phi(t,\cdot)\right|\!\right\|_{p,[a,b]}\|g(\cdot,x_\cdot)\|_{\infty,[a,b]}.\\
&\qquad\qquad\qquad + K\|\Phi(t,\cdot)\|_{\infty,[a,b]}\left\|\!\left|g(\cdot,x_\cdot)\right|\!\right\|_{q,[a,b]}\Big]\\
\leq\ & \left\|\!\left|\omega\right|\!\right\|_{p,[a,b]}\Big[C_Ae^{-\lambda_A(t-a)}(C_g|x_a| + h^*(|a|) + |g(0,0)|)\\
& +KC_A^2\|A\|e^{-\lambda_A(t-b)}(b-a)\Big(C_g\|x\|_{\infty,[a,b]} + (h^*(|a|\vee h^*(|b|)) + |g(0,0)|\Big) + \\
& KC_Ae^{-\lambda_A(t-b)}\Big(C_g\left\|\!\left|x\right|\!\right\|_{p,[a,b]} + h^*(|b-a|)\Big)\Big]\\
\leq\ & KC_A\Big[1 + C_A\|A\|(b-a)\Big]\left\|\!\left|\omega\right|\!\right\|_{p,[a,b]}e^{-\lambda_A(t-b)}\times\\
&\qquad\qquad \times\Big[C_g\|x\|_{p,[a,b]} + (h^*(|a|\vee|b|) + h^*(|b-a|)) + |g(0,0)|\Big].
\end{aligned}
$$

■

Next we denote by Δ_k the inteval $[k,k+1]$, $k\in\mathbb{Z}$ and prove that the solution at time t can be estimates via its norm on consecutive Δ_k that cover $[t_0,t]$.

Lemma 3.2. *Assume that $\lambda := \lambda_A - C_AC_f > 0$. The following estimate hold*

$$
\begin{aligned}
e^{\lambda(t-t_0)}|x_t| \quad &\leq\quad C_A|x_{t_0}| + \int_{t_0}^t C_Ae^{\lambda(s-t_0)}b(s)ds\\
&+M\sum_{k=\lfloor t_0\rfloor}^n \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k}e^{\lambda(k-t_0)}\Big[C_g\|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big], \forall t\in\Delta_n,
\end{aligned}
\tag{3.3}
$$

where $M := KC_Ae^{\lambda_A}(1 + C_A\|A\|)$.

Proof. Firstly note that $\lambda > 0$ by $(\mathbf{H}_4)$. Using $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ we have

$$
\begin{aligned}
|x_t| &\leq |\Phi(t,t_0)x_{t_0}| + \int_{t_0}^t|\Phi(t,s)f(s,x_s)|ds + \Big|\int_{t_0}^t\Phi(t,s)g(s,x_s)d\omega_s\Big|\\
&\leq C_Ae^{-\lambda_A(t-t_0)}|x_{t_0}| + \int_{t_0}^t C_Ae^{-\lambda_A(t-s)}\Big(C_f|x_s| + b(s)\Big)ds +\\
&\qquad\qquad \Big|\int_{t_0}^t\Phi(t,s)g(s,y_s)d\omega_s\Big|
\end{aligned}
$$

$$\leq C_A e^{-\lambda_A(t-t_0)}|x_{t_0}| + \int_{t_0}^t C_A e^{-\lambda_A(t-s)} b(s)ds + \beta_t + L_f \int_{t_0}^t e^{-\lambda_A(t-s)}|x_s|ds$$

where $\beta_t := \left|\int_{t_0}^t \Phi(t,s)g(s,x_s)d\omega_s\right|$, $L_f := C_A C_f$. This implies,

$$e^{\lambda_A(t-t_0)}|x_t| \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda_A(s-t_0)} b(s)ds + e^{\lambda_A(t-t_0)}\beta_t + L_f \int_{t_0}^t e^{\lambda_A(s-t_0)}|x_s|ds.$$

By applying the continuous Gronwall Lemma we obtain

$$\begin{aligned} e^{\lambda_A(t-t_0)}|x_t| &\leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda_A(s-t_0)} b(s)ds + e^{\lambda_A(t-t_0)}\beta_t + \\ &\int_{t_0}^t L_f e^{L_f(t-s)}\Big[C_A|x_{t_0}| + \int_{t_0}^s C_A e^{\lambda_A(u-t_0)} b(u)du + e^{\lambda_A(s-t_0)}\beta_s\Big]ds \\ &\leq C_A e^{L_f(t-t_0)}|x_{t_0}| + \int_{t_0}^t C_A e^{L_f(t-s)+\lambda_A(s-t_0)} b(s)ds + e^{\lambda_A(t-t_0)}\beta_t + \\ &+ \int_{t_0}^t L_f e^{L_f(t-s)+\lambda_A(s-t_0)}\beta_s ds \end{aligned}$$

and then

$$e^{\lambda(t-t_0)}|x_t| \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s)ds + e^{\lambda(t-t_0)}\beta_t + \int_{t_0}^t L_f e^{\lambda(s-t_0)}\beta_s ds. \tag{3.4}$$

Now we use Lemma 3.1 to estimate β_s. Assume $t_0 = n_0 \in \mathbb{Z}$,

$$\begin{aligned} & e^{\lambda(s-t_0)}\beta_s \\ = \quad & e^{\lambda(s-t_0)}\Big|\int_{n_0}^s \Phi(s,u)g(u,x_u)d\omega_u\Big| \\ \leq \quad & e^{\lambda(s-t_0)} \sum_{k=n_0}^{\lfloor s\rfloor -1} \Big|\int_{\Delta_k} \Phi(s,u)g(u,x_u)d\omega_u\Big| + \Big|\int_{\lfloor s\rfloor}^s \Phi(s,u)g(u,x_u)d\omega_u\Big| \\ \leq \quad & e^{\lambda(s-t_0)} \sum_{k=n_0}^{\lfloor s\rfloor} KC_A(1+C_A\|A\|)\, |\!|\!|\omega|\!|\!|_{p,\Delta_k}\, e^{-\lambda_A(s-k-1)} \times \\ & \qquad \times\Big[C_g\|x\|_{p,\Delta_k} + h^*(|k|+1) + |g(0,0)|\Big] \\ \leq \quad & M \sum_{k=n_0}^{\lfloor s\rfloor} |\!|\!|\omega|\!|\!|_{p,\Delta_k}\, e^{\lambda(k-n_0)} e^{-L_f(s-k)}\Big[C_g\|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big]. \qquad (3.5) \end{aligned}$$

Replacing (3.5) into (3.4) and considering $t \in [n, n+1)$ we obtain

$$\begin{aligned}
& e^{\lambda(t-t_0)}|x_t| \\
& \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s) ds \\
& + M \sum_{k=n_0}^{n} \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} e^{\lambda(k-n_0)-L_f(t-k)} \Big[C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big] \\
& + L_f M \int_{t_0}^t \sum_{k=n_0}^{\lfloor s \rfloor} \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} e^{\lambda(k-n_0)-L_f(s-k)} \times \\
& \qquad \times \Big[C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big] ds \\
\leq \ & C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s) ds + M \sum_{k=n_0}^{n} \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} e^{\lambda(k-n_0)} \times \\
& \qquad \times \Big[C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big]. \qquad (3.6)
\end{aligned}$$

The continuity of x at $t = (n+1)$ implies that (3.6) holds for all $t \in [n, n+1]$. Now for $t_0 \in (n_0 - 1, n_0)$, similar to (3.5) and (3.6) we have

$$e^{\lambda(s-t_0)}\beta_s \leq M \sum_{k=n_0-1}^{\lfloor s \rfloor} \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} e^{\lambda(k-t_0)} e^{-L_f(s-k)} \Big[C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big],$$

and by replacing t_0 in the final term of (3.4) by $(n_0 - 1)$ then

$$\begin{aligned}
e^{\lambda(t-t_0)}|x_t| \ \leq \ & C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s) ds \\
& + M \sum_{k=n_0-1}^{n} \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} e^{\lambda(k-t_0)} \Big[C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)\Big].
\end{aligned}$$

This proves (3.3). ■

Proposition 3.1. *Define*

$$\Lambda_k := [2(K+1)C_g] \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k}, \qquad (3.7)$$

$$G_k := \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k} \Big(1 + \Lambda_k^{p-1}\Big) e^{\kappa(1+\Lambda_k^p)+2L}, \qquad (3.8)$$

$$H_k := (1 + \left\|\!\left|\omega\right|\!\right\|_{p,\Delta_k})^2 \Big(1 + \Lambda_k^{2p-1}\Big) e^{\kappa(1+\Lambda_k^p)+2L}, \qquad (3.9)$$

where κ, L in Proposition 2.1 and

$$\zeta := \sum_{k=1}^{\infty} e^{-\lambda k} h^*(|k|+1) H_{-k} \prod_{j=1}^{k} \Big(1 + M C_g G_{-j}\Big) \tag{3.10}$$

(which can be infinity), where λ, M is defined in Lemma 3.2.

Assume further that

$$\lambda > \hat{G} := C_A e^{\lambda_A + 2L}(1 + C_A\|A\|)\Big\{\Big[2(K+1)C_g\Gamma_p\Big]^p + \Big[2(K+1)C_g\Gamma_p\Big]\Big\}, \tag{3.11}$$

where Γ_p is defined in (2.4). Then $\zeta(\omega)$ is finite.

Proof.

Due to the inequality $\log(1 + ae^b) \le a + b$ for $a, b \ge 0$, we have

$$\log\Big(1 + MC_gG_k\Big) \le \Big[Me^{2L+\kappa} + 2\Big][2(K+1)]^{p-1} C_g^p \, |||\omega|||_{p,\Delta_k}^p + Me^{2L+\kappa} C_g \, |||\omega|||_{p,\Delta_k}.$$

It follows that

$$\begin{aligned} &\overline{\lim_{m\to\infty}} \frac{1}{m} \log \prod_{k=1}^{m} \Big(1 + MC_gG_{-k}\Big) = \overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{k=1}^{m} \log\Big(1 + MC_gG_{-k}\Big) \\ \le\ & \Big[Me^{2L}\frac{K+2}{2(K+1)^2} + \frac{1}{K+1}\Big]\Big\{\Big[2(K+1)C_g\Gamma_p\Big]^p + \Big[2(K+1)C_g\Gamma_p\Big]\Big\} \\ \le\ & C_A e^{\lambda_A + 2L}(1 + C_A\|A\|)\Big\{\Big[2(K+1)C_g\Gamma_p\Big]^p + \Big[2(K+1)C_g\Gamma_p\Big]\Big\} = \hat{G}. \end{aligned}$$

Meanwhile, (3.7) and (3.9) yield

$$\log H_k \quad \le \quad D\Big[1 + |||\omega|||_{p,\Delta_k} + |||\omega|||_{p,\Delta_k}^p\Big],$$

where we use the inequalities $\log(1 + a + b) \le \log(1 + a) + \log(1 + b), \forall a, b \ge 0$ and $\log(1 + ab) \le \log(1 + a) + \log b, \forall a \ge 0, b \ge 1$. As a result,

$$\overline{\lim_{m\to\infty}} \frac{\log H_{-m}}{m} = 0$$

and then by assumption on h^*

$$\overline{\lim_{m\to\infty}} \frac{\log h^*(|m|+1)H_{-m}}{m} = 0.$$

Hence, there exists for each $0 < 2\delta < \lambda - \hat{G}$ an $m_0 = m_0(\delta, \omega)$ such that for all $m \geq m_0$,

$$e^{(-\delta+\hat{G})m} \leq \prod_{k=1}^{m} \Big[1 + G_{-k}\Big] \leq e^{(\delta+\hat{G})m}$$

and

$$e^{-\delta m} \leq h^*(|m|+1)H_{-m} \leq e^{\delta m}.$$

Consequently,

$$\zeta \leq \sum_{k=1}^{m_0-1} e^{-\lambda k} h^*(|k|+1)H_{-k} \prod_{j=1}^{k} \Big(1 + MC_g G_{-j}\Big) + \sum_{k=n_0}^{\infty} e^{-(\lambda-2\delta-\hat{G})k}$$

which is finite. ■

We are now in position to state the first main result of this paper.

Theorem 3.2. *Assume that* $(\mathbf{H_0}) - (\mathbf{H_4})$ *are satisfied. Then under the condition* (3.11) *the flow generated by system* (1.1) *possesses a pullback attractor* $\mathcal{A}_t$.

Proof. We first consider $t_0 = n_0 \in \mathbb{Z}^-, n_0 \leq t = n \in \mathbb{Z}$. From Lemma 3.2 we have

$$\begin{aligned}|x_n|e^{\lambda(n-n_0)} \quad \leq \quad & C_A|x_{n_0}| + \int_{n_0}^{n} C_A e^{\lambda(s-t_0)}|b(s)|ds + M \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} \, |\!|\!|\omega|\!|\!|_{p,\Delta_k} \times \\ & \times [C_g \|x\|_{p,\Delta_k} + Dh^*(|k|+1)]\end{aligned}$$

Using $(\mathbf{H_2})(iii)$, $\int_{n_0}^{n} C_A e^{\lambda(s-t_0)}|b(s)|ds \leq D\sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)}$. Then dominating each $\|x\|_{p,\Delta_k}$ by estimation in Proppsition 2.1 with the observation that

$$N_{\Delta_k}^{\frac{p-1}{p}} \leq 1 + \Lambda_k^{p-1}, \quad N_{\Delta_k}^{\frac{2p-1}{p}} \leq 2[1 + \Lambda_k^{2p-1}]$$

we obtain

$$|x_n|e^{\lambda(n-n_0)} \leq C_A|x_{n_0}| + MC_g \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} G_k |x_k| + D \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} h^*(|k|+1)H_k.$$

Fix n_0, put $m = n - n_0$ and $z_k = e^{\lambda k}|x_{k+n_0}|$. We have

$$z_m \leq C_A z_0 + MC_g \sum_{k=0}^{m-1} G_{k+n_0} z_k + D \sum_{k=0}^{m-1} e^{\lambda k} h^*(|k+n_0|+1)H_{k+n_0}$$

holds for all $m \in \mathbb{Z}^+$. Thank to Lemma 4.1

$$z_m \le C_A z_0 \prod_{k=0}^{m-1}\Big(1+MC_gG_{k+n_0}\Big) + D\sum_{k=0}^{m-1} e^{\lambda k} h^*(|k+n_0|+1)H_{k+n_0} \times \times \prod_{j=k+1}^{m-1}\Big(1+MC_gG_{j+n_0}\Big)$$

or

$$\begin{aligned}
&|x(n,n_0,x_{n_0})| \\
&\le C_A|x_{n_0}|e^{-\lambda(n-n_0)} \prod_{k=0}^{n-n_0-1}\Big(1+MC_gG_{k+n_0}\Big) \\
&+D\sum_{k=0}^{n-n_0-1} e^{-\lambda(n-n_0-k)} h^*(|k+n_0|+1)H_{k+n_0} \prod_{j=k+1}^{n-n_0-1}\Big(1+MC_gG_{j+n_0}\Big) \\
&\le C_A|x_{n_0}|e^{-\lambda(n-n_0)} \prod_{k=1-n}^{-n_0}\Big(1+MC_gG_{-k}\Big) + \\
&+De^{-\lambda n}\sum_{k=1-n}^{-n_0} e^{-\lambda k} h^*(|k|+1)H_{-k} \prod_{j=1-n}^{k}\Big(1+MC_gG_{-j}\Big).
\end{aligned}$$

We consider two cases:
If $n \ge 1$,

$$\begin{aligned}
&|x(n,n_0,x_{n_0})| \le \\
&\quad C_Ae^{-\lambda n}\prod_{k=1-n}^{0}\Big(1+MC_gG_{-k}\Big)|x_{n_0}|e^{\lambda n_0}\prod_{k=1}^{-n_0}\Big(1+MC_gG_{-k}\Big)+ \\
&\quad De^{-\lambda n}\sum_{k=1-n}^{0} e^{-\lambda k}h^*(|k|+1)H_{-k}\prod_{j=1-n}^{k}\Big(1+MC_gG_{-j}\Big)+ \\
&\quad De^{-\lambda n}\prod_{j=1-n}^{0}\Big(1+MC_gG_{-j}\Big)\sum_{k=1}^{-n_0} e^{-\lambda k}h^*(|k|+1)H_{-k}\prod_{j=1}^{k}\Big(1+MC_gG_{-j}\Big).
\end{aligned}$$

If $n \le 0$,

$$|x(n,n_0,x_{n_0})| \le \frac{C_Ae^{-\lambda n}}{\prod_{k=1}^{1-n}\Big(1+MC_gG_{-k}\Big)}|x_{n_0}|e^{\lambda n_0}\prod_{k=1}^{-n_0}\Big(1+MC_gG_{-k}\Big)+$$

$$\frac{De^{-\lambda n}}{\prod_{k=1}^{1-n}\Big(1+MC_gG_{-k}\Big)}\sum_{k=1}^{-n_0} e^{-\lambda k}h^*(|k|+1)H_{-k}\prod_{j=1}^{k}\Big(1+MC_gG_{-j}\Big).$$

Hence in both cases,

$$\begin{aligned}
|x(n,n_0,x_{n_0})| &\leq a_n^1|x_{n_0}|e^{\lambda n_0}\prod_{k=1}^{-n_0}\Big(1+MC_gG_{-k}\Big)\\
&\quad +a_n^1\sum_{k=1}^{-n_0} e^{-\lambda k}h^*(|k|+1)H_{-k}\prod_{j=1}^{k}\Big(1+MC_gG_{-j}\Big)+a_n^2\\
&\leq a_n^1|x_{n_0}|e^{\lambda n_0}\prod_{k=1}^{-n_0}\Big(1+MC_gG_{-k}\Big)\\
&\quad +a_n^1\sum_{k=1}^{\infty} e^{-\lambda k}h^*(|k|+1)H_{-k}\prod_{j=1}^{k}\Big(1+MC_gG_{-j}\Big)+a_n^2\\
&\leq a_n^1|x_{n_0}|e^{\lambda n_0}\prod_{k=1}^{-n_0}\Big[1+MC_gG_{-k}\Big]+a_n^1\zeta+a_n^2
\end{aligned}$$

where $\zeta(\omega)$ is given in (3.10)

$$a_n^1 := De^{-\lambda n}\times\begin{cases}\prod_{k=1-n}^{0}\Big[1+MC_gG_{-k}\Big], & \text{if } n\geq 1\\ \Big(\prod_{k=1}^{1-n}\Big[1+MC_gG_{-k}\Big]\Big)^{-1}, & \text{if } n\leq 0,\end{cases}$$

and

$$a_n^2 := \begin{cases} D_2\sum_{k=1-n}^{0} e^{-\lambda(n+k)}h^*(|k|+1)H_{-k}\prod_{j=1-n}^{k}\Big[1+MC_gG_{-j}\Big], & \text{if } n\geq 1,\\ 0, & \text{if } n\leq 0.\end{cases}$$

Hence, for fixed n, if x_{n_0} lies in a tempered set,

$$|x(n,n_0,x_{n_0})|\leq 1+a_n^1\zeta+a_n^2 =: \rho_n$$

when $-n_0$ large enough. It is easy to see that $\lim_{n\to\infty}\frac{\log\rho_n}{n}=0$.

Using Proposition 2.1 again to estimate $|x(t,t_0,x_{t_0})|$ with arbitrary $t,t_0\in\mathbb{R}$, by computation we have a tempered function $\hat{\rho}_t$ is tempered such that

$$|x(t,t_0,x_{t_0})|\leq\hat{\rho}_t \tag{3.12}$$

when $-t_0$ large enough.

Therefore, there exists a family of sets $\mathcal{B}_t = \bar{B}(0, \hat{\rho}_t)$ which absorbs $\tilde{\mathcal{D}}$. This ensures the existence of the pullback attractor $\mathcal{A}_t$ for system (1.1) which is given by (3.1). ■

Remark 3.1. *In [11] we prove that under* (3.11) *the RDS generated by the autonomous equation possesses a random pullback attractor. Thus, the criteria still holds for the nonautonomous case.*

4. Special case: g linear

In this part, we consider (1.1) where $g(t, x)$ is linear in x. For convenience we assume path x valued in $\mathbb{R}$ instead of $\mathbb{R}^m$. Then g has the form $g(t, x) = C(t)x$ where C is a $\mathbb{R}^{d\times d}$-valued, continuous functions. Then, (1.1) becomes

$$dx_t = [A(t)x_t + f(t, x_t)]dt + C(t)x_t d\omega_t, t \in \mathbb{R},\ x(t_0) = x_{t_0} \in \mathbb{R}^d. \tag{4.1}$$

We need the following assumption for C.

$(\mathbf{H}_C)$ C is continuous and $\hat{C} := \sup_k \|C\|_{p,\Delta_k} < \infty$.

Note that in this situation, $(\mathbf{H_3})$ is not fulfilled. However, as proved in [3], (4.1) possesses a unique solution $x(\cdot, t_0, x_0)$ start at t_0 from $x_0 \in \mathbb{R}^d$ which is of bounded $p-$variation on any compact subset of $\mathbb{R}$. Using the estimate

$$\left\| Cx \right\|_{p,[s,t]} \leq \|C\|_{\infty,[s,t]} \left\| x \right\|_{p,[s,t]} + \|x\|_{\infty,[s,t]} \left\| C \right\|_{p,[s,t]}, \tag{4.2}$$

we can treat (4.1) as the general equation in previous section by considering $C_g = \hat{C}$ and omitting $(\mathbf{H_3})(ii), (iii)$. Then one obtains the similar results that the equation generates a two-parameter flow of homeomophism on $\mathbb{R}^d$.

This case is treated in [4] using a kind of Lyapunov function. Here we revise the problem on the existence of the random pullback attractor of the system in such a case by using semi group method as in previous section.

Theorem 4.1. *Assume that* $(\mathbf{H0}), (\mathbf{H1}), (\mathbf{H2}), (\mathbf{H_C})$ *are satisfied. Then there exists* $\varepsilon > 0$ *such that if* $\hat{C} < \epsilon$ *the flow generated by the system* (4.1) *possesses a pullback attractor* $\mathcal{A}_t$.

Proof. The proof is followed step by step of Theorem 3.2. ■

In what follow we impose a stronger condition on f to study the difference between two solutions of the system which facilitates the proof of singleton attractor.

($\mathbf{H_5}$): f is global Lipchitz continuous with Lipchitz constant C_f (here we use an abuse notation for simplicity).

We use the linearity of g to obtain a further result that the pullback attractor in Theorem 4.1 is singleton and moreover forward attractor.

Theorem 4.2. *Under the assumption in Theorem 4.1 and* ($\mathbf{H_5}$)*, the pullback attractor $\mathcal{A}_t$ is singleton for each t and moreover is forward attractor.*

Proof. We fix $\bar{t}$ and consider $\mathcal{A}_{\bar{t}}$. Take $a^1(\bar{t}), a^2(\bar{t}) \in \mathcal{A}_{\bar{t}}$, by the invariance of $\mathcal{A}_{\bar{t}}$, for each $t_0 < \bar{t}$ there exist $b^1 := b^1(t_0), b^2 := b^2(t_0) \in \mathcal{A}_{t_0}$ such that

$$a^i(\bar{t}) = x(\bar{t}, t_0, b^i), \quad i = 1, 2.$$

Put $x^i_\cdot := x^i(\cdot, t_0, b^i)$ and $z_\cdot := x^1_\cdot - x^2_\cdot$ then

$$\begin{aligned}
dz_t &= d(x^1_t - x^2_t) \\
&= [A(t)x^1_t + f(t, x^1_t) - A(t)x^2_t - f(t, x^2_t)]dt + [C(t)x^1_t - C(t)x^2_t]d\omega_t \\
&= [A(t)z_t + f(t, x^1_t) - f(t, x^2_t)]dt + C(t)z_t d\omega_t \\
&=: [A(t)z_t + F(t, z_t)]dt + C(t)z_t d\omega_t, \quad t \geq t_0, \\
z_{t_0} &= b^1 - b^2,
\end{aligned}$$

in which by the definition $|F(t, z_t)| \leq C_f |z_t|$, $F(t, 0) \equiv 0$.

Note that using the estimate in (4.2) one obtains a similar result to that in Proposition 3.1. Then repeat the arguments in Lemma 3.2 with for short $t_0 = n_0 \in \mathbb{Z}$

$$|z_t| e^{\lambda(t-n_0)} \leq C_A |z_{n_0}| + D\hat{C} \sum_{k=n_0}^{n} |||\omega|||_{p,\Delta_k} e^{\lambda(k-n_0)} \|z\|_{p,\Delta_k}, \ \forall t \in \Delta_n,$$

in which the norm $\|z\|_{p,\Delta_k}$ can be estimated similar to x in Proposition 2.1, namely

$$\|z\|_{p,\Delta_k} \leq |z_k| e^{D(1+\hat{C}^p |||\omega|||^p_{p,\Delta_k})}. \tag{4.3}$$

Hence

$$|z_t| e^{\lambda(t-n_0)} \leq C_A |z_{n_0}| + D\hat{C} \sum_{k=n_0}^{n} |||\omega|||_{p,\Delta_k} e^{D|||\omega|||^p_{p,\Delta_k}} e^{\lambda(k-t_0)} |z_k|, \ \forall t \in \Delta_n.$$

This leads to

$$|z_n| \leq C_A |z_{n_0}| e^{-\lambda(n-n_0)} \prod_{k=n_0}^{n-1} \Big[1 + D\hat{C} \, \|\omega\|_{p,\Delta_k} \, e^{D\|\omega\|^p_{p,\Delta_k}}\Big].$$

Since $b^1, b^2 \in \mathcal{A}_{n_0}$, $|z_{n_0}| \leq 2\hat{\rho}_{n_0}$. Note that

$$\lim_{n_0 \to -\infty} \frac{\log \hat{\rho}_{n_0}}{n_0} = 0.$$

For $\bar{t} = \bar{n} \in \mathbb{Z}$, follow the arguments in Theorem 3.2 for $\hat{C}$ small enough, $|a^1(\bar{t}) - a^2(\bar{t})| = |z_{\bar{t}}| \to 0$ as $n_0 \to \infty$ or $a^1(\bar{t}) = a^2(\bar{t})$. Using (4.3) to estimate z_t via z_n which $[n, n+1]$ contain t, this holds for arbitrary $\bar{t} \in \mathbb{R}$. Therefore, $\mathcal{A}_t$ is one point set.

Finally, the above arguments show that the difference of two solutions of the system tends to zero in the forward direction, the attractor is then the forward one. The proof is completed. ■

Appendix

The proof of following Lemmas can be seen in [11]

Lemma 4.1 (Discrete Gronwall Lemma)**.** *Let a be a non negative constant and u_n, α_n, β_n be nonnegative sequences satisfying*

$$u_n \leq a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k, \quad \forall n \geq 1$$

then

$$u_n \leq \max\{a, u_0\} \prod_{k=0}^{n-1} (1 + \alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1} (1 + \alpha_j)$$

for all $n \geq 1$.

Lemma 4.2 (Gronwall-type Lemma)**.** *If y satisfies the following condition*

$$(4.4) \qquad |y_t - y_s| \leq \hat{A}^{1/q}_{s,t} + a_1 \int_s^t |y_u| du + a_2 \, \|\omega\|_{p,[s,t]} \left(|y_s| + a_3 \, \|y\|_{q-\mathrm{var},[s,t]}\right)$$

for all s, t, where a_1, a_2, a_3 are positive real constants, then

$$(4.5) \qquad \|y\|_{p,[u,v]} \leq \Big[|y_u| + 2\hat{A}^{1/q}_{u,v} N_{[u,v]}\Big] \, e^{2a_1(v-u) + \kappa N_{[u,v]}} N^{\frac{p-1}{p}}_{[u,v]}(\omega)$$

with $\kappa = \log \frac{a_3+2}{a_3+1}$, *and*

$$N_{[u,v]} \leq 1 + [2a_2(a_3+1)]^p \left\|\omega\right\|^p_{p,[s,t]}.$$

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