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On Korenblum constants for some weighted function spaces

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Abstract. In this paper, we survey the results on the Korenblum Maximum Principle for some weighted function spaces. Progress and results discussed include the upper bounds and lower bounds of Korenblum constants, as well as the failure of the principle for weighted Bergman spaces, weighted Hardy spaces, weighted Bloch spaces, weighted Fock spaces, and mixed norm spaces. Existing and new open questions are provided.

1. Introduction

The Korenblum Maximum Principle is an important open problem in complex analysis as it acts as one of the fundamental properties of complex function spaces that remains unsolved. First conjectured in 1991, the principle was introduced [15] by Boris Korenblum for the classical Bergman space $A^2(\mathbb{D})$ in the following way.

Conjecture 1.1. *There exists a numerical constant c, $0 < c < 1$, such that if f and g are holomorphic in $\mathbb{D}$ and $|f(z)| \leq |g(z)|$ $(c < |z| < 1)$, then $\|f\|_{A^2} \leq \|g\|_{A^2}$.*

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In [15], Korenblum defined c as the *Korenblum constant* and κ as the largest possible value of c. The exact value of κ remains unknown. In the same paper, Korenblum also proved that $\kappa_{A^2} \leq \frac{1}{\sqrt{2}} \approx 0.7071$.

Initially, only a series of partial results were at first discovered by Korenblum, O'Neil, Richards and Zhu [17], Korenblum and Richards [16], Matero [18], Schwick [22], and others. The existence of Korenblum constant for $A^2(\mathbb{D})$ was first proved in 1999 by Hayman [11] with $\kappa_{A^2} = 0.04$. Thereafter, many results were published by improving the lower bounds and upper bounds of κ_{A^2} (see [21, 24-28, 30]). As time progresses, many interesting results have been obtained by several authors. The regained interest in this problem in the recent years have contributed to many fascinating results for families of function spaces as well as intersections of function spaces. Hence, this calls for a timely review to summarize the key important results concerning the Korenblum Maximum Principle. There are certainly several partial results or results related to modified versions of the Korenblum Maximum Principle, and we apologize to those authors as their work are not explicitly mentioned. Ultimately, we hope that this survey might be of interest to not just complex analysts but also mathematicians from other related fields, and that it might inspire new readers with the interesting results that have been obtained so far. At the same time, we hope both existing and new researchers in this problem can take on existing and new open questions from this survey.

We describe the outline of this survey. First of all, no proofs are provided in this paper. Readers should refer to the original articles for detailed proofs. References are provided for all the results. In next section, we recall all basic definitions and notations for weighted Bergman spaces, weighted Hardy spaces, weighted Bloch spaces and weighted Fock spaces. We also list down specific weight functions that will be discussed in our survey. In fact, different weight functions play an important role in many key results for later sections. The results are organised into four sections, namely Sections 3 to 6. Tables summarizing key results are presented at the end of the section where appropriate. As the original Korenblum constant is defined for Bergman spaces, Section 3 discusses the key results for the Korenblum constants for Bergman spaces first. The results are thus separated into two sections: upper bounds and lower bounds. In Section 4, the Korenblum constants are discussed for other function spaces, namely, weighted Hardy spaces and weighted Fock spaces. Following this, Section 5 is solely dedicated to discuss the extension of results from classical weighted Fock spaces to intersections of weighted Fock spaces. In particular, this section extends from the results for classical weighted Fock spaces in Section 4 by leveraging some preliminaries in the well-known Ramanujan's Master Theorem. Hence, we first recall several key important preliminaries such as the Gamma function, Mellin transform of Dirichlet series and Generalised Hypergeometric function in Section 5.1. Section 5.2 constructs new weighted Fock spaces and reviews the upper bounds of Korenblum constants in the finite and infinite intersections of those spaces. After discussing all the main results in Korenblum constants for the weighted function spaces, Section 6 discusses the remaining results pertaining to the failure of Korenblum Maximum Principle. As the failure of Korenblum Maximum Principle in most function spaces are found using similar methods, we survey all of them together in Section 6. Our final Section 7 describes a possible future direction for the Korenblum Maximum Principle and lists down all existing and new open questions.

2. Basic Notations for Weighted Function Spaces

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{O}(\mathbb{D})$ (resp. $\mathcal{O}(\mathbb{C})$) the space of holomorphic functions (resp. entire functions) on $\mathbb{D}$ (resp. $\mathbb{C}$), endowed with the compact-open topology.

For a domain G, a continuous function $\varphi : G \to [0, \infty)$ can be defined as a *weight function* for weighted function spaces. In this paper, we are only interested in radial weight functions defined on $\mathbb{D}$ or $\mathbb{C}$, i.e. $\varphi(z) = \varphi(|z|)$. To be more precise, we list down the weights that will be used in this paper.

For $G = \mathbb{D}$,

(i) $\varphi(z) = (1 - |z|)^\alpha$, $\alpha \geq 0$, are the standard weights on the disc,

(ii) $\varphi(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, $\alpha > -1$, are the classical Bergman weights,

(iii) $\varphi(z) = e^{-\frac{p\alpha}{2}|z|^2}$, $\alpha > -1$, are the exponential weights defined in [32].

For $G = \mathbb{C}$,

(i) $\varphi(z) = \frac{\alpha}{2}|z|^2$, where $\alpha > 0$, are the classical Fock space weights.

(ii) $\varphi(z) = \frac{\alpha}{2}\lambda|z| - \frac{1}{p}\log|d|$, where $\alpha > 0$, $0 < p < \infty$, $d \in \mathbb{C}\backslash\{0\}$, $\lambda > 0$, are the generalised Fock space weights discussed in [33].

First, we recall the general weighted Hardy space $H^p_\varphi(\mathbb{D})$ where $\varphi : \mathbb{D} \to [0, \infty)$.

Definition 2.1. *Let $\varphi(z) = (1 - |z|)^\alpha$ for $\alpha \geq 0$. For $0 < p < \infty$, the general weighted Hardy space $H^p_\varphi(\mathbb{D})$ consists functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which*

$$\|f\|_{H^p_\varphi} = \sup_{0 \leq r < 1} \left[\varphi(r) \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right] < \infty.$$

If $\varphi(z) = (1-|z|)^\alpha$ where $\alpha \geq 0$, we obtain the weighted Hardy space $H^p_\alpha(\mathbb{D})$. Further, if $\alpha = 0$, we have the Hardy space $H^p(\mathbb{D})$. In the case $p = \infty$, we have the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$, where $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$.

Definition 2.2. *Let $0 < p < \infty$ and $\varphi : \mathbb{D} \to [0,\infty)$. Then the weighted Bergman space $A^p_\varphi(\mathbb{D})$ is the space consisting of analytic functions $f(z) \in \mathcal{O}(\mathbb{D})$ for which*

$$\|f\|_{A^p_\varphi} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p \varphi(z)\, dA(z)\right]^{\frac{1}{p}} < \infty.$$

Here $dA(z) = dxdy = rdrd\theta$, $z = x+iy = re^{i\theta}$, is the Lebesgue measure on $\mathbb{C}$.

Let $\varphi(z) = (\alpha+1)(1-|z|^2)^\alpha$ for $\alpha > -1$. Then we have the classical weighted Bergman space $A^p_\alpha(\mathbb{D})$ which is a Banach space. Further, if $\alpha = 0$, for $0 < p < \infty$, the space becomes the standard Bergman space $A^p(\mathbb{D})$. In particular, for $p = 2$, we have the classical Bergman space $A^2(\mathbb{D})$.

Interestingly, a weighted Bergman space with exponential weights $\varphi(z) = e^{-\frac{p\gamma}{2}|z|^2}$ $(0 < p < \infty, \gamma > -1)$ is introduced in [32]. We shall denote this weighted Bergman space with exponential weights as $A^p_\gamma(\mathbb{D})$, that is, the space of holomorphic functions $f(z) \in \mathcal{O}(\mathbb{D})$ for which

$$\|f\|_{A^p_\gamma} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p e^{\frac{p\gamma}{2}|z|^2}\, dA(z)\right]^{\frac{1}{p}} < \infty.$$

Note that $e^{\frac{p\gamma}{2}|z|^2} \to 1$ as $|z| \to 0$ and $e^{\frac{p\gamma}{2}|z|^2}$ approaches to the constant $e^{\frac{p\gamma}{2}}$ as $|z|$ approaches the boundary of $\mathbb{D}$.

Next, we have the weighted Fock spaces.

Definition 2.3. *Let $\varphi(z) : \mathbb{C} \to [0,\infty)$ be a weight function. For $0 < p < \infty$, the general weighted Fock space $\mathcal{F}^p_\varphi(\mathbb{C})$ with weight $\varphi(z)$, consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$ for which*

$$\|f\|^p_{\mathcal{F}^p_\varphi} = \eta \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)}\, dA(z) < \infty,$$

where the constant η is chosen so that $\|1\|_{\mathcal{F}^p_\varphi} = 1$.

If $\varphi(z) = \frac{\alpha}{2}|z|^2$, $\alpha > 0$, then we have the classical weighted Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$ with norm $\|f\|^p_{\mathcal{F}^p_\alpha} = \dfrac{p\alpha}{2\pi}\displaystyle\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$. for the case $p = \infty$ and $\alpha > 0$, we have the space $\mathcal{F}^\infty_\alpha$ with norm $\|f\|_{\mathcal{F}^\infty_\alpha} = \sup\limits_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha}{2}|z|^2}$.

For $0 < p \leq \infty$ and $\alpha = 1$, we have the Fock space $\mathcal{F}^p(\mathbb{C})$, which is a Banach space if $1 \leq p \leq \infty$ and is a complete metric space with distance $d(f,g) = \|f-g\|_p^p$ if $0 < p < 1$.

Lastly, we have the weighted Bloch spaces.

Definition 2.4. *The weighted Bloch space B_φ with weight $\varphi : \mathbb{D} \to [0,\infty)$, consists of holomorphic functions $f(z) \in \mathcal{O}(\mathbb{D})$ for which*

$$\|f\|_{B_\varphi} = |f(0)| + \sup_{z\in\mathbb{D}} \varphi(z)(1-|z|^2)|f'(z)| < \infty.$$

Note that $\|f\|_{B_\varphi}$ is the weighted Bloch norm and elements of B_φ are known as weighted Bloch functions. If $\varphi(z) \equiv 1$, then we have the classical Bloch space B.

3. Korenblum Constants for Bergman Spaces

In this section, we survey the main results for the Korenblum constants of Bergman spaces. To avoid confusion and provide greater clarity, the results are divided into two sections: Upper bounds and Lower bounds.

3.1. Development of Upper Bounds

Recall from the introduction that Korenblum first discovered the upper bound for κ_{A^2} to be $\frac{1}{\sqrt{2}}$.

Theorem 3.1 ([15])**.** *Let $c > \frac{1}{\sqrt{2}}$. There exist functions f and g in $A^2(\mathbb{D})$ such that $|f(z)| \leq |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^2} > \|g\|_{A^2}$. Therefore, $\kappa_{A^2} \leq \frac{1}{\sqrt{2}}$.*

The next natural question is whether can κ_{A^2} be equal to $\frac{1}{\sqrt{2}}$. An example by Martin [15] shows that $\kappa_{A^2} < \frac{1}{\sqrt{2}}$.

Theorem 3.2 ([15])**.** *Suppose $c = \frac{1}{\sqrt{2}}$. Let*

$$f(z) = \frac{1+(\sqrt{2}-1)z^{20}}{1+(\sqrt{2}-1)z^{-20}}, \quad g(z) = \sqrt{2}z.$$

Then $|f(z)| \leq |g(z)|$ for all $c < |z| < 1$, but

$$\|f\|_{A^2} = \frac{\sqrt{1+(\sqrt{2}-1)^2/21}}{1+(\sqrt{2}-1)2^{-10}} > 1 = \|g\|_{A^2}.$$

In a series of papers [24-28,30], Wang used different pairs of functions f and g to improve the upper bounds for the Korenblum constant of $A^2(\mathbb{D})$. In [24], Wang first used the singular inner function $S_a(z) = \exp(-a\frac{1+z}{1-z})$ in $A^2(\mathbb{D})$, $a \in \mathbb{R}^+$, to prove that the Korenblum constant must be less than 0.69472.

Theorem 3.3 ([24]). *Let*

$$f(z) = e^{-a} S_a(z^n) = e^{-a} \exp\left(-a\frac{1+z^n}{1-z^n}\right) = \exp\left(-\frac{2a}{1-z^n}\right),$$

where a is any positive constant and

$$g(z) = e^{-\frac{2a}{1+c^n}} \frac{z}{c},$$

where $0 < c < 1$, $a = -\frac{1+c^n}{1-c^n} \log c > 0$, $n \in \mathbb{N}$. Then $|f(z)| \leq |g(z)|$ in $c < |z| < 1$. When $n = 14$ and $c = 0.69472$, we have $||f||_{A^2} > ||g||_{A^2}$. Therefore, $\kappa_{A^2} < 0.69472$.

Following that, Wang managed to find sharper upper bounds for κ_{A^2} through the results below.

Theorem 3.4 ([25]). *Let $0 < c < 1$, $a = -\frac{1+c^n}{1-c^n} \log c > 0$ and $n \in \mathbb{N}$. Then define*

$$f(z) = S_{a+b}(z^n), \quad g(z) = zS_b(z^n)$$

and we have $|f(z)| \leq |g(z)|$ in $c < |z| < 1$. Moreover, when $a = 0.3902$, $b = 0.3395$, $n = 11$ and $c = 0.685086$, we have $||f||_{A^2} > ||g||_{A^2}$. Therefore, $\kappa_{A^2} < 0.685086$.

Theorem 3.5 ([26]). *Let $a = \frac{3\sqrt{6}}{11}$ and $n = 10$. Then we define*

$$f(z) = a + z^n, \quad g(z) = z(1 + az^n)$$

and we have $||f||_{A^2} = ||g||_{A^2}$ and $|f(z)| \leq |g(z)|$ in $c < |z| < 1$, where $c = 0.679501$ is the real root in $(0, 1)$ of the equation

$$a + z^{10} = \frac{3\sqrt{6}}{11} + z^{10} = z + \frac{3\sqrt{6}}{11} z^{11} = z(1 + az^{10}). \tag{3.1}$$

Therefore, $\kappa_{A^2} < 0.679501$.

As a result of Theorem 3.5, the upper bound of Korenblum constant is now 0.679501 but $||f||_{A^2} = ||g||_{A^2}$ for this upper bound. Hence, in [26], Wang noted that this bound is not sharp and proceeded to obtain a better upper bound, i.e. $\kappa_{A^2} < 0.67795$.

Theorem 3.6 ([26]). *Let* $0 < a < 1$, $b \geq 0, n \in \mathbb{N}$. *Then we define*

$$f(z) = \frac{a + z^n}{(1 - az^n)^b}, \quad g(z) = \frac{z(1 + az^n)}{(1 - az^n)^b}.$$

Hence, we have $|f(z)| \leq |g(z)|$ *in* $c < |z| < 1$, *where* c *is the real root in* $(0, 1)$ *of the equation*

$$a + z^n = z(1 + az^n),$$

and when $a = 0.666707, b = 0.4768$ *and* $n = 10$, *we have* $c = 0.67795$ *and* $||f||_{A^2} > ||g||_{A^2}$. *Therefore,* $\kappa_{A^2} < 0.67795$.

In 2008, Shen [23] modified the above example to obtain a slightly better upper bound $\kappa_{A^2} < 0.677905$.

Theorem 3.7 ([23]). *Let* $0 < a < 1$ *and* $n \in \mathbb{N}$. *Then we define*

$$f(z) = \frac{a + z^n}{2 - az^n}, \quad g(z) = \frac{z(1 + az^n)}{2 - az^n}.$$

Hence, we have $|f(z)| \leq |g(z)|$ *in* $c < |z| < 1$, *where* c *is the real root in* $(0, 1)$ *of the equation*

$$a + z^n = z(1 + az^n),$$

and when $a = 0.6666714$ *and* $n = 10$, *we have* $c = 0.677905$ *and* $||f||_{A^2} > ||g||_{A^2}$. *Therefore,* $\kappa_{A^2} < 0.677905$.

A final improvement was made by Wang [29] where he obtained $\kappa_{A^2} < 0.6778994$ with the following counter example.

Theorem 3.8 ([29]). *Let* $0 < a < 1$, $b \geq 0, n \in \mathbb{N}$. *Then we define*

$$f(z) = \frac{a + z^n}{(1 - bz^n)^2}, \quad g(z) = \frac{z(1 + az^n)}{(1 - bz^n)^2}.$$

Hence, we have $|f(z)| \leq |g(z)|$ *in* $c < |z| < 1$, *where* c *is the real root in* $(0, 1)$ *of the equation*

$$a + z^n = z(1 + az^n),$$

and when $a = \sqrt{\frac{n-2}{2n-2}}$, $b = \sqrt{\frac{2}{(n-1)(n-2)}}$ *and* $n = 10$, *we have* $c = 0.6778994$ *and* $||f||_{A^2} > ||g||_{A^2}$. *Therefore,* $\kappa_{A^2} < 0.6778994$.

In summary, $\kappa_{A^2} < 0.6778994$ is the best upper bound of Korenblum constant for $A^2(\mathbb{D})$ so far.

Note that the upper bounds by Wang were numerically sharper but it lacks generalisations for the weighted Bergman spaces. In recent years, we obtained explicit expression for the upper bounds in the weighted Bergman spaces with exponential weights, $A^p_\gamma(\mathbb{D})$, $p \geq 1$, $\gamma \geq 0$ [32].

Theorem 3.9 ([32]). *Let $p \geq 1$, $\gamma \geq 0$. Consider the Bergman space $A^p_\gamma(\mathbb{D})$.*

1) For $\gamma = 0$, suppose

$$\left(\frac{2}{p+2}\right)^{\frac{1}{p}} < c < 1.$$

2) For $\gamma > 0$, suppose

$$\sqrt[p]{\frac{\left(\frac{2}{p\gamma}\right)^{\frac{p}{2}} \int_0^{\frac{p\gamma}{2}} u^{\frac{p}{2}} e^{-u} du}{(1 - e^{-\frac{p\gamma}{2}})}} < c < 1.$$

There exist functions f and g in $A^p_\gamma(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^p_\gamma} > \|g\|_{A^p_\gamma}$.

Remark 3.1. *Clearly, in order to have the Korenblum Maximum Principle for $A^p_\gamma(\mathbb{D})$, $p \geq 1$, $\gamma \geq 0$, we must have*

$$\kappa_{A^p_\gamma} \leq \begin{cases} \left(\frac{2}{p+2}\right)^{\frac{1}{p}}, & \gamma = 0, \\ \sqrt[p]{\frac{\left(\frac{2}{p\gamma}\right)^{\frac{p}{2}} \int_0^{\frac{p\gamma}{2}} u^{\frac{p}{2}} e^{-u} du}{(1-e^{-\frac{p\gamma}{2}})}}, & \gamma > 0. \end{cases}$$

The above result acts as a generalization of the initial result $\kappa_{A^2} \leq \frac{1}{\sqrt{2}}$ by Korenblum in Theorem 3.1. Nevertheless, this generalization obtains the upper bounds for rest of the spaces $A^p_\gamma(\mathbb{D})$, $p \geq 1$.

3.2. Development in Lower Bounds

In this subsection, we survey the progress on lower bound of Korenblum constant for Bergman spaces ever since Hinkkanen's result in 1999 [12]. Hinkkanen proved in 1999 that $\kappa_{A^p} \geq 0.15724$ ($p \geq 1$), thereby showing that Korenblum constants exist for all Bergman spaces $A^p(\mathbb{D})$, $p \geq 1$. After that, improvements were made in 2006 when Schuster [21] showed that the Korenblum maximum principle holds for $\kappa_{A^2} = 0.21$, which progresses upon the works of both Hayman and Hinkkanen. In that paper, Schuster made use of the following identities.

Proposition 3.10. *For any z, $w \in \mathbb{C}$,*

$$|z|^2 - |w|^2 \leq 2|z^2 - zw|.$$

Proposition 3.11. *For any subharmonic function h and $0 < r_1 < r_2 < 1$,*

$$\int_0^{2\pi} h(r_1 e^{i\theta})\, d\theta \le \int_0^{2\pi} h(r_2 e^{i\theta})\, d\theta.$$

Using Propositions 3.10 and 3.11, Schuster proved Theorem 3.12.

Theorem 3.12 ([21])**.** *Suppose that $c = 0.21$. Then for any functions $f(z)$ and $g(z)$ holomorphic in $\mathbb{D}$, if $|f(z)| \le |g(z)|$ $(c < |z| < 1)$, then $||f||_{A^2} \le ||g||_{A^2}$. Therefore, $\kappa_{A^2} \ge 0.21$.*

Also in 2006, Wang [27] managed to improve the lower bounds of κ_{A^2} to 0.25018 and κ_{A^p} to 0.1921. Wang used similar methods but a different inequality from Hinkkanen and Schuster. For instance, Wang used the following proposition instead of Proposition 3.10 in proving $\kappa_{A^2} \ge 0.25018$.

Proposition 3.13. *For any $a \in (-1, 1)$ and $z,\ w \in \mathbb{C}$,*

$$|z|^2 - |w|^2 = \frac{|z - aw|^2 - |az - w|^2}{1 - a^2} \le \frac{|z - aw|^2}{1 - a^2}. \tag{3.2}$$

Wang then made his final improvements to the lower bounds in 2011, where he showed that $\kappa_{A^2} \ge 0.28185$ and $\kappa_{A^p} \ge 0.23917$ for $p \ge 1$.

Recently, the Korenblum maximum principle was extended to a large family of function spaces that contains the classical weighted Bergman space $A^p_\varphi(\mathbb{D})$ [4]. In particular, a failure of the principle was proven for the mixed norm space $H^{p,q,s}$ where $0 < p, q, s < \infty$ by Karapetrović [4]. The mixed norm space $H^{p,q,s}$ $(0 < p, q, s < \infty)$ consists of all holomorphic functions in $\mathcal{O}(\mathbb{D})$ for which

$$\|f\|_{H^{p,q,s}} = \left(2sq \int_0^1 r(1 - r^2)^{sq-1} M_p^q(r, f)\, dr\right)^{1/q} < \infty, \tag{3.3}$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right)^{1/p}.$$

Note that when $q = p$ and $s = \frac{\alpha+1}{p}$, then $\|f\|_{H^{p,p,\frac{\alpha+1}{p}}} = \|f\|_{A^p_\alpha}$ which implies that $H^{p,p,\frac{\alpha+1}{p}} = A^p_\alpha(\mathbb{D})$.

Similar to earlier results in Korenblum constants, Karapetrović also proved that the Korenblum constants exist under the mixed norm spaces, $H^{p,q,s}$, $1 \le p \le q < \infty$ and $0 < s < \infty$.

Theorem 3.14 ([4])**.** *Let $1 \le p \le q < \infty$ and $0 < s < \infty$. Then there exists a constant $0 < c < 1$ with the following property: If f and g are holomorphic functions on $\mathbb{D}$ such that $|f(z)| \le |g(z)|$ for all $c < |z| < 1$, then $\|f\|_{H^{p,q,s}} \le \|g\|_{H^{p,q,s}}$.*

Corollary 3.1 ([4]). *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Then the Korenblum maximum principle holds in weighted Bergman space $A^p_\alpha(\mathbb{D})$ if and only if $p \geq 1$.*

In summary, the results for Bergman spaces can be summarised using the following tables.

	α	p	**Lower Bound**	**Upper Bound**
$A^p_\alpha(\mathbb{D})$ $\alpha > -1$ $1 \leq p < \infty$	$\alpha = 0$	$p = 2$	1999: $\kappa_{A^2} \geq \frac{1}{25}$ 2006: $\kappa_{A^2} \geq 0.21$ 2006: $\kappa_{A^2} \geq 0.25018$ 2011: $\kappa_{A^2} \geq 0.28185$	1991: $\kappa_{A^2} < \frac{1}{\sqrt{2}}$ 2003: $\kappa_{A^2} < 0.69472$ 2004: $\kappa_{A^2} < 0.685086$ 2004: $\kappa_{A^2} < 0.67795$ 2008: $\kappa_{A^2} < 0.677905$ 2008: $\kappa_{A^2} < 0.6778994$
		$1 \leq p < \infty$	1999: $\kappa_{A^p} \geq 0.15724$ 2006: $\kappa_{A^p} \geq 0.1921$ 2011: $\kappa_{A^p} \geq 0.23917$	2020: $\kappa_{A^p} \leq \left(\frac{2}{p+2}\right)^{1/p}$
	$\alpha > 0$	$1 \leq p < \infty$	2022: Corollary 2.18	No Specific Development
	$-1 < \alpha < 0$	$1 \leq p < \infty$		

Table 1. Main Development on classical weighted Bergman spaces $A^p_\alpha(\mathbb{D})$

	γ	p	**Lower Bound**	**Upper Bound**
$A^p_\gamma(\mathbb{D})$ $\gamma > -1$ $1 \leq p < \infty$	$\gamma = 0$	-	See Table 1 for $A^p(\mathbb{D})$	
	$\gamma > 0$	$1 \leq p < \infty$	No Specific Development	2020: Theorem 3.9
	$-1 < \gamma < 0$	$1 \leq p < \infty$		No Specific Development

Table 2. Main Development on Bergman spaces with exponential weights $A^p_\gamma(\mathbb{D})$.

Based on the summary, we make some minor progress in the areas with no development so far. In particular, we can apply similar methods from Theorem 3.9 to obtain explicit expressions for the upper bound of Korenblum constants in classical weighted Bergman spaces $A^p_\alpha(\mathbb{D})$, $\alpha \geq 0$ and $p \geq 1$.

Theorem 3.15. *Let $p \geq 1$, $\alpha \geq 0$. Consider the Bergman space $A^p_\alpha(\mathbb{D})$. Suppose*

$$\sqrt[p]{\frac{(\alpha+1)\Gamma(\frac{p}{2}+1)\Gamma(\alpha+1)}{\Gamma(\frac{p}{2}+\alpha+2)}} < c < 1.$$

There exist functions f and g in $A^p_\alpha(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$.

Remark 3.2. *Clearly, in order to have the Korenblum Maximum Principle for $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$, we must have*

$$\kappa_{A^p_\alpha} \leq \sqrt[p]{\frac{(\alpha+1)\Gamma(\frac{p}{2}+1)\Gamma(\alpha+1)}{\Gamma(\frac{p}{2}+\alpha+2)}}.$$

4. Korenblum Constants for Other Weighted Function Spaces

In this section, we survey the results of Korenblum constants for other weighted function spaces. In general, for a function space $\mathcal{L}$, the Korenblum conjecture is as follows,

Conjecture 4.1. *There exists a numerical constant c, $0 < c < 1$, such that for any functions f and g in $\mathcal{L}$, if*

$$|f(z)| \leq |g(z)|, \quad \forall z \in E, \tag{4.1}$$

then

$$\|f\|_{\mathcal{L}} \leq \|g\|_{\mathcal{L}}. \tag{4.2}$$

Here, E is a set of values of z in order for (4.1) to imply (4.2). In [7], the set E satisfying the above conjecture for the function space $\mathcal{L}$ is also known as a dominating set for $\mathcal{L}$.

Weighted Hardy Spaces. Korenblum stated in [15] that for the case where $\mathcal{L}$ is the Hardy-Hilbert space $H^2(\mathbb{D})$, then (4.1) implies (4.2) even for the case $E = \mathbb{D}$. Interestingly, it was only until 1998 that the following criteria was discovered for a dominating set in general Hardy spaces $H^p(\mathbb{D})$ $(0 < p \leq \infty)$.

Theorem 4.1 ([7])**.** *Let $0 < p \leq \infty$. Then E is non-tangentially dense if and only if E is a dominating set for the general Hardy spaces $H^p(\mathbb{D})$ $(0 < p \leq \infty)$.*

As there is no specific development on the set E for $H^p_\alpha(\mathbb{D})$ $(\alpha > 0, \quad 0 < p \leq \infty)$ and even for $H^p_\varphi(\mathbb{D})$ where $\varphi(z) \neq (1-|z|)^\alpha$, then these areas call for investigation. In summary, we have the following table.

		α	p	**Results**
$H^p_\varphi(\mathbb{D})$	$H^p_\alpha(\mathbb{D})$	$\alpha = 0$	$0 < p \leq \infty$	1998: Theorem 4.1
		$\alpha > 0$	$0 < p \leq \infty$	No Specific Development
	-	-	-	No Specific Development

Table 3. Development of Korenblum constants on $H^p_\varphi(\mathbb{D})$.

Weighted Fock Spaces. Throughout the years, some results also showed an extension to the Fock spaces (see [7, 34]). It first began in 2006 when Schuster modified the proof of Theorem 3.12 and obtained $\kappa_{\mathcal{F}^2} \geq 0.54$.

Proposition 4.2 ([21])**.** *Let $f(z)$ and $g(z)$ be entire functions in $\mathcal{F}^2(\mathbb{C})$. Suppose that $|f(z)| \leq |g(z)|$ for any z such that $|z| > c$, where $c_{\mathcal{F}^2} = 0.54$. Then $||f||_{\mathcal{F}^2} \leq ||g||_{\mathcal{F}^2}$.*

Wang [27] then modified his result in $A^2(\mathbb{D})$ for $\mathcal{F}^2(\mathbb{C})$ and obtained the following improved lower bound.

Theorem 4.3 ([27])**.** *Let $f(z)$ and $g(z)$ be entire functions in $\mathcal{F}^2(\mathbb{C})$. Suppose that $|f(z)| \leq |g(z)|$ for any z such that $|z| > c$, where $c_{\mathcal{F}^2} = 0.7248$. Then $||f||_{\mathcal{F}^2} \leq ||g||_{\mathcal{F}^2}$.*

Till today, the lower bound $\kappa_{\mathcal{F}^2} \geq 0.7248$ remains as the best lower bound so far. In 2012, Zhu [34] showed that Korenblum constants actually exist for the classical weighted Fock spaces $\mathcal{F}^p_\alpha(\mathbb{C})$ where $\alpha > 0$ and $p \geq 1$.

Theorem 4.4 ([34])**.** *Let $f(z)$ and $g(z)$ be entire functions in $\mathcal{F}^p_\alpha(\mathbb{C})$. Suppose that $|f(z)| \leq |g(z)|$ for any z such that $|z| > c$. Then $||f||_{\mathcal{F}^p_\alpha} \leq ||g||_{\mathcal{F}^p_\alpha}$.*

To be specific, Zhu proved that the Korenblum Maximum Principle can hold for $\mathcal{F}^p_\alpha(\mathbb{C})$ where $\alpha > 0$ and $p \geq 1$ as there exist a sufficiently small positive c to satisfy

$$2c(1 - e^{-\frac{p\alpha}{2}c^2})\left(\alpha \int_c^\infty e^{-\frac{p\alpha}{2}\rho^2}(\rho^2 - c^2)d\rho\right)^{-1} < 1. \tag{4.3}$$

In general, the outline of the proofs for lower bounds of Korenblum constants are similar to the works of Schuster [21]. The main workhorse for the improvements of the lower bound relies on changing the inequality to manipulate $|f|^p - |g|^p$ in the proof. Consequently, this will lead to changes in the numerical estimate of the lower bound. However, in order to obtain a particular lower bound for the Korenblum constant, several results have used Mathematica to provide a numerical estimate.

On the other hand, the following simple result involving the Gamma function, is proved in 2020 [32]. The result provides an upper bound for $\kappa_{\mathcal{F}^p_\alpha}$, where $\alpha > 0$ and $p \geq 1$.

Theorem 4.5 ([32])**.** *Let $p \geq 1$, $\alpha > 0$ and*

$$c > \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)}.$$

There exist functions f and g in $\mathcal{F}^p_\alpha(\mathbb{C})$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but $\|f\|^p_{\mathcal{F}^p_\alpha} > \|g\|^p_{\mathcal{F}^p_\alpha}$. Therefore,

$$\kappa_{\mathcal{F}^p_\alpha} \leq \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)}.$$

By setting $p = 2$ and $\alpha = 1$ in Theorem 4.5, the following special case is obtained for $\mathcal{F}^2(\mathbb{C})$.

Corollary 4.1 ([32])**.** *Let $c_{\mathcal{F}^2} > 1$. There exist functions f and g in $\mathcal{F}^2(\mathbb{C})$ such that $|f(z)| < |g(z)|$ for all $|z| > c_{\mathcal{F}^2}$, but $\|f\|^2_{\mathcal{F}^2} > \|g\|^2_{\mathcal{F}^2}$. Therefore, $c_{\mathcal{F}^2} \leq 1$.*

For instance, when $p = 1$ and $\alpha = \frac{1}{2}$, we can write the upper bound as

$$\kappa_{\mathcal{F}^1_{0.5}} \leq \sqrt{4}\cdot\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}.$$

Theorem 4.5 inspired us to study the Korenblum constants for other Fock spaces under general weights such as $\frac{\alpha}{2}\lambda|z| - \frac{1}{p}\log|d|$, where $\alpha > 0$, $0 < p < \infty$, $d \in \mathbb{C}\backslash\{0\}$, $\lambda > 0$ in [32]. The interest in these specific set of general weights is that it introduces various weighted Fock spaces where the upper bound of its Korenblum constants involves special functions such as Gamma function, Mellin transform of Dirichlet series and Generalized Hypergeometric function. For example, we applied the Gamma function and obtained a simple generalisation of Theorem 4.5.

For any complex number s with $\mathrm{Re}(s) > 0$, the Gamma function is defined as

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx.$$

Furthermore, with a change of variables, we have

$$\int_0^\infty x^a e^{-bx^c}\,dx = \frac{1}{c}\cdot\left(\frac{1}{b}\right)^{\frac{a+1}{c}}\Gamma\left(\frac{a+1}{c}\right), \quad a,b,c > 0. \tag{4.4}$$

Now, define the weighted Fock spaces $\mathcal{F}^{p,\alpha}_m(\mathbb{C})$, that is,

Definition 4.1. *For $0 < p < \infty$, $\alpha > 0$, $m > 0$, the weighted Fock space $\mathcal{F}^{p,\alpha}_m(\mathbb{C})$ with weight $\frac{\alpha}{2}|z|^m$ consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$ for which*

$$\|f\|^p_{\mathcal{F}^{p,\alpha}_m} = \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^m}\,dA(z) < \infty.$$

Now, the following result [33] generalises Theorem 4.5 for the weighted Fock space $\mathcal{F}^{p,\alpha}_m(\mathbb{C})$.

Theorem 4.6 ([33]). *Let $0 < p < \infty$, $\alpha > 0$, $m > 0$ and let*

$$c > \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{m}} \Gamma\left(\frac{p+2}{m}\right)\left(\Gamma\left(\frac{2}{m}\right)\right)^{-1}}. \tag{4.5}$$

There exist functions f and g in $\mathcal{F}^{p,\alpha}_m(\mathbb{C})$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but $\|f\|^p_{\mathcal{F}^{p,\alpha}_m} > \|g\|^p_{\mathcal{F}^{p,\alpha}_m}$. Therefore,

$$\kappa_{\mathcal{F}^{p,\alpha}_m} \leq \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{m}} \Gamma\left(\frac{p+2}{m}\right)\left(\Gamma\left(\frac{2}{m}\right)\right)^{-1}}.$$

In summary, the results for classical weighted Fock spaces can be summarised using the following table.

	α	p	**Lower Bound**	**Upper Bound**
$\mathcal{F}^p_\alpha(\mathbb{C})$ $\alpha > 0$ $1 \leq p < \infty$	$\alpha = 1$	$p = 2$	$2006 : \kappa_{\mathcal{F}^2} \geq 0.54$ $2006 : \kappa_{\mathcal{F}^2} \geq 0.7248$	$2020 : \kappa_{\mathcal{F}^2} \leq 1$
		$1 \leq p < \infty$	2012: Theorem 4.6	2020: Theorem 4.7
	$\alpha \neq 1$	$1 \leq p < \infty$		

Table 4. Development of Korenblum constants on $\mathcal{F}^p_\alpha(\mathbb{C})$.

5. Extension to Intersections of Weighted Fock spaces

In this section, we survey our extension of results from classical weighted Fock spaces to other weighted Fock spaces. To do this, we directed our attention to the Gamma function, which is one of the many special functions which satisfies the well-known Ramanujan's Master theorem. The Ramanujan's Master Theorem was first reported in Ramanujan's Quarterly Reports [2] and can be satisfied by many special functions, such as Mellin transform of Dirichlet series and Generalized Hypergeometric functions. Further details can be found in [1,2].

5.1. Preliminaries

The Ramanujan's Master theorem was first rigorously treated by G.H. Hardy in [9] whose proof relied on the Cauchy residue theorem and Mellin inversion theorem. Hardy proved that the Ramanujan's Master theorem can be satisfied for a sufficiently large class of functions that satisfies certain growth condition. We shall recall the Ramanujan's Master Theorem for Hardy's class of functions below.

Proposition 5.1 (Ramanujan's Master Theorem [1])**.** *Let $\omega(z)$ be a holomorphic and single-valued function defined on the half-plane $H(\delta) = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq -\delta\}$ for some $0 < \delta < 1$. Suppose that, there exist positive constants C, P and $A < \pi$ such that the growth condition*

$$|\omega(u + iv)| < Ce^{Pu+A|v|}, \tag{5.1}$$

holds for all $z = u + iv \in H(\delta)$. Then for all $0 < \mathrm{Re}(s) < \delta$,

$$\int_0^\infty x^{s-1}\{\omega(0) - x\omega(1) + x^2\omega(2) - \cdots\}\, dx = \frac{\pi}{\sin \pi s}\omega(-s). \tag{5.2}$$

Similar to many previous applications in Ramanujan's Master theorem, (5.2) is more commonly written as

$$\int_0^\infty x^{s-1} \sum_{k=0}^{\infty} \frac{\phi(k)}{k!}(-x)^k\, dx = \Gamma(s)\phi(-s). \tag{5.3}$$

Equation (5.3) acts as a valid integral identity for computing the Mellin transforms for particular functions ϕ, assuming that the series term on the left-hand side holds and the integral is convergent for some values of $\mathrm{Re}(s)$. Over the years, equation (5.3) has been applied conveniently and directly by Ramanujan and several authors.

It is also well-known that this integral transform has close relations to the theory of *Dirichlet series*. Let $0 < (\lambda_n) \uparrow \infty$ be a sequence of positive real numbers and (d_n) be a sequence of complex numbers. We now consider a Dirichlet series, with real frequencies (λ_n),

$$\sum_{n=1}^{\infty} d_n e^{-\lambda_n s}, \quad s \in \mathbb{C}.$$

It is well-known that if we let $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n}$, then in case $L < \infty$, the following inequalities must hold

$$\limsup_{n\to\infty} \frac{\log|d_n|}{\lambda_n} \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq \limsup_{n\to\infty} \frac{\log|d_n|}{\lambda_n} + L,$$

where $\sigma_c, \sigma_a, \sigma_u$ are abscissa of convergence, absolute convergence, or uniform convergence respectively. Readers may refer to the book [10] for more information regarding Dirichlet series.

In the interest of constructing weighted Fock spaces using Dirichlet series, we now restrict the Dirichlet series with non-negative coefficients as follows,

$$g(s) = \sum_{n=1}^{\infty} |d_n| e^{-\lambda_n s}, \; s \in \mathbb{C}, \quad \text{with } \sigma_c \leq 0. \tag{5.4}$$

Then the Dirichlet series $g(s)$ represents a holomorphic function on the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > \sigma_c\}$. Naturally, this class of Dirichlet series can be characterized into two classes [10].

(I) $\displaystyle\sum_{n=1}^{\infty} |d_n| = \infty$ and $\displaystyle\sigma_c = \limsup_{n\to\infty} \frac{\log(|d_1| + |d_2| + \cdots + |d_n|)}{\lambda_n} = 0.$

(II) $\displaystyle\sum_{n=1}^{\infty} |d_n| < \infty$ and $\displaystyle\sigma_c = \limsup_{n\to\infty} \frac{\log(|d_{n+1}| + |d_{n+2}| + \cdots)}{\lambda_n} \leq 0.$

The Dirichlet series itself comprises of several special functions that satisfies the Ramanujan's Master Theorem. Here, we list down the particular cases discussed in [33]:

- If $\lambda_n = F_{n+1}$ for all $n \in \mathbb{N}$, then $g(s)$ becomes the Fibonacci zeta-function $\displaystyle\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}$.
- If $\lambda_n = n$ and $d_n = 1$ for all $n \in \mathbb{N}$, then $g(s)$ is the Riemann Zeta function $\zeta(s)$.
- If $\lambda_n = n$ and $d_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$, then $g(s) = e^{e^{-s}} - 1$.

In addition, one can also consider (d_n) to be arithmetical functions such as the Euler Totient function or the partition function $p(n)$.

The main connection between Dirichlet series and the weighted Fock spaces discussed in [33] is due to the Mellin transform of the Dirichlet series. Historically, Cahen [6] and Perron [9] (see also [5, p. 327]) discovered the Mellin transform of Dirichlet series a long time ago. As an immediate consequence from their early works, we made full use of the fact that $g(s)$ satisfies the Mellin inversion theorem. In fact, the Mellin transform of $g(s)$ is a unique family of special functions satisfying (5.3) and also has many applications in theoretical computer science (see [8]).

Proposition 5.2 ([10]). *Let $s > 0$ and $g(x)$ be series* (5.4)*, $x \in \mathbb{R}$. Then the Mellin transform of $g(x)$ can be computed as*

$$\mathcal{M}(g; s) = \int_0^{\infty} x^{s-1} g(x)\, dx = \Gamma(s) \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^s}, \tag{5.5}$$

provided that the series on the right-hand side converges.

By establishing this connection, we proved the following identity which will be used to deal with our infinite intersections of weighted Fock spaces.

Proposition 5.3 ([33]). *Let s be a complex constant and $m \geq 0$. Then*

$$\sum_{n=1}^{\infty} |d_n| \int_{\mathbb{C}} |sz|^m e^{-\lambda_n |z|}\, dA(z) = \begin{cases} \infty, & \lambda_1 = 0. \\ 2\pi |s|^m \Gamma(m+2) \displaystyle\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{m+2}}, & \lambda_1 > 0, \end{cases}$$

provided that the series in the right-hand side above converges.

The following are immediate consequences of Proposition 5.3.

Corollary 5.1 ([33]). *Let s be a complex constant, (F_n) be the Fibonacci sequence, and $m \geq 0$. Then*

$$\sum_{n=1}^{\infty} \int_{\mathbb{C}} |sz|^m e^{-F_{n+1}} |z|\, dA(z) = 2\pi |s|^m \Gamma(m+2)[\zeta_F(m+2) - 1],$$

where $\zeta_F(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{F_n^s}$ is the Fibonacci zeta-function, provided that the series converges.

Corollary 5.2 ([33]). *Let s be a complex constant and $m \geq 0$. Then*

$$\sum_{n=1}^{\infty} |d_n| \int_{\mathbb{C}} |bz|^m e^{-n|z|}\, dA(z) = 2\pi |s|^m \Gamma(m+2) \sum_{n=1}^{\infty} \frac{|d_n|}{n^{m+2}}.$$

In particular, if $d_n = 1$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \int_{\mathbb{C}} |sz|^m e^{-n|z|}\, dA(z) = 2\pi |s|^m \Gamma(m+2) \zeta(m+2),$$

where $\zeta(\cdot)$ is the Riemann zeta-function, provided that the series converges.

In [33], we presented an interesting connection between Proposition 5.2, Corollary 5.2, and Generalized Hypergeometric functions. To do this, recall that the definition of Generalized Hypergeometric functions ${}_pF_q(\mathbf{c}; \mathbf{d}; z)$, where $p, q \in \mathbb{N}$ (see, e.g., [20]).

Definition 5.1. *Let $\mathbf{c} = (c_1, \cdots, c_p)$, $\mathbf{d} = (d_1, \cdots, d_q)$ be two real p- and q-tuples, $a^{(k)} = a(a+1)\cdots(a+k-1)$ be a real rising factorial. The Generalized Hypergeometric function is*

$$ {}_pF_q(\mathbf{c}; \mathbf{d}; z) = \sum_{k=0}^{\infty} \frac{c_1^{(k)} c_2^{(k)} \cdots c_p^{(k)}}{d_1^{(k)} d_2^{(k)} \cdots d_q^{(k)}} \frac{z^k}{k!}, \ z \in \mathbb{C}. \tag{5.6}$$

Proposition 5.4 ([33]). *Let s be a complex constant and $m \geq 0$. For the function $e^{e^{-r}} - 1$, by Proposition 5.2 and Corollary 5.2, we have*

$$\int_{\mathbb{C}} |sz|^m (e^{e^{-|z|}} - 1)\, dA(z) = \int_0^{2\pi} \int_0^{\infty} |s|^m r^{m+1} \sum_{n=1}^{\infty} \frac{1}{n!} e^{-nr}\, dr\, d\theta$$

$$= 2\pi |s|^m \Gamma(m+2) \sum_{n=1}^{\infty} \frac{1}{n! n^{m+2}} = \sum_{n=1}^{\infty} \int_{\mathbb{C}} |sz|^m e^{-n|z| + \log|\frac{1}{n!}|}\, dA(z).$$

In particular, if m is an integer, we have

$$\int_{\mathbb{C}} |sz|^m (e^{e^{-|z|}} - 1)\, dA(z) = 2\pi |s|^m \Gamma(m+2) \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{1}{k+1} \right)^{m+2}$$

$$= 2\pi |s|^m \Gamma(m+2) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1^{(k)}}{2^{(k)}} \right)^{m+3} = 2\pi |s|^m \Gamma(m+2)\, {}_{m+3}F_{m+3}(\mathbf{1}; \mathbf{2}; 1).$$

5.2. Korenblum Constants for Intersections of Weighted Fock Spaces

Using Proposition 5.3 and its corollaries, we defined the following weighted Fock space in [33].

Definition 5.2. *Let $0 < p < \infty$ and $\alpha > 0$. For a positive real number λ and a non-zero complex number d, we define the weighted Fock space*

$$\mathcal{F}^{p,\alpha}_{\lambda,d}(\mathbb{C}) := \left\{ f(z) \in \mathcal{O}(\mathbb{C}) : \|f\|^p_{\mathcal{F}^{p,\alpha}_{\lambda,d}} = \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}\lambda|z| + \log|d|}\, dA(z) < \infty \right\}.$$

Naturally, we proceeded to obtain an upper bound of Korenblum constants for the above weighted Fock space $\mathcal{F}^{p,\alpha}_{\lambda,d}(\mathbb{C})$.

Theorem 5.5 ([33]). *Let*

$$c > \frac{2}{p\alpha\lambda} \sqrt[p]{\Gamma(p+2)}. \tag{5.7}$$

Then there exist functions f and g in $\mathcal{F}^{p,\alpha}_{\lambda,d}(\mathbb{C})$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but $\|f\|^p_{\mathcal{F}^{p,\alpha}_{\lambda,d}} > \|g\|^p_{\mathcal{F}^{p,\alpha}_{\lambda,d}}$. Therefore,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\lambda,d}} \leq \frac{2}{p\alpha\lambda} \sqrt[p]{\Gamma(p+2)}.$$

With particular values of parameters, Theorem 5.5 gives interesting estimates. For example, if p is a positive integer, then we have

$$\kappa_{\mathcal{F}^{p,\alpha}_{\lambda,d}} \leq \frac{2}{p\alpha\lambda} \sqrt[p]{(p+1)!}.$$

In the case when $\lambda = \pi^{-\frac{2}{3}}$, $p = \frac{3}{2}$ and $\alpha = \frac{\sqrt[3]{225}}{3}$, we also have

$$\kappa_{\mathcal{F}^{p,\alpha}_{\lambda,d}} \le \pi.$$

Finally, we built a sequence for these weighted Fock spaces and consider intersections of these spaces, i.e., let $0 < (\lambda_n) \uparrow \infty$ and (d_n) be a sequence of non-zero complex numbers. Each pair (λ_n, d_n) defines a weighted Fock space $\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}(\mathbb{C})$, for which, by Theorem 5.5, the following is an upper bound estimate for its Korenblum's constant,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}} \le \frac{2}{p\alpha\lambda_n} \sqrt[p]{\Gamma(p+2)},\ n \in \mathbb{N}. \tag{5.8}$$

From here, we first considered the finite intersection of the above weighted Fock spaces. We presented the case for an intersection of two spaces which can be written as

$$\mathcal{F}^{p,\alpha}_{i,j} = \mathcal{F}^{p,\alpha}_{\lambda_i,d_i} \bigcap \mathcal{F}^{p,\alpha}_{\lambda_j,d_j} \quad (i < j),$$

endowed with the topology given by the norm

$$\|f\|_{\mathcal{F}^{p,\alpha}_{i,j}} := \max\left\{\|f\|_{\mathcal{F}^{p,\alpha}_{\lambda_i,d_i}}, \|f\|_{\mathcal{F}^{p,\alpha}_{\lambda_j,d_j}}\right\}. \tag{5.9}$$

Note that the space $\mathcal{F}^{p,\alpha}_{i,j}$ is a Banach space with the norm above.

With standard arguments, we obtained the following upper bounds for Korenblum constant of $\mathcal{F}^{p,\alpha}_{i,j}$.

Theorem 5.6 ([33]). *Let*

$$c > \begin{cases} \frac{2}{p\alpha\lambda_i} \sqrt[p]{\Gamma(p+2)}, & |d_i| \ge |d_j| \\ \frac{2}{p\alpha\lambda_i} \sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} \ge \frac{|d_j|}{\lambda_j^{p+2}} \\ \frac{2}{p\alpha\lambda_j} \sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} < \frac{|d_j|}{\lambda_j^{p+2}}. \end{cases} \tag{5.10}$$

Then there exist functions f and g in $\mathcal{F}^{p,\alpha}_{i,j}$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but

$$\|f\|_{\mathcal{F}^{p,\alpha}_{i,j}} > \|g\|_{\mathcal{F}^{p,\alpha}_{i,j}}.$$

Therefore,

$$\kappa_{\mathcal{F}^{p,\alpha}_{i,j}} \le \begin{cases} \frac{2}{p\alpha\lambda_i} \sqrt[p]{\Gamma(p+2)}, & |d_i| \ge |d_j| \\ \frac{2}{p\alpha\lambda_i} \sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} \ge \frac{|d_j|}{\lambda_j^{p+2}} \\ \frac{2}{p\alpha\lambda_j} \sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} < \frac{|d_j|}{\lambda_j^{p+2}}. \end{cases}$$

Next, we considered the Korenblum constants for an infinite intersection of spaces $\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}$ $(n \in \mathbb{N})$,

$$\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}} = \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\|^p_{\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}} < \infty, \text{ for every } n \in \mathbb{N} \right\},$$

endowed with the topology given by the series of norms

$$\|f\|^p_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}} = \sum_{n=1}^{\infty} \|f\|^p_{\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}} < \infty.$$

For the above intersection, we have to ensure that it is non-empty as otherwise, it would be trivial to consider its Korenblum constants. Fortunately, if we want at least the simple constant function $f(z) = c \in \mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}$, we can have its norm

$$\sum_{n=1}^{\infty} \|f\|^p_{\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}} = 2\pi c^p \left(\frac{2}{p\alpha}\right)^2 \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} < \infty,$$

by assuming that

$$\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} < \infty. \tag{5.11}$$

As mentioned previously, the pair of sequences (λ_n) and (d_n) contributes to a large class of special functions in Dirichlet series. Hence, there exist many pairs (λ_n, d_n) for which condition (5.11) may or may not hold. For example, if $\lambda_n = n$, we take $d_n = n^\rho$, then condition (5.11) is satisfied, if $\rho < 1$, and it is not satisfied, if $\rho \geq 1$.

Note also that since $\exists n_0$ such that $\lambda_{n_0} \geq 1$, then $\lambda_{n_0}^{kp} \geq 1$ for any $0 < p < \infty$ and $k \in \mathbb{N}$. Hence, $\lambda_{n_0}^{kp+2} \geq \lambda_{n_0}^2$. Then for all $n \geq n_0$, $\lambda_n \geq 1$ implies $\lambda_n^{kp+2} \geq \lambda_n^2$. As a result, condition (5.11) implies

$$\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{kp+2}} < \infty. \tag{5.12}$$

This also allows all polynomials $z^k \in \mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}$.

In the rest of this section, we assume that condition (5.11) holds.

Then the space $\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}$ always contain a constant function $f(z) = c$ and all polynomial in z.

By using Proposition 5.3, we obtained an interesting upper bound for $\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}}$ in terms of Dirichlet series and Gamma functions.

Theorem 5.7 ([33]). *Let* $0 < (\lambda_n) \uparrow \infty$ *and* (d_n) *be a sequence of non-zero complex numbers satisfying condition* (5.11). *Suppose*

$$c > \frac{2}{p\alpha} \sqrt[p]{\Gamma(p+2) \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{p+2}} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \right)^{-1}}, \tag{5.13}$$

then there exist functions f *and* g *in* $\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}$, *such that* $|f(z)| < |g(z)|$ *for any* $|z| > c$, *but*

$$\|f\|_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}} > \|g\|_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}}.$$

Therefore,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\Gamma(p+2) \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{p+2}} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \right)^{-1}}.$$

Following the particular cases of Dirichlet series, we also obtained the following special cases of Theorem 5.7 in [33].

Corollary 5.3 ([33]). *Let* $d_n = 1$ *and* (F_n) *be the Fibonacci sequence. If the Korenblum Maximum Principle holds for the space* $\mathcal{F}^{p,\alpha}_{\{F_{n+1},1\}}$, *then*

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{F_{n+1},1\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\Gamma(p+2) \frac{\zeta_F(p+2) - 1}{\zeta_F(2) - 1}},$$

where $\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}$ *is the Fibonacci zeta function.*

Corollary 5.4 ([33]). *Let* $\lambda_n = n$ *for all* $n \in \mathbb{N}$. *If the Korenblum Maximum Principle holds for the space* $\mathcal{F}^{p,\alpha}_{\{n,d_n\}}$, *then*

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{n,d_n\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\Gamma(p+2) \sum_{n=1}^{\infty} \frac{|d_n|}{n^{p+2}} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{n^2} \right)^{-1}}, \tag{5.14}$$

provided that a series $\sum_{n=1}^{\infty} \frac{|d_n|}{n^2}$ *converges.*

For Corollary 5.4, if $d_n = 1$ for all $n \in \mathbb{N}$, we get a corollary which involves the Riemann zeta function $\zeta(z)$.

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{n,1\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\frac{6}{\pi^2} \Gamma(p+2) \zeta(p+2)}. \tag{5.15}$$

On the other hand, if $d_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$, we use Proposition 5.4 to get this corollary which involves the Generalized Hypergeometric functions.

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{n,\frac{1}{n!}\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\frac{\Gamma(p+2)}{{}_3F_3(\mathbf{1};\mathbf{2};1)} \sum_{n=1}^{\infty} \frac{1}{n! n^{p+2}}}.$$

In addition, when p is an integer, we have

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{n,\frac{1}{n!}\}}} \leq \frac{2}{p\alpha} \sqrt[p]{\frac{\Gamma(p+2)_{p+3}F_{p+3}(\mathbf{1};\mathbf{2};1)}{{}_3F_3(\mathbf{1};\mathbf{2};1)}}.$$

Combining definitions 4.1 and 5.2 of two types of weighted Fock spaces together with Theorem 5.7 led us to generalise them to the following slightly more complicated weighted Fock spaces in [33].

Definition 5.3. *Let $0 < p < \infty$, $\alpha > 0$ and $m > 0$. For each $n \in \mathbb{N}$, the weighted Fock spaces $\mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}(\mathbb{C})$ consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$ for which*

$$\|f\|^p_{\mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}} = \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}\lambda_n |z|^m + \log|d_n|} \, dA(z) < \infty.$$

Similarly, we extended our results to the infinite intersections of weighted Fock spaces $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}} = \bigcap_{n=1}^{\infty} \mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}(\mathbb{C})$. $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}$ consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$ such that $\|f\|^p_{\mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}} < \infty$ for all $n \in \mathbb{N}$, endowed with the topology given by the series of norms

$$\|f\|^p_{\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}} := \sum_{n=1}^{\infty} \|f\|^p_{\mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}} < \infty.$$

Following condition (5.11), we shall assume that the triple $\{m, d_n, \lambda_n\}$ must fulfill the condition

$$\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{2/m}} < \infty, \tag{5.16}$$

which would imply that for any $k \in \mathbb{N}$ and $0 < p < \infty$,

$$\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{\frac{kp+2}{m}}} < \infty.$$

This would mean that the space $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}$ is also non-empty, as it contains the constant function c and all polynomial in z.

Our final upper bound in [33] is for the case $\lambda_n = n^m$ and $d_n = 1$ for all $n \in \mathbb{N}$. Clearly, for any $m > 0$, $0 < p < \infty$ and $\alpha > 0$, condition (5.16) must remain satisfied here. With a combination of proofs from both Theorem 4.6 and Theorem 5.7, the upper bound for the Korenblum constant of the space $\mathcal{F}^{p,\alpha}_{\{m,n^m,1\}}$ is

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{m,n^m,1\}}} \leq \left(\frac{2}{p\alpha}\right)^{\frac{1}{m}} \sqrt[p]{\frac{6}{\pi^2}\zeta(p+2)\Gamma\left(\frac{p+2}{m}\right)\Gamma\left(\frac{2}{m}\right)^{-1}}.$$

We summarize the results surveyed in this section. In this section, the following main results are surveyed.

(i) For $\mathcal{F}^{p,\alpha}_m(\mathbb{C})$,

$$\kappa_{\mathcal{F}^{p,\alpha}_m} \leq \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{m}}\Gamma\left(\frac{p+2}{m}\right)\left(\Gamma\left(\frac{2}{m}\right)\right)^{-1}}.$$

(ii) For $\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}(\mathbb{C})$,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}} \leq \frac{2}{p\alpha\lambda_n}\sqrt[p]{\Gamma(p+2)}.$$

(iii) For $\mathcal{F}^{p,\alpha}_{i,j}(\mathbb{C})$,

$$\kappa_{\mathcal{F}^{p,\alpha}_{i,j}} \leq \begin{cases} \frac{2}{p\alpha\lambda_i}\sqrt[p]{\Gamma(p+2)}, & |d_i| \geq |d_j| \\ \frac{2}{p\alpha\lambda_i}\sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} \geq \frac{|d_j|}{\lambda_j^{p+2}} \\ \frac{2}{p\alpha\lambda_j}\sqrt[p]{\Gamma(p+2)}, & |d_i| < |d_j| \quad \text{and} \quad \frac{|d_i|}{\lambda_i^{p+2}} < \frac{|d_j|}{\lambda_j^{p+2}}. \end{cases}$$

(iv) For $\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}$,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}} \leq \frac{2}{p\alpha}\sqrt[p]{\Gamma(p+2)\sum_{n=1}^{\infty}\frac{|d_n|}{\lambda_n^{p+2}}\left(\sum_{n=1}^{\infty}\frac{|d_n|}{\lambda_n^2}\right)^{-1}}.$$

(v) Lastly, for $\mathcal{F}^{p,\alpha}_{\{m,n^m,1\}}$, $m > 0$ and $\alpha > 0$,

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{m,n^m,1\}}} \leq \left(\frac{2}{p\alpha}\right)^{\frac{1}{m}} \sqrt[p]{\frac{6}{\pi^2}\zeta(p+2)\Gamma\left(\frac{p+2}{m}\right)\Gamma\left(\frac{2}{m}\right)^{-1}}.$$

6. Failure of Korenblum Maximum Principle

In this section, we survey the remaining results from all function spaces having a failure of Korenblum Maximum Principle. The failure of Korenblum Maximum Principle was first discovered for the Bloch space B by Jiang, Prajitura and Zhao in [14]. In fact, the Korenblum constant does not exist even if E is the whole unit disc $\mathbb{D}$ in the Bloch space B. In other words, the whole unit disc $\mathbb{D}$ cannot even be a dominating set for the Bloch space B.

Theorem 6.1 ([14]). *Let $g(z) = z^3 + z$ and $f(z) = zg(z) = z^4 + z^2$. Then $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{D}$, but $\|f\|_B > \|g\|_B$.*

In the recent years, there is a regained interest in Korenblum constants which should be largely attributed to the failure of Korenblum Maximum Principle for $A^p(\mathbb{D})$, $0 < p < 1$ reported in 2018. In 2018, Vladimir Božin and Karapetrović [3] discovered a complete failure in Korenblum Maximum Principle for Bergman space $A^p(\mathbb{D})$, $0 < p < 1$. In other words, no Korenblum constant exist to satisfy the Korenblum Maximum Principle for Bergman spaces as long as $0 < p < 1$. This result closes a research gap from Hinkkanen where he first proved the existence of κ_{A^p} for $A^p(\mathbb{D})$ but for $p \geq 1$. Note that Proposition 6.3 is a consequence of Proposition 6.2.

Proposition 6.2 ([3]). *Let $0 < p < 1$ and $0 < c < 1$. Then, there exist $n \in \mathbb{N}$ and $0 < \varepsilon < 1$ such that*

$$1 + \frac{np}{2}\varepsilon^{np+2} > \left(1 + \left(\frac{\varepsilon}{c}\right)^n\right)^p.$$

Theorem 6.3 ([3]). *Let $0 < p < 1$ and $0 < c < 1$. Then there exist functions f and g in $A^p(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$ and*

$$||f||_{A^p} > ||g||_{A^p}.$$

Shortly after, Lou and Hu [13] disproved the principle for classical weighted Fock spaces $\mathcal{F}^p_\alpha(\mathbb{C})$ where $0 < p < 1$, $\alpha > 0$ using similar methods. In particular, the following proposition is used to disprove the principle instead.

Proposition 6.4 ([13]). *Let $0 < p < 1$ and $\alpha > 0$. Suppose $c > 0$. Then there* *exist positive integer n and $0<\rho<\infty$, such that*

$$2\rho^{np+2}\left(\int_0^1 ue^{-\frac{p\alpha}{2}\rho^2u^2}\,du+\int_1^\infty u^{np+1}e^{-\frac{p\alpha}{2}\rho^2u^2}\,du\right)$$
$$>\left(1+\left(\frac{\rho}{c}\right)^n\right)^p\left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1}\Gamma\left(\frac{np}{2}+1\right).$$

Lou and Hu [13] then obtained the following result.

Proposition 6.5 ([13])**.** *Let $0<p<1$ and $\alpha>0$. Suppose $c>0$. Then there exist functions f and g in $\mathcal{F}^p_\alpha(\mathbb{C})$ such that $|f(z)|<|g(z)|$ for all $|z|>c$ and*

$$||f||_{\mathcal{F}^p_\alpha}>||g||_{\mathcal{F}^p_\alpha}.$$

The results about the failures of Korenblum Maximum Principle in $A^p(\mathbb{D})$, $0<p<1$ [3] and in $\mathcal{F}^p_\alpha(\mathbb{C})$, $0<p<1$, $\alpha>0$ [13] inspired us to investigate whether there are any failures of the Korenblum Maximum Principle for the weighted Bergman space with exponential weights $A^p_\gamma(\mathbb{D})$, $0<p<1$, $\gamma\neq 0$. Similarly, we showed that such a failure exist and the result below not only proves this fact, but also generalizes Theorem 6.3 for any $\gamma>0$ [32].

Proposition 6.6 ([32])**.** *Let $0<p<1$, $\gamma>0$ and $0<c<1$. Then there exist positive integer n and $0<\delta<1$, such that*

$$2\delta^{np+2}\left(\int_0^1 ue^{-\frac{p\gamma}{2}\delta^2u^2}\,du+\int_1^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\gamma}{2}\delta^2u^2}\,du\right)$$
$$>\left(1+\left(\frac{\delta}{c}\right)^n\right)^p\left(\frac{2}{p\gamma}\right)^{\frac{np}{2}+1}\int_0^{\frac{p\gamma}{2}} u^{\frac{np}{2}}e^{-u}\,du.$$

Theorem 6.7 ([32])**.** *Let $0<p<1$ and $\gamma>0$. Suppose $0<c<1$. Then there exist functions f and g in $A^p_\gamma(\mathbb{D})$ such that $|f(z)|<|g(z)|$ for any z with $c<|z|<1$ and $\|f\|_{A^p_\gamma}>\|g\|_{A^p_\gamma}$.*

Recently, we turned our attention back to the weighted Fock spaces and answer whether the Korenblum constant exist even for small intersections of weighted Fock spaces. However, we found that the principle fails for the infinite intersection $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}(\mathbb{C})$ when $0<p<1$ and $m\geq 2$ [33]. This also left a new open problem as to whether the principle fails for $0<m<2$.

The following proposition is a generalisation of Propostion 6.4.

Proposition 6.8 ([33])**.** *Let $0<p<1$, $\alpha>0$, $m\geq 2$, $0<(\lambda_n)\uparrow\infty$ and $c>0$. Then there exist a positive integer k and $0<\delta<1$ such that for all*

$n \in \mathbb{N}$,

$$m\delta^{kp+2}\bigg(\int_0^1 ue^{-\frac{p\alpha}{2}\lambda_n\delta^m u^m}\,du + \int_1^\infty u^{kp+1}e^{-\frac{p\alpha}{2}\lambda_n\delta^m u^m}\,du\bigg) > \bigg(1+\Big(\frac{\delta}{c}\Big)^k\bigg)^p\Big(\frac{2}{p\alpha\lambda_n}\Big)^{(\frac{kp+2}{m})}\Gamma\Big(\frac{kp+2}{m}\Big).$$

Thus, we have the following result.

Theorem 6.9 ([33]). *Let $0 < p < 1$, $\alpha > 0$, $0 < (\lambda_n) \uparrow \infty$, $m \geq 2$ and (d_n) be a sequence of non-zero complex numbers. Suppose $c > 0$. Then there exist functions f and g in $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}(\mathbb{C})$ such that $|f(z)| < |g(z)|$ for any z with $|z| > c$ and $\|f\|_{\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}} > \|g\|_{\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}}$.*

Remark 6.1. *In [33], we make note that Lemma 6.8 can be slightly modified or relaxed accordingly for finite intersection or individual spaces such as $\mathcal{F}^{p,\alpha}_{m,\lambda_i,d_i}(\mathbb{C})\bigcap\mathcal{F}^{p,\alpha}_{m,\lambda_j,d_j}(\mathbb{C})$ and $\mathcal{F}^{p,\alpha}_{m,\lambda_n,d_n}(\mathbb{C})$ respectively. For these spaces, the proof for the failure of Korenblum constants when $0 < p < 1$ and $m \geq 2$ can be similarly proven just like Theorem 6.9. We refer the reader to [4] for further details.*

Instead of working towards smaller spaces, a recent failure of Korenblum maximum principle was also extended to the mixed norm space $H^{p,q,s}$ where $0 < p, q, s < \infty$ by Karapetrović [4]. Recall that the mixed norm space $H^{p,q,s}$ $(0 < p, q, s < \infty)$ consists of all holomorphic functions in $\mathcal{O}(\mathbb{D})$ for which

$$\|f\|_{H^{p,q,s}} = \bigg(2sq\int_0^1 r(1-r^2)^{sq-1}M_p^q(r,f)\,dr\bigg)^{1/q} < \infty, \tag{6.1}$$

where

$$M_p(r,f) = \bigg(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\bigg)^{1/p}.$$

Interestingly, this latest failure of Korenblum Maximum Principle was extended for the space $H^{p,q,s}$ where $0 < p, s < \infty$ and $0 < q < 1$.

Theorem 6.10 ([4]). *Let $0 < p, s < \infty$, $0 < q < 1$ and $0 < c < 1$. Then there exist functions f and g holomorphic on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{H^{p,q,s}} > \|g\|_{H^{p,q,s}}$.*

7. Future Directions and Open Questions

7.1. Future Directions

In [31], a proposed future direction is to study the Korenblum constants for general function spaces with series norms. Recall that G is a domain and we shall write $\mathcal{O}(G)$ as the set of holomorphic functions defined on G. Then we define $\beta = (\beta_k)$ as a sequence of positive real numbers. This allows us to define the function space

$$\mathcal{H}(G,\beta) := \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{O}(G) : ||f||^2_{O(G)} = \sum_{k=0}^{\infty} |a_k|^2 \beta_k^2 < \infty \right\}.$$

When $G = \mathbb{C}$, we have the Hilbert space of entire functions. Further, if $\beta = (\sqrt{k!})_{k\in\mathbb{N}}$, we have the classical Fock space $\mathcal{F}^2(\mathbb{C})$. If $G = \mathbb{D}$, we have $H^2(\beta)$. In the case $\beta = (1)_{k\in\mathbb{N}}$, it becomes the classical Hardy space $H^2(\mathbb{D})$; if $\beta = (\frac{1}{\sqrt{k+1}})_{k\in\mathbb{N}}$, it becomes the classical Bergman space $A^2(\mathbb{D})$; if $\beta = (\sqrt{k+1})_{k\in\mathbb{N}}$, we then have the Dirichlet space $\mathcal{D}(\mathbb{D})$. If we consolidate the development of Korenblum constants under $\mathcal{H}(G,\beta)$, we obtain Table 5.

	G	-	$\beta = (\beta_k)$	**Function Space**	**Current Development**
$\mathcal{H}(G,\beta)$	$\mathbb{C}$	Hilbert Space of Entire Functions	$\sqrt{k!}$	$\mathcal{F}^2(\mathbb{C})$	$0.7248 \leq \kappa_{\mathcal{F}^2} \leq 1$
	$\mathbb{D}$	$H^2(\beta)$	1	$H^2(\mathbb{D})$	(see Table 4)
			$\frac{1}{\sqrt{k+1}}$	$A^2(\mathbb{D})$	$0.28185 \leq \kappa_{A^2} < 0.67795$
			$\sqrt{k+1}$	$\mathcal{D}(\mathbb{D})$	No Specific Development but $\mathcal{D}(\mathbb{D}) \subset H^2(\mathbb{D})$

Table 5. Current Development on $\mathcal{H}(G,\beta)$

7.2. Open Questions

1. For the weighted Bergman spaces, the following open questions call for investigation. The open questions were previously mentioned in [32].

Question 7.1 ([32])**.** *Let $p \geq 1$, let $\gamma \geq 0$ and let*

$$c = \begin{cases} \left(\frac{2}{p+2}\right)^{\frac{1}{p}}, & \gamma = 0, \\ \sqrt[p]{\dfrac{\left(\frac{2}{p\gamma}\right)^{\frac{p}{2}} \int_0^{\frac{p\gamma}{2}} u^{\frac{p}{2}} e^{-u} du}{(1-e^{-\frac{p\gamma}{2}})}}, & \gamma > 0. \end{cases}$$

Does there exist functions $f(z)$ and $g(z)$ in $A^p_\gamma(\mathbb{D})$ for which $|f(z)| < |g(z)|$ with $c < |z| < 1$ and $\|f\|_{A^p_\gamma} > \|g\|_{A^p_\gamma}$?

Question 7.2 ([32])**.** *Let $-1 < \gamma < 0$ and $1 \le p < \infty$. Does there exist functions $f(z)$ and $g(z)$ in $A^p_\gamma(\mathbb{D})$ for which $|f(z)| < |g(z)|$ with $c < |z| < 1$ and $\|f\|_{A^p_\gamma} > \|g\|_{A^p_\gamma}$?*

2. For the weighted Hardy spaces, the following question calls for investigation.

Question 7.3. *Can Theorem 4.1 be generalised for $H^p_\alpha(\mathbb{D})$, $\alpha > 0$, $0 < p \le \infty$? In addition, how does Theorem 4.1 change with respect to φ?*

3. For the intersection of weighted Fock spaces, we have the following open questions from [33].

Question 7.4. *It would be interesting to know whether the upper bound of $\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}}$ in Theorem 5.7 is always less than $\kappa_{\mathcal{F}^{p,\alpha}_{\lambda_1,d_1}}$. This is because the upper bound of $\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}}$ is always less than the upper bound of $\kappa_{\mathcal{F}^{p,\alpha}_{\lambda_1,d_1}}$. To see this, for $0 < \lambda_1 < \lambda_n$ for all $n \ge 2$, $\lambda_n^{p+2} > \lambda_1^p \lambda_n^2$ which shows that*

$$\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{p+2}} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \right)^{-1} < \frac{1}{\lambda_1^p} \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \right)^{-1} = \frac{1}{\lambda_1^p} < \infty.$$

Hence, we have

$$\kappa_{\mathcal{F}^{p,\alpha}_{\{\lambda_n,d_n\}}} \le \frac{2}{p\alpha} \sqrt[p]{\Gamma(p+2) \sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^{p+2}} \left(\sum_{n=1}^{\infty} \frac{|d_n|}{\lambda_n^2} \right)^{-1}} < \frac{2}{p\alpha\lambda_1} \sqrt[p]{\Gamma(p+2)}.$$

Question 7.5. *Does there exist any relationship between $\kappa_{\mathcal{F}^{p,\alpha}_{\{m,n^m,1\}}}$ and $\kappa_{\mathcal{F}^{p,\alpha}_m}$?*

Question 7.6. *What is an upper bound for finite intersection of more than two spaces $\mathcal{F}^{p,\alpha}_{\lambda_n,d_n}(\mathbb{C})$, etc?*

Question 7.7. *Is it true that the principle still fails for $\mathcal{F}^{p,\alpha}_{\{m,\lambda_n,d_n\}}(\mathbb{C})$ when $0 < p < 1$ and $0 < m < 2$, in particular, $m = 1$?*

4. Following Theorem 6.1, we propose the following open question for the weighted Bloch spaces B_φ.

Question 7.8. *Does there exist functions $f(z)$ and $g(z)$ in B_φ such that $|f(z)| \le |g(z)|$ for all $z \in \mathbb{D}$, but $\|f\|_{B_\varphi} > \|g\|_{B_\varphi}$?*

5. For the function space $\mathcal{H}(G, \beta)$, we propose the following open question.

   Question 7.9. *Is it possible to generalise the results for Korenblum constants under the spaces $H^2(\mathbb{D})$, $A^2(\mathbb{D})$ and $\mathcal{D}(\mathbb{D})$ under $H^2(\beta)$?*

   In general, the main challenge along this direction would be that the space $\mathcal{H}(G, \beta)$ involves norms in series notations while most of the function spaces discussed in this survey deals with integral norms.

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