A generalization of product inequality for the higher topological complexity

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(Received Oct. 22, 2022)

Abstract. In [4] M. Farber defined the topological complexity $TC(X)$ of a path-connected space X. Generalizing this notion, ten years later, Yu. Rudyak introduced a sequence of invariants, called the higher topological complexities $TC_n(X)$, for any path-connected space X in [7]. These invariants have their origin in the notion of the Schwarz genus of a fibration defined in [8]. One of the tools used to calculate these invariants is the product inequality for the Schwarz genus. In this paper, we will give a generalization of the product inequality of the higher topological complexity.

1. Introduction

Let X be a path-connected topological space, PX the space of all continuous paths $\gamma : I = [0,1] \to X$ with the usual compact-open topology.

Let's consider the map

$$\begin{array}{rccc} \pi: & PX & \longrightarrow & X \times X \\ & \gamma & \longmapsto & (\gamma(0), \gamma(1)) \end{array}.$$

Key words and phrases: higher topological complexity, configuration space, Schwarz genus, product inequality.
2020 Mathematics Subject Classification: 55R80, 55M30
The first author is supported by the grant NCVCC01.16/22-22

In [4], M. Farber defined the topological complexity $TC(X)$ of X as the smallest number k such that there exists an open covering $\{U_i, i = 1, ..., k\}$ of $X \times X$ with a continuous section $s_i : U_i \to PX$ of π on each U_i, i.e. $\pi \circ s_i = id_{U_i}$.

In 2010 Yu. Rudyak (see [7]) introduced a series of invariants, denoted by $TC_n(X)$, $n \geq 2$, for any path-connected space X. $TC_n(X)$ is called the higher topological complexity of X. It coincides with the topological complexity $TC(X)$ defined by M. Farber when $n = 2$.

Since then, the higher topological complexity has been computed for many topological spaces like spheres in [7] and product of spheres in [1], wedge sum of spheres of different dimensions in [3], configuration spaces on Euclidean spaces in [5], configuration space of distinct ordered points on compact Riemann surfaces of genus g in [6], the complement of some classes of complex hyperplane arrangements in [2].

Being closely related to the notion of the Schwarz genus and the Lusternik-Shnirelman category, the higher topological complexity inherited many interesting properties of these invariants. Some of these properties such as some evaluations from above or from below are used in computing the higher topological complexity TC_n.

One of the important evaluations of higher topological complexity is the product inequality saying that if X and Y are path-connected spaces, then

$$TC_n(X \times Y) \leq TC_n(X) + TC_n(Y) - 1.$$

In this paper, we give a generalization of this inequality. The paper is organized as follows. In section 2 we investigate the higher topological complexity and some of its properties. We formulate and prove our main result in section 3.

2. Higher topological complexity

For $n = 2, 3, \ldots$ let J_n denote the wedge sum of n closed unit intervals $[0, 1]_i$, $i = 1, \ldots, n$ with 0 as attached point. Suppose that X^{J_n} denotes the space of all continuous maps $\gamma : J_n \to X$ with compact-open topology. Consider the map

$$\begin{array}{rccc} e_n^X : & X^{J_n} & \longrightarrow & X^n , \\ & \gamma & \longmapsto & (\gamma(1_1), \ldots, \gamma(1_n)) \end{array}$$

where 1_i is the unit in $[0, 1]_i$ respectively.

Definition 2.1 (see [7])**.** *The higher topological complexity $TC_n(X)$ of the space X is the smallest number k such that there is an open covering $\{U_i, i = 1, \ldots, k\}$ of X^n and there exists a continuous section $s_i : U_i \to X^{J_n}$ of e_n^X, on each U_i, i.e., $e_n^X \circ s_i = id_{U_i}$.*

Obviously, when $n = 2$, $TC_2(X)$ coincides the topological complexity $TC(X)$ defined by M.Faber.

Remark 2.1. *It is known that the map e_n^X is a fibration in the sense of Serre. By definition, the higher topological complexity $TC_n(X)$ of the space X is exactly the Schwarz genus of the fibration e_n^X (see [8]). Moreover, as it is indicated in [7], e_n^X is a fibrational substitute of the diagonal map d_n, i.e. there exists a homotopy equivalence $h : X \to X^{J_n}$ such that $d_n = e_n^X \circ h$. Therefore, $TC_n(X)$ is also called the Schwarz genus of d_n.*

Similar to the topological complexity, the higher topological complexity $TC_n(X)$ is a homotopy invariant. This property has been proved for the topological complexity $TC(X)$ in [4]. We present here proof of this important property for the $TC_n(X)$.

Proposition 2.1. *Suppose that X is homotopic to Y. Then $TC_n(X) = TC_n(Y)$.*

Proof. Assume that there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$. We will prove that $TC_n(Y) \leq TC_n(X)$.

Let U be a open set in X^n such that there exists a continuous section $s : U \to X^{J_n}$ of e_n^X.

For $(B_1, \ldots, B_n) \in U$, we have $s(B_1, \ldots, B_n)$ is a map $\gamma : J_n \to X$. For any $i = 1, \ldots, n$, let γ_i be the path in X defined by $\gamma_i(t) = \gamma|_{[0,1]_i}(t)$, where $[0, 1]_i$ denotes the i^{th} unit interval in the wedge $J_n = [0, 1] \vee \ldots \vee [0, 1]$.

Let's consider the set

$$V = (g \times \ldots \times g)^{-1}(U) = \{(A_1, ..., A_n) \in Y^n | (g(A_1), ..., g(A_n)) \in U\}.$$

We are going to construct a continuous section $\sigma : V \to Y^{J_n}$ of e_n^Y on this open set V of Y^n.

Suppose that $H_t : Y \to Y$ is the homotopy $id_Y \simeq f \circ g$ with $H_0 = id_Y$, $H_1 = f \circ g$. For $(A_1, A_2, ..., A_n) \in V$ we have $(g(A_1), ..., g(A_n)) \in U$ and therefore there exists a continuous section s of e_n^X. As mentioned above, $s(g(A_1), ..., g(A_n))$ is a path $\gamma : J_n \to X^n$. Now, we define $[A_1, A_i]$ to be the path in Y connecting A_1 to A_i , $i = 1, \ldots, n$, by

$$[A_1, A_i](t) = \begin{cases} H_{3t}(A_1), & \text{if } 0 \leq t < \frac{1}{3} \\ f(s_{(g(A_1), g(A_i))}(3t - 1)), & \text{if } \frac{1}{3} \leq t < \frac{2}{3} \\ H_{3(1-t)}(A_i), & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}.$$

Here $s_{(g(A_1),g(A_i))}$ denotes the path in X, connecting $g(A_1)$ to $g(A_i)$, defined from s by

$$s_{(g(A_1),g(A_i))}(t) = \begin{cases} \gamma_1(1-2t), & \text{if } 0 \leq t < \dfrac{1}{2} \\ \gamma_i(2t-1), & \text{if } \dfrac{1}{2} \leq t \leq 1 \end{cases}.$$

The section σ of e_n^Y on V is defined by putting

$$\sigma(A_1, ..., A_n) = ([A_1, A_1], [A_1, A_2], ..., [A_1, A_n]).$$

Here the right hand side denotes a map from J^n to X^n having its restriction on the i^{th} summand $[0,1]_i$ of J^n to be the path $[A_1, A_i]$. It is easy to see that σ defined above is continuous and $e_n^Y \circ \sigma = id_V$. Thus, $TC_n(Y) \leq TC_n(X)$.

Similar arguments will prove $TC_n(X) \leq TC_n(Y)$. And these imply the Proposition.

The following property shows the relation between the higher topological complexity $TC_n(X)$ of a space X and homotopy properties of X.

Proposition 2.2. *If X is a finite r-connected polyhedron, then*

$$TC_n(X) < \frac{ndimX + 1}{r+1} + 1.$$

In particular $r = 0$ (i.e. X is path-connected) then

$$TC_n(X) \leq n \dim X + 1.$$

Proof. As it is mentioned in the Remark after Definition 2.1, the higher topological complexity $TC_n(X)$ is nothing but the Schwarz genus of the fibration e_n^X. The proposition is a consequence of the similar property of the Schwarz genus (see [8, Theorem 5]).

A lower bound for the higher topological complexity of the space X is given in terms of its cohomology with coefficient in any field $\mathbb{K}$.

Definition 2.2. *Let X be a finite path-connected polyhedron. Suppose that n is an integer, d_n is the diagonal map $d_n : X \longrightarrow X^n$ and $\mathbb{K}$ is a field.*

1. *The kernel of the homomorphism $d_n^* : H^*(X^n; \mathbb{K}) \longrightarrow H^*(X; \mathbb{K})$ is called the n-zero divisor of X.*

2. *d_n-zero divisor cup length of X, denoted by $cl(X, n)$ (see[1]), is the maximal number k such that there exists k elements of the n-zero divisor of X satisfying $u_1 \cup u_2 \ldots \cup u_k \neq 0$.*

Then $cl(X,n)$ will be a lower bound of $TC_n(X)$. Precisely, we have the following proposition, which follows from [8, Theorem 4]. Detailed proofs can be found in [7] and [1].

Proposition 2.3. *Suppose that n is an integer, $n \geq 2$.*

i) Let X be a path-connected topology space. Then

$$TC_n(X) \geq cl(X,n) + 1.$$

ii) For finite path-connected polyhedra X and Y we have

$$cl(X \times Y, n) \geq cl(X,n) + cl(Y,n).$$

The next property, usually called product inequality, gives us an estimate of the higher topological complexity of the product space by those of its factors. This inequality is very useful in computing the higher topological complexity of many spaces.

Proposition 2.4. *For path-connected spaces X and Y. If $(X \times Y)^n$ is normal, then we have*

$$TC_n(X \times Y) \leq TC_n(X) + TC_n(Y) - 1. \tag{2.1}$$

Similar property for the Schwarz genus is known in [8, proposition 22] and for the topological complexity $TC(X)$ in [4]. A detailed proof for the case of the higher topological complexity can be found in [1].

3. A generalization of inequality product

In this section, we will generalize the inequality product for the higher topological complexity.

Theorem 3.1. *Suppose that E, X are finite path-connected CW-complexes, (Y, y_0) is a pointed space and $p : E \to X$ is a continuous map such that the following conditions hold*

i) For all $x \in X$, the fiber $p^{-1}(x)$ is homotopic to Y.

ii) The map p accepts a section $s : X \to E$, i.e., $p \circ s = id_X$.

iii) There exists a family of homotopy equivalences $h_x : p^{-1}(x) \to Y$ depending continuously on $x \in X$ such that $h_x(s(x)) = y_0$.

Then,

$$TC_n(E) \leq TC_n(X) + TC_n(Y) - 1.$$

To prove our main result, we first need some technical lemmas.

Lemma 3.1. *Let Y be a finite path-connected CW-complex, U an open set in Y^n on which there is a continuous section $s_U : U \to Y^{J_n}$ of e_n^Y. Then, for any $y_0 \in Y$ there exists always a continuous section $s'_U : U \to Y^{J_n}$ of e_n^Y such that $s'_U(A_1, ..., A_n)(0) = y_0$ for any $(A_1, ..., A_n) \in U$.*

Proof. We first consider the case when $y_0 \in U$.

For $(A_1, ..., A_n) \in U$ we have $s_U(A_1, ..., A_n)$ is a map $\gamma : J^n \longrightarrow Y^n$. Denote $\gamma(0) = P \in Y$. Define γ_i to be the restriction of γ on the i^{th} unit interval of J^n. That is, γ_i is a path connecting P and A_i.

By means of the section s_U, it implies that for any point $P \in U$, there exists a path connecting y_0 to P and this path depends continuously on P. Let denote this path by ℓ_P.

Now the section s'_U is defined by putting $s'_U(A_1, ..., A_n)$ to be a map $\gamma' : J^n \longrightarrow Y^n$, where γ' has its restriction on the i^{th} unit interval of J^n as $\ell_P * \gamma_i$. It is easy to see that $s'_U(A_1, ..., A_n)$ is a continuous section of e_n^Y and $s'_U(A_1, ..., A_n)(0) = y_0$.

Now suppose that $y_0 \notin U$. Let fix a point $y_1 \in U$. By the above arguments, we can construct a continuous section $\tilde{s}_U$ of e_n^Y such that $\tilde{s}_U(A_1, ..., A_n)(0) = y_1$. Suppose that $\tilde{s}_U(A_1, ..., A_n)$ is a map $\tilde{\gamma} : J^n \longrightarrow Y^n$. Then its restriction $\tilde{\gamma}_i$ on the i^{th} unit interval of J^n is a path connecting y_1 to A_i. Fix a path ℓ in Y connecting y_0 and y_1. Now we define the section s' by putting $s'_U(A_1, ..., A_n)$ to be a map $J^n \longrightarrow Y^n$ having its restriction on the i^{th} unit interval of J^n to be $\ell *\tilde{\gamma}_i$. Obviously, the defined map σ' is a section of e_n^Y having $s'_U(A_1, ..., A_n)(0) = y_0$.

The following lemma is implied from Proposition 20 of [8] and the fact that $TC_n(X)$ coincides with the Schwarz genus of the fibration e_X^n.

Lemma 3.2. *Given the topological space X, let $C = \{C_1, ..., C_p\}$ and $D = \{D_1, ..., D_q\}$ be open covering of X^n such that on each $C_i \cap D_j$ there exists local section of e_n. Then*

$$TC_n(X) \leq p + q - 1.$$

Proof. [Proof of Theorem 3.1] Suppose that $TC_n(X) = p$. By definition, there is an open covering $U = \{U_1, \dots, U_p\}$ of X^n such that there exists a section of e_n^X on each U_i. Put $C_i = \{(A_1, \dots, A_n) \in E^n | (p(A_1), \dots, p(A_n)) \in U_i\}$. Then $C = \{C_1, \dots, C_p\}$ is an open covering of E^n

Suppose that $TC_n(Y) = q$, and $V = \{V_1, \ldots, V_q\}$ is an open covering of Y^n such that there exists a section of e_n^Y on each V_j. Let

$$D_j = \{(A_1, \ldots, A_n) \in E^n | (h_{p(A_1)}(A_1), \ldots, h_{p(A_n)}(A_n)) \in V_j\}.$$

Then $D = \{D_1, \ldots, D_q\}$ is an open covering of E^n. Let we fix a section of e_n^Y on each V_j as that in the Lemma 3.1.

We will now construct a section of e_n^E on each $C_i \cap D_j,\ i = 1, \ldots, p,\ j = 1, \ldots, q$. Suppose that $(A_1, \ldots, A_n) \in C_i \cap D_j \subset E^n$.

By definition $(A_1, \ldots, A_n) \in C_i$ means that $(p(A_1), \ldots, p(A_n)) \in U_i$. Since there exists a continuous section s_1 of e_n^X on U_i, there is a path going from $p(A_1)$ to $p(A_i)$ defined by this section s_1. Let denote γ_2 the image of this path by the section s of p.

Since $(A_1, \ldots, A_n) \in D_j$ it implies that $(h_{p(A_1)}(A_1), \ldots, h_{p(A_n)}(A_n)) \in V_j$. By assumption, there exists a section s_2 of e_n^Y on this V_j. Let's choose s_2 to be the one defined in the Lemma 3.1. This section s_2 defines a path in Y connecting $h_{p(A_1)}(A_1)$ to the point y_0. We denote by γ_1 the inverse image of this path by the homotopy equivalence $h_{p(A_1)}$.

Similarly, the section s_2 defines a path in Y connecting y_0 to the point $h_{p(A_i)}(A_i)$, $i = 1, \ldots, n$. We denote by γ_3 the inverse image of this path by the homotopy equivalence $h_{p(A_i)}$.

We denote by $[A_1, A_i]$ the path $\gamma_1 * \gamma_2 * \gamma_3$.

Now we can define a map $\sigma : C_i \cap D_j \longrightarrow E^n$ by putting $\sigma(A_1, ..., A_n) = ([A_1, A_1], ..., [A_1, A_n])$ for any $(A_1, ..., A_n) \in C_i \cap D_j$.

This map σ is obviously a continuous section of e_n^E. Applying the Lemma 3.2 we have $TC_n(E) \leq p + q - 1 = TC_n(X) + TC_n(Y) - 1$.

Remark 3.1. *Suppose that X, Y are path-connected CW-complexes. Put $E = X \times Y$ and the map $p : E \to X$ to be the projection in the first component. The map $s : X \to E$ defined by $s(x) = (x, y_0)$ is obviously a section of p and $h_x : p^{-1}(x) \to Y$, $h_x(x, y) = y$ is a homotopy equivalence. Then, all the assumptions of Theorem 3.1 are satisfied. And we get again the product inequality (2.1) as stated in Proposition 2.4.*

Theorem 3.1 has been used to compute the higher topological complexity of configuration spaces on some topological manifolds in [3]. We briefly recall the case of configuration space $F(\mathbb{T}, k)$ on the two-dimensional torus.

The configuration space of k distinct ordered points in $\mathbb{T}$ is a subset of $\mathbb{T}^k$, defined by

$$F(\mathbb{T}, k) = \{(x_1, \cdots, x_k) \in \mathbb{T}^k |\ x_i \neq x_j \text{ with } 1 \leq i \neq j \leq k\}.$$

Proposition 3.2. *Let k be an integer with $k \geq 2$. The higher topological complexity of the configuration space of k instinct ordered points on the 2-dimensional torus $\mathbb{T}$ is*

$$TC_n(F(\mathbb{T},k)) = n(k+1) - 1.$$

Observe that the projection on the first $k-1$ coordinates $\pi_k : F(\mathbb{T},k) \longrightarrow F(\mathbb{T},k-1)$ is a fibration with the fiber homotopic to the bouquet of k circles $Y_k = \underbrace{\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1}_{k}$. It is proved in [3] that this fibration π_k admits a section σ_k for any $k = 1, \ldots$ and satisfies all assumptions of Theorem 3.1. Combining properties of the higher topological complexity mentioned in the previous section and Theorem 3.1 we get

$$TC_n(F(\mathbb{T},k)) \leq TC_n(F(\mathbb{T},k-1)) + TC_n(Y_k) - 1.$$

Moreover, it follows from [3, Theorem 3] that $TC_n(Y_k) \leq n+1$ for $k \geq 2$. So,

$$TC_n(F(\mathbb{T},k)) \leq TC_n(F(\mathbb{T},k-1)) + n \text{ for all } k \geq 2.$$

By induction on k, we get

$$TC_n(F_n(\mathbb{T},k)) \leq TC_n(F(\mathbb{T},1)) + n(k-1) = 2n - 1 + n(k-1) = n(k+1) - 1.$$

To prove that $n(k+1)-1$ is the lower bound of $TC_n(F(\mathbb{T},k))$ we need to use the lower bound stated in Proposition 2.3 and spectral sequence arguments. We will skip it here. A detailed proof can be found in [3]

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