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A survey on solution approaches for the equilibrium problem defined by the Nikaido-Isoda-Fan inequality

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Abstract. We provide a brief survey on basic solution approaches for solving the equilibirum problem defined by the Nikaido-Isoda-Fan inequality. Namely, first we state the problem and consider its most important special cases including the optimization, inverse optimization, Kakutani fixed point, variational inequality, Nash equilibrium problems. Next, we present some basic solution approaches for the problem. Finally, as an application, we consider the famous Cournot-Nash oligopolistic equilibrium model and discuss algorithms for solving it.

1. Introduction

Thoughout the paper let $\mathbb{H}$ be a real Hilbert space. In what follows we mainly work on the weak topology of $\mathbb{H}$. Let $C \subseteq \mathbb{H}$ be a closed convex set and $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$. We suppose that $f(x,y) \in \mathbb{R}$ for every $x, y \in C$. As usual, we call f an equilibirum bifunction if $f(x,x) = 0$ for every $x \in C$. The problem to be considered in this paper is formulated as follows.

$$(EP) \qquad \text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C.$$

The inequality appeared in problem (EP) was first used by Nikaido and Isoda in 1955 [37] in a non-cooperative convex game. In the seminal paper [19] in 1972, Fan called

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problem (EP) a minimax inequality and established solution existence results for it when C is convex, compact and f is quasiconvex on C. To our best knowledge, up to now there does not exist an algorithm for finding a solution of the problem considered in [19]. This result by Fan was extended by Brezis, Nirenberg, and Stampachia in [11]. In 1984, Muu [29] called (EP) a variational inequality and studied its some stability properties. In 1992, Muu and Oettli [32] called problem (EP) an equilibrium one, and a penalty algorithm was proposed for finding a solution of (EP) when f possesses certain monotonicity properties. After the appearance of the paper [10] by Blum and Oettli in 1994, problem (EP) attracted much attention of many authors, see e.g. the interesting monographs by Bigi et al. [8], mainly for solution technique issues in Hilbert spaces, and by Kassay et al. [24], mainly for theoretical aspects in vector topological spaces.

It worth mentioning that when $f(x,\cdot)$ is convex and subdifferentiable on C, the equilibrium problem (EP) can be reformulated as the following multivalued variational inequality.

$$(MultiVI) \qquad \text{Find } x^* \in C, v^* \in F(x^*) \text{ such that } \langle v^*, x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

where $F(x^*) = \partial_2 f(x^*,x^*)$ with $\partial_2 f(x^*,x^*)$ being the diagonal subdifferential of f at x^*, that is the subdifferential of the convex function $f(x^*,\cdot)$ at x^*. In the case $f(x,\cdot)$ is semi-strictly quasiconvex rather than convex, problem (EP) can take the form of $(MultiVI)$ with $F(x) := Na_{f(x,x)} \setminus \{0\}$, where $Na_{f(x,x)}$ is the normal cone of the adjusted sublevel set of the function $f(x,\cdot)$ at the level $f(x,x)$, see [5]. More details about the links between equilibrium problems and variational inequalities can be found in [6].

In this paper we provide a brief survey on solution approaches for problem (EP) in real Hilbert spaces. Namely, in the next section we present some important special cases of problem (EP) such as optimization, reverse optimization, multivalued variational inequality, the Kakutani fixed point, the Walras and Nash equilibrium problems. The third section is devoted to the discussion of some basic solution approaches for problem (EP) involving bifunctions having certain monotonicity properties. We close the paper by showing a formulation of the Cournot-Nash oligopolistic equilibrium model in the form of equilibrium problem (EP) and discuss algorithms for solving the model for the convex and quasiconcave cost functions.

2. Special cases

Although the formulation of problem (EP) is very simple, it contains a lot number of important problems as special cases. In [32] it has been shown that the opimization, Kakutani fixed point and multivalued variational inequality problems can be formulated in the form of (EP). In [10] the Nash equilibrium problem has been formulated equivalently as a problem of the form (EP). Some other problems such as inverse optimization, vector optimization have been converted equivalently into problems of the form (EP), see e.g. [8]. For more detail, below we present the form (EP) for the Kakutani fixed point, multivalued variational inequality, Nash equilibrium and inverse optimization problems. The other ones can be found in e.g. [8, 10, 24].

- *Optimization.* Consider the minimization problem

$$(OP) \qquad \min_{x \in C} g(x)$$

in which $g : \mathbb{H} \to \mathbb{R}$ is a single-valued function and $C \subset \mathbb{R}$ is a closed convex set. Let $f(x,y) := g(y) - g(x)$ and consider the equilibrium problem

$$(EP_1) \qquad \text{Find } x^* \in C \text{ such that } f(x^*,y) \geq 0 \text{ for all } y \in C.$$

It is clear that $x^* \in C$ a global minimizer of (OP) if and only if

$$g(x^*) \leq g(y) \quad \forall y \in C,$$

or equivalently,

$$f(x^*,y) = g(y) - g(x^*) \geq 0 \quad \forall y \in C,$$

i.e., x^* is a solution to (EP_1). It means that the minimization problem (OP) is equivalent to the equilibrium problem (EP_1) in the sense that their solution sets coincide.

- *Kakutani fixed point*. Let $C \subset \mathbb{R}^n$ be a compact convex set, $F : C \rightrightarrows C$ an upper semicontinuous multi-valued mapping with convex, compact values. The Kakutani fixed point problem asks:

$$(KP) \qquad \text{Find } x^* \in C \text{ such that } x^* \in F(x^*).$$

The Kakutani fixed point theorem, which is one of famous fixed point ones, states that such a point x^* exists. In a special case, when F is single valued, this theorem becomes the Brouwer theorem that was proved in 1910. Up to now there does not exist an efficient algorithm for finding a fixed point for the Brouwer mapping. In 1967 Scarf, an economist, first developed an algorithm for finding a Brouwer fixed point in $\mathbb{R}^n$ (see [43]), but many computational experiments show that this algorithm and its modifications can only solve the problem with moderate dimension n. In order to formulate the Kakutani fixed point problem in the form of problem (EP), we define the bifunction $f : C \times C \to \mathbb{R}$ by taking

$$f_2(x,y) := \max_{u \in F(x)} \langle x - u, y - x \rangle$$

for each $x, y \in C$. It is shown in [32] that a point x^* solves the problem (KP), i.e. $x^* \in F(x^*)$, if and only if it is a solution to the following equilibrium problem

$$(EP_2) \qquad \text{Find } x^* \in C \text{ such that } f_2(x^*, y) \geq 0 \text{ for all } y \in C.$$

In other words, the Kakutani fixed point problem (KP) is equivalent to the equilibrium problem (EP_2) in the sense that their solution sets coincide.

- *Variational inequality problem.* Let $F : C \rightrightarrows \mathbb{H}$ be a (multivalued) operator with convex, (weakly) compact values and $\varphi : \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(x)$ is finite on C. The mixed variational inequality problem stated in [18] is formulated as $(MixedVI)$
  Find $x^* \in C$ such that $\exists u^* \in F(x^*) : \langle u^*, y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \; \forall y \in C$.

  Clearly, when F is single valued, this problem is reduced to the following one:

  $$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \; \forall y \in C.$$

  It is worth noting that, when C is a convex cone and φ is a constant, problem $(MixedVI)$ becomes the following complementarity one.

  $$(MC) \qquad \text{Find } x^* \in C \text{ such that } \exists u^* \in F(x^*) : \; \langle u^*, x^* \rangle = 0.$$

  Let
  $$f_3(x, y) := \max_{u \in F(x)} \langle u, y - x \rangle + \varphi(y) - \varphi(x)$$
  and consider the following equilibrium problem

  $$(EP_3) \qquad \text{Find } x^* \in C \text{ such that } f_3(x^*, y) \geq 0 \text{ for all } y \in C.$$

  It was proved in [32] that a point x^* is a solution of problem $(MixedVI)$ if and only if it is also a solution to (EP_3). Therefore, $(MixedVI)$ is equivalent to the equilibrium problem (EP_3) in the sense that they share the same solution set.

- *Nash equilibria*. In a noncooperative game with N players, each player i has a set of possible strategies $C_i \subseteq \mathbb{R}^{n_i}$ and aims at minimizing a cost function $g_i : C \to \mathbb{R}$ with $C := C_1 \times \ldots \times C_N$. By definition, a *Nash equilibrium point* is any point in C such that no player can reduce her/his cost by unilaterally changing her/his strategy. The Nash equilibrium problem is to find such a Nash equilibrium point, i.e. a point $x^* \in C$ such that

  $$g_i(x^*) \leq g_i(x^*[y_i]) \quad \forall y_i \in C_i, i = 1, \ldots, N,$$

  where $x^*[y_i]$ stands for the vector obtained from x^* by replacing the component x_i^* with y_i. If we take $f_4 : C \times C \to \mathbb{R}$ defined as

  $$f_4(x, y) := \sum_{i=1}^{N} \Big[g_i(x[y_i]) - g_i(x) \Big],$$

and consider the following equilibrium problem

$$(EP_4) \qquad \text{Find } x^* \in C \text{ such that } f_4(x^*, y) \geq 0 \text{ for all } y \in C,$$

then it is not hard to see that x^* is a solution to the Nash equilibrium problem if and only if it is a solution to (EP_4).

- *An inverse optimization*. Let $C_1 \subseteq \mathbb{R}^n$ and $C_2, C_3 \subseteq \mathbb{R}^m$ be convex sets and $g_j : C_2 \to \mathbb{R} (j = 1, \ldots, m)$. Let

$$h_2(p,q) := \sum_{j=1}^{m} p_j g_j(q).$$

The inverse problem reads as
$(InvP)$
Find $p^* = (p_1^*, \ldots, p_n^*)^T \in C_1$ such that $\arg\min\{h_2(p^*, q) \mid q \in C_2\} \cap C_3 \neq \emptyset$.

In some economics models p^* plays the role of a price that is required to be found such that the latter inclusion is satisfied. Clearly, this inverse problem can be formulated as a noncooperative game with three players. The first player controls p by choosing a point $p^* \in C_1$, the second one solves the problem $\min_{q \in C_2} f_2(p^*, q)$, while the third player controls her/his strategy in C_3. Of course one can extend this model by assuming that the first and third players have more general lost functions, say, $h_1(p,q,r)$ and $h_3(p,q,r)$. Following the equivalent of the Nash equilibrium problem in noncooperative game with (EP_4) as discussed above, the inverse optimization problem $(InvP)$ in turn can take the form of an equilibrium problem (EP).

3. Solution approaches

3.1. Basic solution existence

Under the condition $f(x,x) = 0$ for every $x \in C$, it follows immediately that x^* is a solution to problem (EP) if and only if $x^* \in \text{argmin}\{f(x^*, y) \mid y \in C\}$, i.e. x^* is a fixed point of the mapping $S(\cdot)$ with $S(x) = \text{argmin}\{f(x,y) \mid y \in C\}$. The first result for solution existence of the equilibrium problem (EP) is due to Fan [19] in 1972. There, Fan called (EP) a minimax inequality and established the following theorem. His proof was based upon the KKM Lemma (a variant of a fixed point theorem).

Theorem 3.1. *(see [19]). Let C be a compact, convex set in a Hausdorff topological vector space. Let $f : C \times C \to \mathbb{R}$ be a continuous bifunction such that for every $x \in C$*

we have $f(x,x) = 0$ *and* $f(x,\cdot)$ *is quasiconvex on* C*. Then, the equilibrium problem* (EP) *is solvable, i.e., there exists* $x^* \in C$ *such that* $f(x^*,y) \geq 0$ *for every* $y \in C$*.*

To our best knowledge, up to now there does not exist an efficient algorithm for approximating the solution mentioned in this theorem.

3.2. Some solution approaches

A key assumption for equilibrium problem (EP), that we assume in what follows, is that the bifunction f is convex with respect to its second variable on the feasible convex set C, i.e., $f(x,\cdot)$ is convex on C for any fixed $x \in C$. Under this main assumption, we have the following auxiliary problem principle.

3.2.1. Auxiliary problem principle and fixed point

The auxiliary problem principle first was introduced by Cohen [12] for the optimization and variational inequality problems and extended to the equilibrium problem [28].

Theorem 3.2. *(Auxiliary problem principle). Suppose that* $f(x,\cdot)$ *is subdifferentiable on* C *for every* $x \in C$*. Then a point* x^* *is a solution of problem* (EP) *if and only if it is also a solution to the following regularized equilibrium one*

$$(REP) \qquad \text{Find } x^* \in C \text{ such that } f_\rho(x^*,y) := f(x^*,y) + \frac{1}{2\rho}\|y - x^*\|^2 \geq 0 \ \forall y \in C,$$

where $\rho > 0$*.*

A main advantage of the regularized problem is that the bifunction $f_\rho(x,\cdot)$ is strongly convex on C, which implies that the mathematical program $\min\{f_\rho(x,y) \mid y \in C\}$ always admits a unique solution. Thus x^* is a solution of (EP) if and only if $x^* = s(x^*)$, where $s(x^*)$ is the unique solution of the strongly convex mathematical programming problem $\min\{f_\rho(x^*,y) \mid y \in C\}$, that means x^* is a fixed point of $s(\cdot)$.

It is worth noting that, under some continuity property of the bifunction f, the solution-map $s : C \to C$ is continuous, and therefore, by the Brouwer fixed point theorem, it has a fixed point whenever C is compact. In order to find a fixed point of this mapping, one needs additional assumptions to ensure that the mapping has a certain Lipschitz property such as contractive or nonexpansive. Under the properties, one can derive iterative scheme for approximating a fixed point of the map s.

For this purpose the following monotonicity concepts for a bifunction are commonly used [10], see also [7] Section 20.

Definition 3.1. *Let* $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ *and* $D \subseteq \mathbb{H}$ *such that* f *is finite on* D*. The bifunction* f *is said to be* *(i) strongly monotone on D with modulus $\mu > 0$ (shortly μ-strongly monotone) if*

$$f(x,y) + f(y,x) \le -\mu\|x-y\|^2 \quad \forall x,y \in D;$$

(ii) μ-strongly pseudomonotone on C if

$$f(x,y) \ge 0 \quad \Rightarrow \quad f(y,x) \le -\mu\|x-y\|^2;$$

(iii) monotone on D if

$$f(x,y) + f(y,x) \le 0 \quad \forall x,y \in D;$$

(iv) pseudomonotone on D if for all $x,y \in D$ we have

$$f(x,y) \ge 0 \quad \Rightarrow \quad f(y,x) \le 0.$$

More types of monotonicity can be found in e.g. [9].

The monotonicity notions of a bifunction are generalizations of those for a (multi-valued) operator. We recall from [7, 42] that a (multi-valued) operator F with compact values is said to be

(i) strongly monotone on C with modulus $\mu > 0$ (shortly μ-strongly monotone) if

$$\langle u-v, x-y\rangle \ge \mu\|x-y\|^2 \quad \forall x,y \in C, u \in F(x), v \in F(y);$$

(ii) μ-strongly pseudomonotone on C if for all $x,y \in C, u \in F(x), v \in F(y)$ we have

$$\langle u, y-x\rangle \ge 0 \quad \Rightarrow \quad \langle v, y-x\rangle \le \mu\|x-y\|^2;$$

(iii) monotone on C if

$$\langle u-v, x-y\rangle \ge 0 \quad \forall x,y \in C, u \in F(x), v \in F(y);$$

(iv) pseudomonotone on C if for all $x,y \in C, u \in F(x), v \in F(y)$ we have

$$\langle u, y-x\rangle \ge 0 \quad \Rightarrow \quad \langle v, y-x\rangle \ge 0.$$

Clearly the strongly monotonicity implies the monotonicity, which in turn implies the pseudomonotonicity.

The following Lipschitz-type for a bifunction, which is an extension of the Lipschitz property of a map, is often used.

Definition 3.2. *(see [27]) The bifunction f is said to be Lipschitz-type on D with the constants L_1, L_2 if*

$$f(x,y) + f(y,z) \ge f(z,x) - L_1\|x-y\|^2 - L_2\|y-z\|^2 \quad \forall x,y,z \in D.$$

Clearly, in the optimization case, when $f(x,y) = g(y) - g(x)$, then f is monotone and Lipschitz-type for any function g.

For the statement of the next lemma, we need the following definition.

Definition 3.3. *We say that a multi-valued mapping $F : \mathbb{H} \rightrightarrows \mathbb{H}$ is Lipschitz with Hausdorff distance on a closed convex set $C \subset \mathbb{H}$ if there exists a so-called Lipschitz constant $L > 0$ such that*

$$d_H(F(x), F(y)) \leq L\|x - y\| \quad \forall x, y \in C,$$

in which

$$d_H(F(x), F(y)) = \max\{\sup_{u \in F(x)} \inf_{v \in F(y)} \|u - v\|, \sup_{v \in F(y)} \inf_{u \in F(x)} \|u - v\|\}$$

is the Hausdorff distance between two sets $F(x)$ and $F(y)$.

For the multi-valued mixed variational inequality problem $(MixedVI)$, by taking

$$f(x,y) := \max_{u \in F(x)} \langle u, y - x\rangle + \varphi(y) - \varphi(x)$$

we have the following relationships.

Lemma 3.1. *(see [41]) (i) If F is Lipschitz with Hausdorff distance on C with Lipschitz constant L, then f is Lipschitz-type on C with constants $L_1 = \frac{L\xi}{2}, L_2 = \frac{L}{2\xi}$ with any $\xi > 0$.*

(ii) If F is monotone (resp., strongly monotone, pseudomonotone), then f is strongly monotone (resp., monotone, pseudomonotone) on C.

The following problem is called Minty (or dual) problem for (EP).

(DEP) Find $y^* \in C$ such that $f(x, y^*) \leq 0 \; \forall x \in C$.

The following theorem provides a relationship between (EP) and (DEP).

Theorem 3.3. *(see [29]). (i) If f is pseudomonotone on C, then every solution (if exists) of problem (EP) is also a solution of problem (DEP).*

(ii) Conversely, if $f(x,\cdot)$ is lower semicontinuous and for any $y \in C$ the function $f(\cdot, y)$ is hemicontinuous at zero (i.e., for any $x' \in C$ one has $\lim_{t\to 0^+} f(tx + (1-t)x', y) = f(x', y)$ for all $y \in C$), then every solution of problem (DEP) is also a solution of (EP).

Note that, since $f(x,\cdot)$ is convex, the solution set of the Minty problem is a convex set as it is the intersection of a family of convex sets of the type $\{y \in C \mid f(x,y) \leq 0\}$.

3.2.2. Contraction fixed point method

This method is based upon the Banach contraction fixed point theorem. Namely, we have the following theorem.

Theorem 3.4. *(see [33]). Suppose that*

(i) for each $x \in C$, the function $f(x,\cdot)$ is convex, subdifferentiable on C;

(ii) f is μ-strongly monotone and Lipschitz type with constants L_1, L_2 on C.
Then one can choose regularization $\rho > 0$ (depending on μ and the Lipschitz constants L_1, L_2) such that the mapping $s(\cdot): C \to C$ defined by $s(x) = argmin\{f_\rho(x,y) \mid y \in C\}$ is contractive on C. Consequently, for any starting point $x^0 \in C$, the sequence $\{x^k\}$ is defined by $x^{k+1} = s(x^k)$ satisfying

$$\|x^{k+1} - x^*\| \le \alpha \|x^k - x^*\| \quad \forall k \ge 0,$$

provided that $0 < \rho < 1/(2L_2)$ and $\alpha = 1 - 2\rho(\mu - L_1)$, where x^* is the unique solution of problem (EP).

Note that we can replace the regularization function $\|\cdot\|^2$ in (REP) by any strongly differentiable convex one (Bregman function, for instance). This contraction method can be extended to the case f is strongly pseudomonotone in e.g. [16]. Note furthermore that, under the assumption of the above theorem, problem (EP) always admits a unique solution even the feasible set C may not be compact (see [10]). Some other results for solution existence of problem (EP) can be found in e.g. [10] and the monographs [8, 24, 25].

For the variational inequality problem concerning single-valued mapping F:

$$(VI) \qquad \text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \ge 0 \ \forall y \in C,$$

we have $s(x) = P_C(x - \frac{1}{2\rho}F(x))$, where P_C stands for the metric projection onto the closed convex set C. In this case we have $x^{k+1} = P_C(x^k - \frac{1}{2\rho}F(x^k))$. It is well known that if F is merely monotone (not strongly monotone or not strongly pseudomonotone), the sequence of the iterates $\{x^k\}$ may not be convergent. For multivalued monotone variational inequality problems, an algorithm by coupling the Banach iterative scheme and the proximal point method was developed in [2].

3.2.3. Extragradient method

The extragradient method was introduced by Korpelevich [26] for optimization and saddle point problems. Then it has been extended to the equilibrium problem see e.g. [41]. Namely, we have the following results.

Theorem 3.5. *(see [41]). Suppose that the bifunction f is subdifferentiable, pseudomonotone, and Lipschitz-type on C with constants L_1, L_2, while $f(\cdot, y)$ is upper* *semicontinuous for each $y \in C$. The sequence $\{x_k\}$ of iterates defined by*

$$y^k = argmin\{f_\rho(x^k, y) := f(x^k, y) + \frac{1}{2\rho}\|y - x^k\|^2 \mid y \in C\},$$
$$x^{k+1} = argmin\{f_\rho(y^k, y) := f(y^k, y) + \frac{1}{2\rho}\|y - x^k\|^2 \mid y \in C\}$$

converges to a solution of (EP) provided $0 < \rho < \min\{1/(2L_1), 1/(2L_2)\}$.

Note that, as before, we can replace the regularization function $\|\cdot\|^2$ by any Bregman one.

In order to avoid the Lipschitz-type condition, a linesearch extragradient algorithm has been described in [41] and its convergence has been proved. Recently, in [21], an algorithm, where the stepsize is updated at each iteration (without linesearch), for solving pseudomonotone equilibrium problem has been developed.

As we have seen, equilibrium problem (EP) can be formulated equivalently as a fixed point problem. When the bifunction is strongly monotone, the fixed point map is contractive. The following results in [4] show that when f posseses certain monotonicity property, problem (EP) can be formulated as a fixed point problem with the map having certain nonexpansive or generalized nonexpansive property. For this purpose, let us define two mappings, the proximal mapping and the composited mapping. The proximal mapping is denoted by T_ρ and defined as the solution set of the regularized strongly monotone equilibrium problem

$$\text{Find } z \in C \text{ such that } f(z, y) + \frac{1}{2\rho}\langle y - z, z - x\rangle \geq 0 \;\forall y \in C.$$

For this mapping we have the following theorem

Theorem 3.6. *(see [7]) Suppose that*

(i) the solution set $S(EP)$ of problem (EP) is not empty;

(ii) $f(\cdot, y)$ is upper semicontinuous and $f(x, \cdot)$ is lower semicontinuous, convex on C for every $x, y \in C$.
Then for any $\rho > 0$, the mapping T_ρ is defined everywhere, single valued, and firmly nonexpansive, i.e.,

$$\|T_\rho(x) - T_\rho(y)\|^2 \leq \langle T_\rho(x) - T_\rho(y), y - x\rangle \quad \forall x, y \in C.$$

Moreover the solution set of (EP) coincides with the fixed point set of T_ρ.

The composited mapping is defined for each $x \in C$ by taking

$$C_\rho(x) := \text{argmin}\{f(B_\rho(x), y) + \frac{1}{2\rho}\|y - z\|^2 \mid y \in C\},$$

where

$$B_\rho(x) := \text{argmin}\{f(x, y) + \frac{1}{2\rho}\|y - x\|^2\}.$$

Theorem 3.7. *(see [4]). Assume that*

(i) The solution set $S(EP) \neq \emptyset$;

(ii) f is subdifferentiable and satisfies the Lipschitz-type with constants L_1, L_2.

(iii) f is jointly continuous on an open set containing $C \times C$.
Then C_ρ is quasinonexpansive and demiclosed at 0 provided that

$$0 < \rho < \min\{1/(2L_1), 1/(2L_2)\}.$$

A survey on the relationship between the fixed point and the equilibrium problem (EP) can be found in [30].

3.2.4. The proximal and Tikhonov regularization methods

The equilibrium problem (EP), in general, has many solutions, so it is a ill-posed one. The two main regularization methods commonly used to handle ill-posedness are the Tikhonov and proximal ones. The Tikhonov and proximal point regularization methods have been used to various problems in different fields of applied mathematics. These methods have been extended by Moudafi in [28]. The key idea of these methods is of the use of a suitable regularization bifunction to define regularized equilibrium problems depending on regularization parameters, thereby to obtain a trajector that converges to a solution of the original problem whenever the parameter tends to a suitable value.

In a regularization method a sequence of regularized equilibrium problems is defined, at each iteration k, as

$$\text{(3.1)} \quad \text{Find } x^{\rho_k} \in C \text{ such that } f_{\rho_k}(x^{\rho_k}, y) := f(x^{\rho_k}, y) + \frac{1}{2\rho_k} g_k(x^{\rho_k}, y) \geq 0 \ \forall y \in C,$$

where $\rho_k > 0$ (regularization parameter) and g_k is a strongly monotone bifunction.

First we consider the Tikhonov regularization method, where the regularized problem is defined with $\frac{1}{2\rho_k} = c_k$ and $g_k := \langle x - x^g, y - x \rangle$ (does not depend on k) and x^g is a guessed solution. Then the regularized problem takes the form

$$(TREP_k) \quad \text{Find } x^k \in C \text{ such that } f_{c_k}(x^k, y) := f(x^k, y) + c_k \langle x^k - x^g, y - x^k \rangle \geq 0 \ \forall y \in C.$$

We make use the following assumptions.

(A1) $f(\cdot, y)$ is (weakly) upper semicontinuous for each $y \in C$;

(A2) $f(x, \cdot)$ is lower semicontinuous and convex for each $x \in C$.
Then we have the following result.

Theorem 3.8. *(see [23]) Suppose that f is monotone on C. Then problem $(TREP_k)$ is strongly monotone (hence always admits a unique solution x^k) and x^k converges strongly to some x^* with $c_k \searrow 0$ as $k \to +\infty$.*

Unlike the Tikhonov regularization, in the proximal regularization the regularized bifunction at each iteration k depends on the previous iterate, such a bifunction often used is

$$f_k(x,y) := f(x,y) + c_k\langle x - x^{k-1}, y - x\rangle, \text{ where } c_k > 0.$$

For the proximal regularization method we have the following convergence result.

Theorem 3.9. *([23]) Suppose that f is monotone on C and Assumptions (A1), (A2) are satisfied. Then, for each k, the regularized problem*

$$(PREP_k) \quad \textit{Find } x^k \in C \textit{ such that } f_k(x^k,y) := f(x^k,y) + c_k\langle x^k - x^{k-1}, y - x^k\rangle \geq 0 \; \forall y \in C$$

is strongly monotone (hence always admits a unique solution). Furthermore, x^k converges weakly to some x^* as $k \to +\infty$ and $c_k \to c < +\infty$.

In the case f is pseudomonotone, but not monotone, since the sum of a pseudomonotone and a strongly monotone bifunctions may not be monotone, even not pseudomonotone, the regularized problems for both Tikhonov and proximal regularization methods may have many solutions. However, any trajector converges to the same solution as shown in the following theorem.

Theorem 3.10. *(see [22]). Suppose that f is pseudomonotone on C and satisfies Assumptions (A1), (A2). Suppose furthermore that the solution sets of the original problem (EP) and each regularized problem $(TREP_k)$ are nonempty. Let x^k be any solution of the regularized problem $(TREP_k)$. Then the sequence $\{x^k\}$ converges strongly to the unique solution of the strongly monotone equilibrium problem*

$$(BEP) \qquad \textit{Find } x \in S \textit{ such that } g(x,y) \geq 0 \; \forall y \in S$$

that is nearest to x^g, where S denotes the solution set of the original problem (EP) and $g(x,y) := \langle x - x^g, y - x\rangle$.

This result allows the bilevel level methods can be applied to the regularized pseudomonotone problem (EP) by solving the strongly monotone equilibrium problem (BEP). Since S is closed convex and g is strongly monotone, problem (BEP) always admits a unique solution. An algorithm for solving (BEP) was developed in [14].

For the proximal regularization method, we have a similar result, namely as follows.

Theorem 3.11. *(see [22]). Under the assumptions of Theorem 3.10, the sequence $\{x^k\}$ with x^k being any solution of the regularized problem $(PREP_k)$ converges weakly to a solution of problem (EP) provided $0 < c_k \to c < +\infty$.*

3.2.5. The gap function method

An important solution approach to equilibrium problem is based upon formulations of it in the form of a mathematical programming problem by using a gap function. We recall that $g : C \to \mathbb{R}$ is called a *gap function* for problem (EP) if $g(x) \geq 0$ for every $x \in C$ and $g(x) = 0$ if and only if x solves (EP). The first gap function is called the Auslender gap function which is defined as $g(x) := -\min\{f(x,y) \mid y \in C\}$. Clearly it is a gap function thanks to the condition $g(x,x) = 0$ for every $x \in C$. The main disadvantage of this gap function is that the problem defining it may not be solvable, and if yes, its solutions may not be unique. In order to overcome this disadvantage, Fukushima in [20] defined the following gap function that is called regularization gap function by taking for $x \in C, \rho > 0$, the function

$$g(x) := -\min\{f(x,y) + \frac{1}{2\rho}\|y-x\|^2 \mid y \in C\}.$$

Since the objective function of this optimization problem is strongly convex, it is always uniquely solvable. It is easy to see that it indeed is a gap function for (EP). The gap functions allow that methods of mathematical programming could be applied to solve equilibrium problems. However, since the gap function is not convex in general, finding its global minimum is a difficult task. Algorithms using a gap function for Minty equilibrium problem were proposed in [39, 40]. Other algorithms for Minty equilibrium problem can be found e.g. [15]. Some algorithms by coupling the extragradient method with the bundle, inertial (ball heavy), interior, ergodic and splitting techniques have been proposed in [1, 3, 17, 36, 38, 44] for solving pseudomonotone problem (EP).

4. Cournot-Nash oligopolistic equilibrium model

An important model for the Nash equilibria in economics is the Cournot-Nash oligopolistic model. This model was first introduced in [13] in 1838 by Cournot, a French economist, then it has been extended by using the famous Nash equilibrium concept.

An oligopolistic market model concerns with n firms (producers) that produce a common homogeneous commodity. Each firm has a profit function which is the difference between the income defined by the price and the cost. Each firm attempts to maximize its profit by choosing the corresponding production level on its strategy set.

In the classical model the price for all firms and the cost for each firm are given respectively as

$$p(x) := \alpha - \beta \sum_{j=1}^{n} x_j \text{ and } c_j(x_j) = \xi_j x_j + \eta_j,$$

where $\alpha > 0$ (in general is large) and $\beta > 0$ (often is small), $\xi_j > 0, \eta_j > 0$). So the price depends on the sum of the commodity, while the cost for each firm depends only on the amount of the commodity that it produces. Then the profit of each firm j is

$$f_j(x) := p(x)x_j - c_j(x_j) \quad (j = 1, \ldots, n). \tag{4.1}$$

Actually, each firm seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept. A point (strategy) $x^* = (x_1^*, \ldots, x_n^*)^T \in C$ is said to be a Nash equilibrium point of this Cournot-Nash oligopolistic market model if

$$f_j(x^*) \geq f_j(x^*[x_j]) \quad \forall j = 1, \ldots, n, \ \forall x_j \in C_j. \tag{4.2}$$

It has been shown [31] that, mathematically, the problem of finding a Nash equilibrium strategy for an oligopolistic market model with the profit function of each firm being given by (4.1) can be formulated in the following mixed equilibrium problem.

(MEP) Find $x^* \in C$ such that $f(x,y) := \langle B_1 x - \alpha, y - x\rangle + \beta \left[\sum_{j=1}^{n} y_j^2 - x_j^2 \right] \geq 0 \ \forall y \in C,$

where B_1 is the $(n \times n)$-matrix whose every diagonal entry is zero and the others are all β. It is well known (see [31]) that for this classical model, problem (MEP) can be formulated equivalently in the form of a strongly convex quadratic program. In the case there is a convex cost function rather than all are affine, the model can be formulated as a monotone equilibrium problem (see e.g. [33]). In practice, since the cost for producing a unit commodity does decrease as the amount of the commodity gets larger, the cost is a concave increasingly function, the model then can be formulated as the mixed equilibrium problem

$(MEP1)$

Find $x^* \in C$ such that $f(x,y) := \langle B_1 x - \alpha, y - x\rangle + \beta \left[\sum_{j=1}^{n} c_j(y_j) - c_j(x_j) \right] \geq 0 \ \forall y \in C.$

When c_j is concave even for only one j, the function $f(x,\cdot)$, in general, is neither convex nor quasiconvex, and therefore a local equilibrium point may not be a global one. An algorithm for finding a stationary point of this nonconvex equilibrium problem was developed in [34], whereas a branch-and-bound algorithm using global optimization techniques for approximating the equilibrium problem $(MEP1)$ was proposed in [35]. Recently in [45] an algorithm for solving problem $(MEP1)$ with the bifunction f is quasiconvex and pseudo-paramonotone.

5. Conclusion

We have outlined some basic solution methods for solving equilibrium problems under the two main assumptions that the bifunction involved possesses certain monotonicity property and is convex in its second variable. Namely, we have shown how to formulate the problem in the form of a fixed point one that satisfies a suitable contraction property or its generalized nonexpansiveness. We have also presented the auxiliary problem principle, the regularization techniques as well as extragradient and gap function methods. Unfortunately, these solution methods may fail to apply directly to the problems, where the bifunction involved is quasiconvex rather than convex. In our opinion, research on efficient algorithms for finding a solution of the equilibrium problem whose solution existence has been proved by Ky Fan would be very interesting. Further research for the following subjects might be of interest.

- Development of new more efficient algorithms for problem (EP);
- Solution algorithms for convex split feasibility problem of finding $x^* \in C, F(x^*) \in Q$ with C and/or Q being given implicitly as the solution sets of certain equilibrium problems;
- Extensions of the above mentioned methods to vector and set-valued equilibrium problems;
- Applications of problem (EP) to study models in game theory and in optimal control;
- Solution algorithms for practical equilibrium models encounted in economics, environments, and other fields by using the form of problem (EP);
- Solution methods for equilibrium problems where the bifunction is monotone (not paramonotone) and quasiconvex with respect to its second variable.

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