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ON THE EQUATION $F(n^2+m^2+k)=H(n)+H(m)+K$

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Abstract. We give all solutions of the equation

$$F(n^2+m^2+k)=H(n)+H(m)+K \quad (\forall n,m\in\mathbb{N}),$$

where $k\in\mathbb{N}$ is the sum of two fixed squares, $K\in\mathbb{C}$ and F,H are completely multiplicative functions.

1. Introduction.

Let $\mathcal{P},\mathbb{N}_0,\mathbb{N},\mathbb{Z}$ and $\mathbb{C}$ be the set of primes, non-negative integers, positive integers, integers and complex numbers, respectively. Let $\mathcal{M}$ $(\mathcal{M}^*)$ be the set of all multiplicative (completely multiplicative) functions, respectively. For $D\in\mathbb{N}, D\geq 2$ we denote by $\chi_D^*(n)$ the principal Dirichlet character and by $\chi_D(n)$ the non-principal Dirichlet character $\pmod{D}$. For numbers $x,y\in\mathbb{Z}$ we denote by (x,y) the greatest common divisor of x and y. For each $n\in\mathbb{N}$ let $\overline{n}\in\{0,1,2,3\}$ be such that $\overline{n}\equiv n \pmod 4$.

Furthermore, we define the sets $\mathcal{B}$ and $\mathcal{E}$ as follows:

$$\mathcal{B}=\{n^2+m^2|\ \ n,m\in\mathbb{N}\}=\{2,5,8,10,13,17,18,20,25,\cdots\}$$

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and

$$\mathcal{E} = \{n^2 + m^2 > 0 | \ \ n, m \in \mathbb{N}_0\} = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, \cdots\}.$$

It is obvious that

$$\mathcal{B} \subset \mathcal{E}.$$

In 1996 P. V. Chung [2] characterized all multiplicative functions f satisfying the equation

$$f(m^2 + n^2) = f(n^2) + f(m^2) \quad \text{for every} \ \ n, m \in \mathbb{N}.$$

P. V. Chung proved that there are only two possible categories of solutions, the first of which contains the identity function

In 2014 B. Bojan [1] determined all $f : \mathbb{N} \to \mathbb{C}$ for which

$$f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every} \ \ n, m \in \mathbb{N}$$

holds.

Poo-Sung Park in [14] and [15] proved that if $f \in \mathcal{M}$ and $k \in \mathbb{N}, \ k \geq 3$ satisfy one of following two conditions

$$f(x_1^2 + \cdots + x_k^2) = f(x_1)^2 + \cdots + f(x_k)^2$$

or

$$f(x_1^2 + \cdots + x_k^2) = f(x_1^2) + \cdots + f(x_k^2)$$

for all positive integers $x_1, \cdots, x_k$, then f is the identity function.

I. Kátai and B. M. Phong proved in [5] that if the sets

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{S} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A}+\mathcal{S} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ and $h : \mathcal{S} \to \mathbb{C}$ satisfy the equation

$$f(a + n^2) = g(a) + h(n^2) \quad \text{for every} \quad a \in \mathcal{A}, n \in \mathbb{N},$$

then the assumption $8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A}$ implies that there is a complex number A such that

$$g(a) = Aa + \overline{g}(a), \ \ h(n^2) = An^2 + \overline{h}(n) \ \text{ and } \ f(a+n^2) = A(a+n^2) + \overline{g}(a) + \overline{h}(n)$$

hold for every $a \in \mathcal{A}, n \in \mathbb{N}$. Furthermore

$$\begin{aligned} &\overline{g}(a) = \overline{g}(b) \ \text{ if } \ a \equiv b \pmod{120}, \quad (a, b \in \mathcal{A}), \\ &\overline{h}(n) = \overline{h}(m) \ \text{ if } \ n \equiv m \pmod{60}, \quad (n, m \in \mathbb{N}) \end{aligned}$$

are true.

B.M.M.Khanh [12] determined all solutions of the equation

$$f(n^2+m^2+k)=f(n)^2+f(m)^2+K \quad (\forall n,m\in\mathbb{N}),$$

where $k\in\mathbb{N}_0$ and $K\in\mathbb{C}$. In [10] and [11] she gave all functions $f:\mathbb{N}\to\mathbb{C}$ which satisfy the equation

$$f(n^2+Dm^2+k)=f(n)^2+Df(m)^2+k \quad \text{for every} \quad n,m\in\mathbb{N}.$$

The conjecture of I. Kátai and B. M. Phong formulated in [6] was proved by B. M. M. Khanh in [10].

In [7] and [8] I. Kátai and B. M. Phong gave all arithmetical functions $f,h:\mathbb{N}\to\mathbb{C}$, which satisfy the relations

$$f(a^2+b^2+c^2+d^2+k)=h(a)+h(b)+h(c)+h(d)+K$$

for every $a,b,c,d\in\mathbb{N}$, where $k\in\mathbb{N}_0$ and $K\in\mathbb{C}$.

Recently, in [9] I. Kátai and B. M. Phong gave all functions $f,h:\mathbb{N}\to\mathbb{C}$ which satisfy the relation

$$f(a^2+b^2+c^2+k)=h(a)+h(b)+h(c)+K$$

for every $a,b,c\in\mathbb{N}$, where $k\in\mathbb{N}_0$ and $K\in\mathbb{C}$.

In this paper we prove the following result.

Theorem 1. *The numbers $k\in\mathcal{E},K\in\mathbb{C}$ and the functions $F,H\in\mathcal{M}^*$ satisfy the equation*

$$F(n^2+m^2+k)=H(n)+H(m)+K \quad \text{for every} \quad n,m\in\mathbb{N} \tag{1.1}$$

if and only if one of the following assertions holds:

- $(T1)$ $K=k, \quad H(m)=m^2, F(n)=n,$
- $(T2)$ $K=-1, \quad H(m)=1, F(n)=1,$
- $(T3)$ $K=-2, \quad H(m)=1, \quad F(n^2+m^2+k)=0,$
- $(T4)$ $K=-1, \quad k\equiv 2 \pmod 3, \quad H(m)=\chi_3^*(m), \quad F(n)=\chi_3(n)$

for every $n,m\in\mathbb{N}$, where $\chi_3^*(m) \pmod 3$ is the principal Dirichlet character and $\chi_3(m) \pmod 3$ is the non-principal Dirichlet character, i.e $\chi_3^*(0)=0$, $\chi_3^*(1)=1$, $\chi_3^*(2)=1$, $\chi_3(0)=0$, $\chi_3(1)=1$, $\chi_3(2)=-1$.

By Theorem 1, we have

Corollary 1. *If the numbers $k \in \mathcal{E}, K \in \mathbb{C} \setminus \{-1,-2\}$ and the functions $F, H \in \mathcal{M}^*$ satisfy the equation*

$$F(n^2+m^2+k) = H(n) + H(m) + K \quad \text{for every} \quad n, m \in \mathbb{N},$$

then

$$K = k, \quad H(m) = m^2 \quad \text{and} \quad F(n) = n \quad \text{for every} \quad n, m \in \mathbb{N}.$$

2. Lemmas.

Assume that the functions $F, H \in \mathcal{M}^*$ and the numbers $k \in \mathbb{N}_0, K \in \mathbb{C}$ satisfy the equation (1.1). Since $H \in \mathcal{M}^*$, we have $H(1) = 1$, $H(4) = H(2)^2$ and $H(6) = H(2)H(3)$.

We shall use the following results due to the second author:

Lemma 1. (B. M. M. Khanh [13], Lemma 6). *We have*

$$\begin{cases} H(7) &= 2H(5) - 1 \\ H(8) &= 2H(5) + H(2)^2 - 2 \\ H(9) &= H(2)H(3) + 2H(5) - H(2) - 1 \\ H(10) &= H(2)H(3) + 3H(5) - H(3) - 2 \\ H(11) &= H(2)H(3) + 4H(5) - H(3) - H(2) - 2 \\ H(12) &= H(2)H(3) + 4H(5) + H(2)^2 - H(2) - 4 \end{cases}$$

and

$$H(\ell+12m) = H(\ell+9m) + H(\ell+8m) + H(\ell+7m) - \\ - H(\ell+5m) - H(\ell+4m) - H(\ell+3m) + H(\ell)$$

holds for every $\ell, m \in \mathbb{N}$.

Lemma 2. (B. M. M. Khanh [13], Lemma 7). *Let*

$$\begin{cases} A &= \frac{1}{120}\Big(H(2)H(3) + 4H(5) - H(3) - H(2) - 3\Big), \\ \Gamma_2 &= \frac{-1}{8}\Big(H(2)H(3) - 4H(5) + 4H(2)^2 - H(3) + 3H(2) - 3\Big), \\ \Gamma_3 &= \frac{-1}{3}\Big(H(2)H(3) - 2H(5) + 2H(3) - H(2)\Big), \\ \Gamma_4 &= \frac{1}{4}\Big(H(2)H(2) - 2H(2)^2 - H(3) + H(2) + 1\Big), \\ \Gamma_5 &= \frac{1}{5}\Big(H(2)H(3) - H(5) - H(3) - H(2) + 2\Big), \\ \Gamma &= \frac{1}{4}\Big(H(2)H(3) - 4H(5) + 2H(2)^2 + 3H(3) + H(2) + 1\Big), \end{cases}$$

$$\mathcal{S}(\ell)=\Gamma_2\chi_2^*(\ell)+\Gamma_3\chi_3^*(\ell)+\Gamma_4\chi_4(\ell-1)+\Gamma_5\chi_5(\ell)+\Gamma,$$

where $\chi_2^*(\ell) \pmod 2$, $\chi_3^*(\ell) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(\ell) \pmod 4$, $\chi_5(\ell) \pmod 5$ *are the real, non-principal Dirichlet characters.*

Then we have

$$H(n)=An^2+\mathcal{S}(n) \quad \textit{for every} \;\; n\in\mathbb{N}. \tag{2.1}$$

Lemma 3. *Assume that the numbers* $k\in\mathcal{E}$ *and the function* $G\in\mathcal{M}^*$ *satisfy the equation*

$$G(n^2+m^2+k)=1 \quad \textit{for every} \;\; n,m\in\mathbb{N}.$$

Then $G(n)=1$ *for every* $n\in\mathbb{N}$.

Proof. This lemma follows from Theorem 1.1 of Fehér J., K.-H. Indlekofer and N. M. Timofeev [4] (see also Fehér J. and I. Kátai [3]).

Lemma 4. *If the numbers* $k\in\mathcal{E}, K\in\mathbb{C}$ *and the functions* $F,H\in\mathcal{M}^*$ *satisfy the equations*

$$H(m)=1 \quad \textit{for every} \;\; m\in\mathbb{N}$$

and

$$F(n^2+m^2+k)=H(n)+H(m)+K=2+K \quad (\forall \;\; n,m\in\mathbb{N}), \tag{2.2}$$

then one of the assertions $(T2)$, $(T3)$ *of Theorem 1 holds.*

Proof. Assume that $F,H\in\mathcal{M}^*$ satisfy (2.2). Since

$$(k+25)^2+25+k=(k+25)(k+26) \;\text{ and }\; 25=5^2=3^2+4^2, \;\; 26=1^2+5^2,$$

by using the facts $F\in\mathcal{M}^*$, we infer from (2.2) that

$$\begin{aligned} 2+K=F\Big((k+25)^2+5^2+k\Big)&=F(3^2+4^2+k)F(1^2+5^2+k)=\\ &=(2+K)^2, \end{aligned}$$

which implies that $K\in\{-1, \; -2\}$.

If $K=-1$, then we infer from (2.2) that

$$F(n^2+m^2+k)=2+K=1 \quad \text{for every} \;\; n,m\in\mathbb{N},$$

and so Lemma 3 with $G=F\in\mathcal{M}^*$ proves that $F(n)=1$ for every $n\in\mathbb{N}$. Thus the case (T2) holds.

If $K=-2$, then we infer from (2.2) that

$$F(n^2+m^2+k)=2+K=0 \quad \text{for every} \quad n,m\in\mathbb{N},$$

and so the case (T3) holds.

Lemma 4 is thus proved.

Lemma 5. *Let $p\in\{2,3\}$. Assume that the numbers $k\in\mathcal{E}, K\in\mathbb{C}$ and the functions $F,H\in\mathcal{M}^*$ satisfy the equations*

$$F(n^2+m^2+k)=H(n)+H(m)+K \quad \textit{and} \quad H(m)=\chi_p^*(m), \tag{2.3}$$

where χ_p^* (mod p) is the principal Dirichlet character. Then $p=3$ and the assertion $(T4)$ of Theorem 1 holds.

Proof. Since

$$(k+25)^2+25+k=(k+25)(k+26) \quad \text{and} \quad (k+900)^2+900+k=(k+900)(k+901),$$

we infer from $F\in\mathcal{M}^*$ and (2.3) that

$$F((k+25)^2+5^2+k)=F(3^2+4^2+k)F(1^2+5^2+k)$$

and

$$F((k+900)^2+30^2+k)=F(18^2+24^2+k)F(1^2+30^2+k),$$

which with (2.3) imply that

$$\begin{cases} H(k+1) & =-\chi_p^*(5)-K+(\chi_p^*(3)+\chi_p^*(4)+K)(1+\chi_p^*(5)+K)= \\ & =-1-K+(1+K)(2+K)=(K+1)^2 \\ H(k) & =-\chi_p^*(30)-K+(\chi_p^*(18)+\chi_p^*(24)+K)(1+\chi_p^*(30)+K)= \\ & =-K+K(1+K)=K^2. \end{cases}$$

In the above steps we use the fact $\chi_p^*(3)+\chi_p^*(4)=1,\ \chi_p^*(5)=1$ if $p\in\{2,3\}$. Since $H(m)=\chi_p^*(m)\in\{0,1\}$, we have $H(k)=K^2\neq(K+1)^2=H(k+1)$, consequently

$$\Big(H(k),H(k+1)\Big)=\Big(K^2,(K+1)^2\Big)\in\Big\{(1,0),(0,1)\Big\}.$$

These imply that either

$$k\equiv-1 \pmod p \quad \text{and} \quad K=-1$$

or

$$k\equiv 0 \pmod p \quad \text{and} \quad K=0.$$

We distinguish the proof for four cases according to $p\in\{2,3\}$.

The case I: $p=2,\ K=-1,\ k\equiv 1 \pmod 2,\ H(m)=\chi_2^*(m)$.

We shall prove that this case does not occur.

In this case, we have

$$F(n^2+m^2+k)=\chi_2^*(n)+\chi_2^*(m)-1 \quad \text{for every} \quad n,m\in\mathbb{N}.$$

It is clear to check that

$$\chi_2^*(n)+\chi_2^*(m)=\overline{n^2+m^2} \quad \text{for every} \quad n,m\in\mathbb{N},$$

consequently

$$F(\eta+k)=\overline{\eta}-1 \quad \text{for every} \quad \eta\in\mathcal{B}, \tag{2.4}$$

where $\overline{\eta}\in\{0,1,2,3\}$ such that $\eta\equiv\overline{\eta} \pmod 4$.

Since $k\in\mathcal{E}$ and $k\equiv 1 \pmod 2$, we have $k\eta\in\mathcal{B}$ for every $\eta\in\mathcal{B}$, and so it follows from (2.4) that

$$F(k)F(\eta+1)=F(k\eta+k)=\overline{k\eta}-1=\overline{\eta}-1 \quad \text{for every} \quad \eta\in\mathcal{B}. \tag{2.5}$$

In the last relation we use $k\equiv 1 \pmod 4$, because $k\equiv 1 \pmod 2$ and $k\in\mathcal{E}$. Since $2=1^2+1^2\in\mathcal{B}$ and $8=2^2+2^2\in\mathcal{B}$, we obtain from (2.5) that

$$F(k)F(3)=\overline{2}-1=1 \quad \text{and} \quad F(k)F(3)^2=F(k)F(8+1)=\overline{8}-1=-1.$$

These imply that

$$F(3)=F(k)=-1 \tag{2.6}$$

and

$$F(\eta+1)=1-\overline{\eta} \quad \text{for every} \quad \eta\in\mathcal{B}. \tag{2.7}$$

In the next part we deduce from (2.7) that

$$F(n)=\chi_4(n) \quad \text{for every} \quad n\in\mathbb{N}, \tag{2.8}$$

where $\chi_4(n)$ denotes the non-principal Dirichlet character $\pmod 4$. This relation with $k\equiv 1 \pmod 4$ implies that $F(k)=1$, which contradicts to (2.6).

Since $F(3)=-1$, it follows from (2.7) that

$$\begin{aligned}
&F(2)=-F(2)F(3)=-F(6)=-F(1^2+2^2+1)=-(1-\overline{5})=0,\\
&F(7)=-F(3)F(7)=-F(21)=-F(2^2+4^2+1)=-(1-\overline{20})=-1,\\
&F(5)=-F(5)F(7)=-F(35)=-F(3^2+5^2+1)=-(1-\overline{34})=1.
\end{aligned}$$

Thus we have proved that $F(n) = \chi_4(n)$ for $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Now assume that $F(n) = \chi_4(n)$ for all $n < P$, where $P \geq 11$. We will prove that $F(P) = \chi_4(P)$. Since $F \in \mathcal{M}^*$, we may assume that $P = p \in \mathcal{P}$. One can check that there are $n, m \in \mathbb{N}$ such that

$$n^2 + m^2 + 1 = pQ, \text{ where } \ Q < p, \ (Q, 2) = 1.$$

Then $n \equiv m \pmod 2$, $\chi_4(pQ) = (-1)^n$ and

$$F(n^2 + m^2 + 1) = 1 - \overline{n^2 + m^2} = (-1)^n = \chi_4(pQ),$$

consequently we infer from our assumptions and from (2.7) that

$$F(p) = \frac{F(pQ)}{F(Q)} = \frac{F(n^2 + m^2 + 1)}{F(Q)} = \frac{\chi_4(pQ)}{\chi_4(Q)} = \chi_4(p).$$

This relation shows that $F(n) = \chi_4(n)$ for all $n \in \mathbb{N}$

Thus the case I does not occur.

The case II: $p = 3, \ K = -1, \ k \equiv 2 \pmod 3, \ H(m) = \chi_3^*(m)$.

In this case, we have

$$F(n^2 + m^2 + k) = \chi_3^*(n) + \chi_3^*(m) - 1 \quad \text{for every} \quad n, m \in \mathbb{N}.$$

It is clear to check that

$$\chi_3^*(n) + \chi_3^*(m) - 1 = \chi_3(n^2 + m^2 + 2) \quad \text{for every} \quad n, m \in \mathbb{N},$$

consequently

$$F(\eta + k) = \chi_3(\eta + 2) = \chi_3(\eta + k) \quad \text{for every} \quad \eta \in \mathcal{B}, \tag{2.9}$$

where $\chi_3(n)$ is the non-principal character $\pmod 3$, i. e. $\chi_3(0) = 0, \chi_3(1) = 1, \chi_3(2) = -1$.

Let us note that for $\alpha \in \mathcal{B}$ and $\eta \in \mathcal{E}$ the condition $\alpha\eta \notin \mathcal{B}$ may hold only in the case $\alpha = 2u^2$ and $\eta = 2v^2$. Thus, we have

$$\eta(n^2 + m^2) \in \mathcal{B} \quad \text{for every} \quad \eta \in \mathcal{E} \quad \text{and} \quad n, m \in \mathbb{N}, n \neq m. \tag{2.10}$$

Consequently, we have $13k = (2^2 + 3^2)k \in \mathcal{B}$, $25k = (4^2 + 5^2)k \in \mathcal{B}$, $90k = (3^2 + 9^2)k \in \mathcal{B}$ and $97k = (4^2 + 9^2)k \in \mathcal{B}$. Thus, we infer from (2.9) that

$$\begin{aligned} F(k)F(2)F(7) = F(k)F(14) = F(13k + k) = \chi_3(13k + k) = \\ = \chi_3(k)\chi_3(14) = \chi_3(2)\chi_3(2)\chi_3(7) = 1, \end{aligned}$$

$$F(k)F(2)F(13)=F(k)F(26)=F(25k+k)=\chi_3(25k+k)=$$
$$=\chi_3(k)\chi_3(26)=\chi_3(2)^2\chi_3(13)=1,$$

$$F(k)F(7)F(13)=F(k)F(91)=F(90k+k)=\chi_3(90k+k)=$$
$$=\chi_3(k)\chi_3(91)=\chi_3(2)\chi_3(1)=-1$$

and

$$F(k)F(2)F(7)^2=F(k)F(98)=F(97k+k)=\chi_3(97k+k)=$$
$$=\chi_3(k)\chi_3(98)=\chi_3(2)\chi_3(2)=1.$$

We infer from these relations that

$$F(7)=1,\ F(2)=-1,\ F(13)=1 \text{ and } F(k)=-1,$$

which with (2.9) and (2.10) implies

(2.11)
$$F(n^2+m^2+1)=-F(k)F(n^2+m^2+1)=-F\Big(k(n^2+m^2)+k\Big)=$$
$$=-\chi_3\Big(k(n^2+m^2)+k\Big)=-\chi_3(k)\chi_3(n^2+m^2+1)=$$
$$=\chi_3(n^2+m^2+1)\quad\text{for every}\quad n,m\in\mathbb{N},n\neq m.$$

Since $F(2)=-1$ and $F(7)=1$, it follows from (2.11) that

$$F(3)=-F(2)F(3)=-F(6)=-F(1^2+2^2+1)=-\chi_3(6)=0$$

and

$$F(5)=F(5)F(7)=F(35)=F(3^2+5^2+1)=\chi_3(35)=-1.$$

Thus we have proved that $F(n)=\chi_3(n)$ for $n\in\{1,2,3,4,5,6,7,8,9,10\}$.

Now assume that $F(n)=\chi_3(n)$ for all $n<P$, where $P\geq 11$. We will prove that $F(P)=\chi_3(P)$. Since $F\in\mathcal{M}^*$, we may assume that $P=p\in\mathcal{P}$. One can check that there are $n,m\in\mathbb{N},n\neq m$ such that

$$n^2+m^2+1=pQ,\ \text{where}\ \ Q<p,\ (Q,3)=1.$$

Then we infer from our assumptions and from (2.11) that

$$F(p)=\frac{F(pQ)}{F(Q)}=\frac{F(n^2+m^2+1)}{F(Q)}=\frac{\chi_3(pQ)}{\chi_3(Q)}=\chi_3(p).$$

This relation shows that $F(n) = \chi_3(n)$ for all $n \in \mathbb{N}$, therefore (T4) is thus proved.

The case III: $p = 2,\ K = 0,\ k \equiv 0 \pmod 2,\ H(m) = \chi_2^*(m)$.

We prove that this case does not occur. Indeed, we have

$$F(8 + k) = F(2^2 + 2^2 + k) = \chi_2^*(2) + \chi_2^*(2) + K = 0,$$

which implies that $F(2) = 0$ or $F(Q) = 0$, where $Q|k + 8, (Q, 2) = 1$. The case $F(2) = 0$ is not occur, because

$$0 = F(2)F\Big(1 + \frac{k}{2}\Big) = F\Big(1^2 + 1^2 + k\Big) = \chi_2^*(1) + \chi_2^*(1) + K = 2.$$

If $F(Q) = 0, (Q, 2) = 1$, then there are $n, m \in \mathbb{N}$ such that

$$n^2 + m^2 + k \equiv 0 \pmod Q \quad \text{and} \quad (n, 2) = (m, 2) = 1.$$

Then

$$0 = F(Q)F\Big(\frac{n^2 + m^2 + k}{Q}\Big) = F\Big(n^2 + m^2 + k\Big) = \chi_2^*(n) + \chi_2^*(m) + K = 2,$$

which is impossible.

The case IV: $p = 3,\ K = 0,\ k \equiv 0 \pmod 3,\ H(m) = \chi_3^*(m)$.

We prove that this case does not occur.

Indeed, we have

$$(k + 3)^2 + 1 + k = (k + 2)(k + 5),$$

which implies that

$$\begin{aligned} 0 =&\chi_3^*(k + 3) + \chi_3^*(1) + K - (\chi_3^*(1) + \chi_3^*(1) + K)(\chi_3^*(1) + \chi_3^*(2) + K) = \\ &= 1 - 2^2 = -3. \end{aligned}$$

This is impossible.

Lemma 5 is proved.

Lemma 6. *Assume that $k \in \mathbb{N}, K \in \mathbb{C}$. Then there isn't a function $F \in \mathcal{M}^*$ such*

$$F(n^2 + m^2 + k) = \chi_5(n) + \chi_5(m) + K \quad \textit{for every} \quad n, m \in \mathbb{N}, \tag{2.12}$$

where $\chi_5(n)$ is the Dirichlet non-principal character (mod 5).

Proof. Assume in the contradiction that there is $F\in\mathcal{M}^*$ such that (2.12) is true.

We infer from the following relations

$$\begin{aligned}(k+25)^2+25+k&=(k+25)(k+26),\\(k+169)^2+169+k&=(k+169)(k+170),\\(k+676)^2+676+k&=(k+676)(k+677)\end{aligned}$$

that

$$\begin{aligned}&F\Big((k+25)^2+25+k\Big)-F(k+25)F(k+26)=\\&=\chi_5(k)+\chi_5(5)+K-(\chi_5(3)+\chi_5(4)+K)(\chi_5(1)+\chi_5(5)+K)=\\&=\chi_5(k)+K-K(1+K)=\chi_5(k)-K^2=0,\end{aligned}$$

$$\begin{aligned}&F\Big((k+169)^2+169+k\Big)-F(k+169)F(k+170)=\\&=\chi_5(k+4)+\chi_5(13)+K-(\chi_5(5)+\chi_5(12)+K)(\chi_5(1)+\chi_5(13)+K)=\\&=\chi_5(k+4)-1+K-(-1+K)K=\chi_5(k+4)-(K-1)^2=0\end{aligned}$$

and

$$\begin{aligned}&F\Big((k+676)^2+676+k\Big)-F(k+676)F(k+677)=\\&=\chi_5(k+1)+\chi_5(26)+K-(\chi_5(10)+\chi_5(24)+K)(\chi_5(1)+\chi_5(26)+K)=\\&=\chi_5(k+1)+1+K-(1+K)(2+K)=\chi_5(k+1)-(K+1)^2=0.\end{aligned}$$

These imply that

$$\Big(\chi_5(k),\chi_5(k+1),\chi_5(k+4)\Big)=\Big(K^2,(K+1)^2,(K-1)^2\Big).$$

Since $\chi_5(n)\in\{1,-1,-1,1,0\}$ for every $n\in\mathbb{N}$, we have

$$\begin{aligned}&\Big(\chi_5(k),\chi_5(k+1),\chi_5(k+4)\Big)=\Big(K^2,(K+1)^2,(K-1)^2\Big)\in\\&\in\Big\{(1,-1,0),(-1,-1,1),(-1,1,-1),(1,0,-1),(0,1,1)\Big\}.\end{aligned}$$

This relation is true in the following case:

$$k\equiv 0 \pmod 5,\ K=0,\ \chi_5(k)=0,\ \chi_5(k+1)=\chi_5(k+4)=1. \tag{2.13}$$

Since

$$(k+10)^2+4+k=(k+13)(k+8),$$

and so from (2.12) and (2.13) we have

$$\begin{aligned}0 &= F\Big((k+10)^2+2^2+k\Big) - F(2^2+3^2+k)F(2^2+2^2+k) = \\ &= \Big(\chi_5(k+10)+\chi_5(2)+K\Big) - (\chi_5(2)+\chi_5(3)+K)(2\chi_5(2)+K) = \\ &= -1-(-2)^2 = -5,\end{aligned}$$

which is impossible.

Lemma 6 is proved.

3. Proof of Theorem 1.

It is easy to check that the functions defined in (T1), (T2), (T3) and (T4) satisfy the functional equation (1.1). Now we prove the "only if " part.

Assume that $k \in \mathcal{E}$, $K \in \mathbb{C}$ and $F, H \in \mathcal{M}^*$ satisfy (1.1). Since $H \in \mathcal{M}^*$, using (2.1), we have

$$\begin{aligned}0 &= H(nm) - H(n)H(m) = \\ &= A(nm)^2 + \mathcal{S}(nm) - \Big((An^2+\mathcal{S}(n))(Am^2+\mathcal{S}(m))\Big) = \\ &= (A-A^2)n^2m^2 - A\mathcal{S}(m)n^2 - A\mathcal{S}(n)m^2 + \mathcal{S}(nm) - \mathcal{S}(n)\mathcal{S}(m)\end{aligned}$$

holds for every $n, m \in \mathbb{N}$. Since $\mathcal{S}(n)$ is an bounded function, the above equation shows that

$$(3.1) \qquad \begin{cases} A^2 &= A \\ A\mathcal{S}(n) &= 0 \quad \text{for every} \quad n \in \mathbb{N} \\ \mathcal{S}(nm) &= \mathcal{S}(n)\mathcal{S}(m) \quad \text{for every} \quad n, m \in \mathbb{N}, (n,m) = 1. \end{cases}$$

The first equation implies $A \in \{0, 1\}$.

a) Assume that $A = 0$. Then (2.1) implies that $H(n) = \mathcal{S}(n)$. It follows from $A = 0$ and from the definitions of A, we have

$$H(5) = \frac{1}{4}(-H(2)H(3) + H(3) + H(2) + 3).$$

Since $H \in \mathcal{M}^*$, we have $H(1) = 1$, $H(4) = H(2)^2$, $H(6) = H(2)H(3)$, $H(8) = H(2)^3$, $H(9) = H(3)^2$, $H(10) = H(2)H(5)$ and $H(12) = H(2)^2H(3)$. It can check from Lemma 1 that

$$(3.2)\qquad \begin{cases} H(7) &= \frac{1}{2}(-H(2)H(3)+H(3)+H(2)+1),\\ H(8) &= \frac{1}{2}(-H(2)H(3)+H(3)+H(2)-1+2H(2)^2),\\ H(9) &= \frac{1}{2}(H(2)H(3)+H(3)-H(2)+1),\\ H(10) &= \frac{1}{4}(H(2)H(3)-H(3)+3H(2)+1),\\ H(11) &= 1,\\ H(12) &= H(3)-1+H(2)^2. \end{cases}$$

Furthermore, we infer from Lemma 2 that

$$(3.3)\qquad \begin{cases} \Gamma_2 &= \frac{1}{4}(-H(2)H(3)+H(3)-H(2)+3-2H(2)^2),\\ \Gamma_3 &= \frac{1}{2}(-H(2)H(3)-H(3)+H(2)+1),\\ \Gamma_4 &= \frac{1}{4}(H(2)H(3)-2H(2)^2-H(3)+H(2)+1),\\ \Gamma_5 &= \frac{1}{4}(H(2)H(3)-H(3)-H(2)+1),\\ \Gamma &= \frac{1}{2}(H(2)H(3)+H(3)-1+H(2)^2) \end{cases}$$

and

$$(3.4)\qquad H(\ell) = \mathcal{S}(\ell) := \Gamma_2\chi_2^*(\ell) + \Gamma_3\chi_3^*(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell) + \Gamma.$$

These imply that

$$\begin{cases} H(8)-H(2)^3 &= -\frac{1}{2}(H(2)-1))(2H(2)^2+H(3)-1) = 0\\ H(9)-H(3)^2 &= \frac{1}{2}(H(3)-1)(-2H(3)+H(2)-1) = 0\\ H(10)-H(2)F(5) &= \frac{1}{4}(H(2)-1)(H(2)+1)(H(3)-1) = 0. \end{cases}$$

This system has four solutions

$$\Big(H(2), H(3)\Big) \in \Big\{(1,1),(0,1),(1,0),(-1,-1)\Big\}$$

In order to prove Theorem 1, we distinguish the proof for four cases.

Case (I): Assume that $H(2)=1,\ H(3)=1$. Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0 \ \text{ and } \ \Gamma = 1,$$

therefore it follows from (3.4) that $H(m) = 1 \quad$ for every $\ m \in \mathbb{N}$.

Thus, Lemma 4 implies the proof of (T2) and (T3) of Theorem 1.

Case (II): Assume that $H(2) = 0, H(3) = 1$. Then we infer from (3.2) and (3.3) that

$$A = \Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma = 0 \ \text{ and } \ \Gamma_2 = 1,$$

therefore it follows from (3.4) that $H(m) = \chi_2^*(m) \quad \text{for every} \quad m \in \mathbb{N}$.

Lemma 5 implies that this case does not occur.

Case (III): Assume that $H(2) = 1, H(3) = 0$. Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_4 = \Gamma_5 = \Gamma = 0 \ \text{ and } \ \Gamma_3 = 1,$$

therefore it follows from (3.4) that $H(m) = \chi_3^*(m) \quad \text{for every} \quad m \in \mathbb{N}$.

Thus, Lemma 5 implies the proof of (T4) of Theorem 1.

Case (IV): Assume that $H(2) = -1, H(3) = -1$. Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma = 0 \ \text{ and } \ \Gamma_5 = 1,$$

therefore it follows from (3.4) that $H(m) = \chi_5(m) \quad \text{for every} \quad m \in \mathbb{N}$

Thus, Lemma 6 implies that this case does not occur.

b) Assume now that $A = 1$. Then (3.1) implies that $\mathcal{S}(n) = 0$ for every $n \in \mathbb{N}$ and so $H(m) = m^2 \quad \text{for every} \quad m \in \mathbb{N}$. We obtain from (1.1) that

$$F(n^2 + m^2 + k) = n^2 + m^2 + K \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{3.5}$$

We shall prove that $k = K$. Since

$$(k+25)^2 + 25 + k = (k+25)(k+26) \ \text{ and } \ 25 = 5^2 = 3^2 + 4^3, 26 = 1^2 + 5^2,$$

we infer from (3.5), using the fact $F \in \mathcal{M}^*$ that

$$\begin{aligned} 0 &= F\Big((k+25)^2 + 5^2 + k\Big) - F(k+25)F(k+26) = \\ &= (k+25)^2 + 5^2 + K - (25+K)(26+K) = -(K+50+k)(K-k). \end{aligned} \tag{3.6}$$

We also have

$$(k+100)^2 + 100 + k = (k+100)(k+101) \ \text{ and } \ 100 = 10^2 = 6^2 + 8^2, 101 = 1^2 + 10^2,$$

which with (3.5) implies

$$
\begin{aligned}
0 &= F\Big((k+100)^2+100+k\Big) - F\Big((k+100)(k+101)\Big) = \\
&= (k+100)^2+100+K-(100+K)(101+K) = \qquad (3.7)\\
&= -(K+200+k)(K-k).
\end{aligned}
$$

It is obvious from (3.6) and (3.7) that $k=K$.

Now let

$$G(n) := \frac{F(n)}{n},\ G\in\mathcal{M}^* \quad \text{for every} \quad n\in\mathbb{N}.$$

Then we infer from (3.5) and $k=K$ that $G(n^2+m^2+1)=1$ for every $n,m\in\mathbb{N}$, therefore Lemma 3 implies that $G(n)=1$ and $F(n)=n$ for every $n\in\mathbb{N}$. The proof of (T1) is finished.

Theorem 1 is proved.

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