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On the real-analytic infinitesimal CR automorphism of hypersurfaces of infinite type

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Abstract. We consider a real smooth hypersurface $M \subset \mathbb{C}^2$, which is of D'Angelo infinite type at $p \in M$. The purpose of this paper is to show that the real vector space of tangential holomorphic vector field germs at p vanishing at p is either trivial or of real dimension 1.

1. Introduction

Let (M, p) be a real $\mathcal{C}^1$-smooth hypersurface germ at $p \in \mathbb{C}^n$. A smooth vector field germ (X, p) on M is called *a real-analytic infinitesimal CR automorphism germ at p of M* if there exists a holomorphic vector field germ (H, p) in $\mathbb{C}^n$ such that H is tangent to M, i.e. Re H is tangent to M, and $X = \text{Re } H\,|_M$. We denote by $\text{hol}_0(M, p)$ the real vector space of holomorphic vector field germs (H, p) vanishing at p which are tangent to M.

For a real hypersurface in $\mathbb{C}^n$, the real-analytic infinitesimal CR automorphism is not easy to describe explicitly; besides, it is unknown in most cases. For instance, the study of $\text{hol}_0(M, p)$ of various hypersurfaces is given in [1, 3, 7, 10, 11]. However, these results are known for Levi nondegenerate hypersurfaces or more generally for Levi degenerate hypersurfaces of finite

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type. For various real $\mathcal{C}^\infty$-smooth hypersurfaces of D'Angelo infinite type in $\mathbb{C}^2$, explicit descriptions of $\mathrm{hol}_0(M,p)$ are given in [2, 8, 9].

In this paper we shall prove that $\mathrm{hol}_0(M,p)$ of a certain hypersurface of D'Angelo infinite type in $\mathbb{C}^2$ is either trivial or of real dimension 1. To state the result explicitly, we need some notations and a definition. Taking the risk of confusion we employ the notations

$$P'(z) = P_z(z) = \frac{\partial P}{\partial z}(z),\ f_z(z,t) = \frac{\partial f}{\partial z}(z,t),\ f_t(z,t) = \frac{\partial f}{\partial t}(z,t)$$

throughout the article. Also denote by $\Delta_r = \{z \in \mathbb{C} \colon |z| < r\}$ for $r > 0$ and by $\Delta = \Delta_1$. A function f defined on Δ_r $(r > 0)$ is called to be *flat* at the origin if $f(z) = o(|z|^n)$ for each $n \in \mathbb{N}$ (cf. Definition 2.1).

The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let $(M,0)$ be a real $\mathcal{C}^1$-smooth hypersurface germ at 0 defined by the equation $\rho(z) := \rho(z_1,z_2) = \mathrm{Re}\ z_1 + P(z_2) + \mathrm{Im}\ z_1 Q(z_2, \mathrm{Im}\ z_1) = 0$ satisfying the conditions:*

(1) P,Q are $\mathcal{C}^1$-smooth with $P(0) = Q(0,0) = 0$,

(2) $P(z_2) > 0$ for any $z_2 \neq 0$, and

(3) $P(z_2), P'(z_2)$ are flat at $z_2 = 0$.

Then $\dim_{\mathbb{R}} \mathrm{hol}_0(M,p) \leq 1$.

Remark 1.1. *When P,Q are $\mathcal{C}^\infty$-smooth, the condition* (3) *simply says that P vanishes to infinite order at 0 and moreover 0 is a point of D'Angelo infinite type.*

In the case M is *radially symmetric in* z_2, i.e. $P(z_2) = P(|z_2|)$ and $Q(z_2,t) = Q(|z_2|,t)$ for any z_2 and t, it is well-known that $iz_2\frac{\partial}{\partial z_2}$ is tangent to M (see cf. [2]). Therefore, by Theorem 1.1 one gets the following corollary, which is a slight generalization of the main result in [2].

Corollary 1.1. *Let $(M,0)$ be a real $\mathcal{C}^1$-smooth hypersurface germ at 0 defined by the equation $\rho(z) := \rho(z_1,z_2) = \mathrm{Re}\ z_1 + P(z_2) + \mathrm{Im}\ z_1 Q(z_2, \mathrm{Im}\ z_1) = 0$ satisfying the conditions:*

(1) P,Q are $\mathcal{C}^1$-smooth with $P(0) = Q(0,0) = 0$,

(2) $P(z_2) = P(|z_2|),\ Q(z_2,t) = Q(|z_2|,t)$ for any z_2 and t,

(3) $P(z_2) > 0$ for any $z_2 \neq 0$, and

(4) $P(z_2), P'(z_2)$ are flat at $z_2 = 0$.

Then $\mathrm{hol}_0(M,0)=\{i\beta z_2\frac{\partial}{\partial z_2}:\beta\in\mathbb{R}\}$.

Next, we shall give an explicit description for real-analytic infinitesimal CR automorphisms of another class of real hypersurfaces in $\mathbb{C}^2$.

Let $a(z)=\sum_{n=1}^\infty a_n z^n$ be a nonzero holomorphic function defined on Δ_{ϵ_0}, ($\epsilon_0>0$) and let p,q be $\mathcal{C}^1$-smooth functions defined respectively on $(0,\epsilon_0)$ and $[0,\epsilon_0)$ satisfying that $q(0)=0$ and that $g(z),g'(z)$ are flat at 0, where g is a $\mathcal{C}^1$-smooth function given by

$$g(z)=\begin{cases} e^{p(|z|)} & \text{if } 0<|z|<\epsilon_0\\ 0 & \text{if } z=0.\end{cases}$$

Denote by $M(a,\alpha,p,q)$ the germ at $(0,0)$ of a real hypersurface defined by

$$\rho(z_1,z_2):=\mathrm{Re}\ z_1+P(z_2)+f(z_2,\mathrm{Im}\ z_1)=0,$$

where f and P are respectively defined on $\Delta_{\epsilon_0}\times(-\delta_0,\delta_0)$ ($\delta_0>0$ small enough) and Δ_{ϵ_0} by

$$f(z_2,t)=\begin{cases} -\frac{1}{\alpha}\log\left|\frac{\cos\big(R(z_2)+\alpha t\big)}{\cos(R(z_2))}\right| & \text{if } \alpha\neq 0\\ \tan(R(z_2))t & \text{if } \alpha=0,\end{cases}$$

where $R(z_2)=q(|z_2|)-\mathrm{Re}\big(\sum_{n=1}^\infty\frac{a_n}{n}z_2^n\big)$ for all $z_2\in\Delta_{\epsilon_0}$, and

$$P(z_2)=\begin{cases} \frac{1}{\alpha}\log\Big[1+\alpha P_1(z_2)\Big] & \text{if } \alpha\neq 0\\ P_1(z_2) & \text{if } \alpha=0,\end{cases}$$

where

$$P_1(z_2)=\exp\Big[p(|z_2|)+\mathrm{Re}\Big(\sum_{n=1}^\infty\frac{a_n}{in}z_2^n\Big)-\log\big|\cos\big(R(z_2)\big)\big|\Big]$$

for all $z_2\in\Delta^*_{\epsilon_0}$ and $P_1(0)=0$.

It is easily checked that $M(a,\alpha,p,q)$ is $\mathcal{C}^1$-smooth and moreover $P(z_2),P'(z_2)$ are flat at 0. Furthermore, we note that q,p can be chosen, e.g. $q(t)=0$ and $p(t)=-\frac{1}{t^\alpha}$ ($\alpha>0$) for all $t>0$, so that P,R are $\mathcal{C}^\infty$-smooth in Δ_{ϵ_0} and P is flat at 0, and hence $M(a,\alpha,p,q)$ is $\mathcal{C}^\infty$-smooth and of D'Angelo infinite type.

It follows from Theorem 4.1 in Appendix that the holomorphic vector field

$$H^{a,\alpha}(z_1,z_2):=L^\alpha(z_1)a(z_2)\frac{\partial}{\partial z_1}+iz_2\frac{\partial}{\partial z_2},$$

where

$$L^\alpha(z_1)=\begin{cases} \frac{1}{\alpha}\big(\exp(\alpha z_1)-1\big) & \text{if } \alpha\neq 0\\ z_1 & \text{if } \alpha=0,\end{cases}$$

is tangent to $M(a,\alpha,p,q)$. Hence, by Theorem 1.1 we obtain following corollary.

Corollary 1.2. $\operatorname{hol}_0\big(M(a,\alpha,p,q),0\big)=\{\beta H^{a,\alpha}\colon \beta\in\mathbb{R}\}$.

This paper is organized as follows. In Section 2, we recall several definitions and give several technical lemmas. Next, the proof of Theorem 1.1 is given in Section 3. Finally, a theorem is pointed out in Appendix.

2. Preliminaries

In this section, we shall recall several definitions and introduce two technical lemmas used in the proof of Theorem 1.1. In what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant. In addition, we use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

Definition 2.1. *A function $f:\Delta_{\epsilon_0}\to\mathbb{C}$ ($\epsilon_0>0$) is called to be* flat *at $z=0$ if for each $n\in\mathbb{N}$ there exist positive constants $C,\epsilon>0$, depending only on n, with $0<\epsilon<\epsilon_0$ such that*

$$|f(z)|\leq C|z|^n$$

for all $z\in\Delta_\epsilon$.

We note that in the above definition we do not need the smoothness of the function f. For example, the following function

$$f(z)=\begin{cases}\frac{1}{n}e^{-\frac{1}{|z|^2}} & \text{if } \frac{1}{n+1}<|z|\leq\frac{1}{n}\ , n=1,2,\ldots\\ 0 & \text{if } z=0\end{cases}$$

is flat at $z=0$ but not continuous on Δ. However, if $f\in\mathcal{C}^\infty(\Delta_{\epsilon_0})$ then it follows from the Taylor's theorem that f is flat at $z=0$ if and only if

$$\frac{\partial^{m+n}}{\partial z^m\partial\bar z^n}f(0)=0$$

for every $m,n\in\mathbb{N}$, i.e., f vanishes to infinite order at 0. Consequently, if $f\in\mathcal{C}^\infty(\Delta_{\epsilon_0})$ is flat at 0 then $\frac{\partial^{m+n}f}{\partial z^m\partial\bar z^n}$ is also flat at 0 for each $m,n\in\mathbb{N}$.

Let F be a $\mathcal{C}^1$-smooth complex-valued function defined in a neighborhood U of the origin in the complex plane. We consider the autonomous dynamical system

$$\frac{dz}{dt}=F(z),\ z(0)=z_0\in U. \tag{2.1}$$

First of all, let us recall several definitions.

Definition 2.2. *A state $\hat{z} \in U$ is called an equilibrium of (2.1) if $F(\hat{z}) = 0$.*

Definition 2.3. *An equilibrium, $\hat{z}$, of (2.1) is called locally asymptotically stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|z_0 - \hat{z}| < \delta$ implies that $|z(t) - \hat{z}| < \epsilon$ for all $t \geq 0$ and $\lim_{t\to+\infty} z(t) = 0$.*

The following lemma is a generalization of Lemma 3 in [8] and plays a key role in the proof of Theorem 1.1.

Lemma 2.1. *Let $P : \Delta_{\epsilon_0} \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function satisfying that $P(z) > 0$ for any $z \in \Delta^*_{\epsilon_0}$ and that P is flat at 0. If a, b are complex numbers and if g_0, g_1, g_2 are $\mathcal{C}^1$-smooth functions defined on Δ_{ϵ_0} satisfying:*

(A1) $g_0(z) = O(|z|)$, $g_1(z) = O(|z|^\ell)$, and $g_2(z) = o(|z|^m)$, and

(A2) $Re\Big[az^m + \frac{1}{P^n(z)}\Big(bz^\ell\big(1 + g_0(z)\big)\frac{P'(z)}{P(z)} + g_1(z)\Big)\Big] = g_2(z)$ for every $z \in \Delta^*_{\epsilon_0}$

for any nonnegative integers ℓ, m and n except for the following two cases

(E1) $\ell = 1$ and $Re\, b = 0$, and

(E2) $m = 0$ and $Re\, a = 0$

then $ab = 0$.

Proof. We shall prove the lemma by contradiction. Suppose that there exist non-zero complex numbers $a, b \in \mathbb{C}^*$ such that the identity in (A2) holds with the smooth functions g_0, g_1, and g_2 satisfying the growth conditions specified in (A1).

Denote by $F(z) := \dfrac{1}{2} \log P(z)$ for all $z \in \Delta^*_{\epsilon_0}$ and by $f(z) := bz^\ell(1 + g_0(z))$ for all $z \in \Delta_{\epsilon_0}$.

Case 1. $\ell = 0$:

Let $\gamma : [0, \delta_0) \to \Delta_{\epsilon_0}$ ($\delta_0 > 0$) be the solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = b + bg_0(\gamma(t)), \quad \gamma(0) = 0.$$

Let us denote by $u(t) := F(\gamma(t))$, $0 < t < \delta_0$. By (A2), it follows that $u'(t)$ is bounded on the interval $(0, \delta_0)$. Integration shows that $u(t)$ is also bounded on $(0, \delta_0)$. But this is impossible since $u(t) \to -\infty$ as $t \downarrow 0$.

Case 2. $\ell = 1$:

By (E1), we have $b_1 := \mathrm{Re}(b) \neq 0$. Assume momentarily that $b_1 < 0$. Let $\gamma : [t_0, +\infty) \to \Delta^*_{\epsilon_0}$ ($t_0 > 0$) be the solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = b\gamma(t)\Big(1 + g_0(\gamma(t))\Big),\ \gamma(t_0) = z_0 \in \Delta^*_{\epsilon_0}. \tag{2.2}$$

Thanks to [4, Theorem 5], the system (2.2) is locally trajectory equivalent at the origin to the system

$$\frac{dz}{dt} = bz(t).$$

It is well-known that the origin is a locally asymptotically stable equilibrium of the above diffenrential equation. Therefore, we have $\gamma(t) \to 0$ as $t \to +\infty$. Moreover, we can assume that $|\gamma(t)| < r_1$ for every $t_0 < t < +\infty$, where $r_1 := 1/2$ if $\mathrm{Im}(b) = 0$ and $r_1 := \min\{1/2, |b_1|/(4|\mathrm{Im}(b)|)\}$ if otherwise. This implies that $\mathrm{Re}\Big(b(1 + g_0(\gamma(t)))\Big) < b_1/4 < 0$. Integration and a simple estimation tell us that

$$|\gamma(t)| \leq |\gamma(t_0)| \exp\Big(b_1 t/4\Big),\ \forall t > t_0.$$

Consequently, this in turn yields that $t \lesssim \log \frac{1}{|\gamma(t)|}$.

Denote by $u(t) := F(\gamma(t))$ for $t \geq t_0$. Then, it follows from (A2) that $u'(t)$ is bounded on $(t_0, +\infty)$, and thus $|u(t)| \lesssim t$. Therefore, there exists a constant $A > 0$ such that $|u(t)| \leq A \log \dfrac{1}{|\gamma(t)|}$ for all $t > t_0$. Hence we obtain, for all $t > t_0$, that $\log P(\gamma(t)) = 2u(t) \geq -2A \log \dfrac{1}{|\gamma(t)|}$, and thus

$$P(\gamma(t)) \geq |\gamma(t)|^{2A},\ t \geq t_0.$$

Hence we arrive at

$$\lim_{t\to+\infty} \frac{P(\gamma(t))}{|\gamma(t)|^{2A+1}} = +\infty,$$

which is impossible since P is flat at 0. The case $b_1 > 0$ is similar, with considering the side $t < 0$ instead.

Case 3. $\boldsymbol{\ell = k + 1 \geq 2}$:

Let $\gamma : [t_0, +\infty) \to \Delta^*_{\epsilon_0}$ ($t_0 > 0$) be a solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = f(\gamma(t)) = b\gamma^{k+1}(t)\Big(1 + g_0(\gamma(t))\Big),\ \gamma(t_0) = z_0 \in \Delta^*_{\epsilon_0}. \tag{2.3}$$

According to [4, Theorem 5], the system (2.3) is locally trajectory equivalent at the origin to the system

$$\frac{dz}{dt} = bz^{k+1}(t).$$

Hence, it follows from [12, Theorem 1] that $\gamma(t)\to 0$ as $t\to+\infty$.

Now we shall estimate $\gamma(t)$. Indeed, integration shows that

$$\frac{1}{\gamma^k(t)}=c-kbt\big(1+\epsilon(t)\big),\ \forall t>t_0, \tag{2.4}$$

where c is a constant depending only on the initial condition and

$$\epsilon(t)=\frac{\int_{t_0}^t g_o(\gamma(s))ds}{t-t_0}\ \text{for every } t>t_0.$$

Choose $\delta>0$ such that either $\arg\big(b(1+z)\big)\in(0,2\pi)$ for all $z\in\Delta_\delta$ (for the case $\mathrm{Im}(b)\neq 0$) or $\arg\big(b(1+z)\big)\in(-\pi/2,3\pi/2)$ for all $z\in\Delta_\delta$ (for the case $\mathrm{Im}(b)=0$). Moreover, without loss of generality we can assume that $|g_0(\gamma(t))|<\delta$ for all $t>t_0$ and hence $|\epsilon(t)|<\delta$ for all $t>t_0$. Therefore, by changing the initial condition $\gamma(t_0)=z_0$ if necessary, we may assume that either $c-kbt(1+\epsilon(t)), c-kbt\in\mathbb{C}\setminus[0,+\infty)$ for all $t\in[t_0,+\infty)$ or $c-kbt(1+\epsilon(t)), c-kbt\in\mathbb{C}\setminus(-\infty i,0]$ for all $t\in[t_0,+\infty)$. Without loss of generality, we can assume that the first case occurs.

Notice that $\omega_j(t):=\tau^{-j}\sqrt[-k]{c-kbt},\ j=0,\ldots,k-1$, are solutions of the equation

$$\frac{dz}{dt}=bz^{k+1},$$

where $\tau:=e^{i2\pi/k}$. Furthermore, for each $j\in\{0,1,\ldots,k-1\}$ let $\theta_j(t)$ $(t\geq t_0)$ be the solution of the equation

$$\theta_j'(t)=f(\omega_j(t)+\theta_j(t))-b\omega_j^{k+1}(t)$$

satisfying $\theta_j(t_0)=0$. Then $\gamma_j(t):=\omega_j(t)+\theta_j(t)$ $(t>t_0)$, $j=0,1,\ldots,k-1$, are solutions of

$$\frac{dz}{dt}=f(z).$$

Moreover, again by changing the initial condition $\gamma(t_0)=z_0$ if necessary we can assume that $|g_0(\gamma_j(t))|<\delta$ for every $j=0,1,\ldots,k-1$ and for every $t>t_0$. In addition, integeration shows that

$$\frac{1}{\gamma_j^k(t)}=c-kbt\big(1+\epsilon_j(t)\big),\ \forall t>t_0, \tag{2.5}$$

where $\epsilon_j(t)=\frac{\int_{t_0}^t g_o(\gamma_j(s))ds}{t-t_0}$ for every $t>t_0$ and for every $j=0,1,\ldots,k-1$. Hence, we obtain the following.

$$\begin{aligned}\gamma_j(t)&=\tau^{-j}\sqrt[-k]{c-kbt\big(1+\epsilon_j(t)\big)}\\&=\tau^{-j}\sqrt[-k]{|c-kbt(1+\epsilon(t))|}e^{-i\arg\big(c-kbt(1+\epsilon(t))\big)/k}\\&=\sqrt[-k]{|c-kbt(1+\epsilon(t))|}e^{-i\arg\big(c-kbt(1+\epsilon(t))\big)/k-i2\pi j/k},\end{aligned}$$

where $0 < \arg\big(c - kbt(1+\epsilon_j(t))\big)/k < 2\pi$, for every $j = 0, 1, \ldots, k-1$. Consequently, $|\gamma_j(t)| \approx \dfrac{1}{|t|^{1/k}}$ for all $t \geq t_0$ and for all $j = 0, 1, \ldots, k-1$.

Let $u_j(t) := F(\gamma_j(t))$ for $j = 0, 1, \ldots, k-1$. It follows from (A2) that

$$u_j'(t) = -P^n(\gamma_j(t))\Big(\mathrm{Re}\big(a\gamma_j^m(t) + o(|\gamma_j(t)|^m)\big)\Big) + O(|\gamma_j(t)|^{k+1}) \tag{2.6}$$

for all $t > t_0$ and for all $j = 0, 1, \ldots, k-1$.

We now consider the following.

Subcase 3.1: $n \geq 1$.

Since P is flat at the origin, (2.6) and the discussion above imply

$$\begin{aligned}
|u_0'(t)| &\lesssim P^n(\gamma_0(t))|\gamma_0(t)|^m + \frac{1}{t^{1+1/k}} \\
&\lesssim P^n(\gamma_0(t)) + \frac{1}{t^{1+1/k}} \\
&\lesssim \frac{P^n(\gamma_0(t))}{|\gamma_0(t)|^{2k}} \frac{1}{t^2} + \frac{1}{t^{1+1/k}} \\
&\lesssim \frac{1}{t^2} + \frac{1}{t^{1+1/k}} \\
&\lesssim \frac{1}{t^{1+1/k}}
\end{aligned}$$

for all $t \geq t_0$. This in turn yields

$$\begin{aligned}
|u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^{t} \frac{1}{s^{1+1/k}} ds \\
&\lesssim |u_0(t_0)| + k\Big(\frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}}\Big) \\
&\lesssim 1
\end{aligned}$$

for all $t > t_0$. This is a contradiction, because $\lim_{t\to\infty} u_0(t) = -\infty$.

Subcase 3.2: $n = 0$.

We again divide the argument into 4 sub-subcases.

Subcase 3.2.1: $m/k > 1$.

It follows from (2.6) that

$$|u_0'(t)| \lesssim \frac{1}{t^{m/k}} + \frac{1}{t^{1+1/k}}$$

for all $t \geq t_0$. Hence, we get

$$\begin{aligned}|u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^{t}\Big(\frac{1}{s^{m/k}}+\frac{1}{s^{1+1/k}}\Big)ds\\ &\lesssim |u_0(t_0)| + \frac{k}{m-k}\Big(\frac{1}{t_0^{m/k-1}}-\frac{1}{t^{m/k-1}}\Big)+k\Big(\frac{1}{t_0^{1/k}}-\frac{1}{t^{1/k}}\Big)\\ &\lesssim 1\end{aligned}$$

for all $t > t_0$, which contradicts $\lim_{t\to+\infty} u_0(t) = -\infty$.

Subcase 3.2.2: $m/k = 1$.

Here, (2.6) again implies

$$|u_0'(t)| \lesssim \frac{1}{t} + \frac{1}{t^{1+1/k}} \lesssim \frac{1}{t}$$

for all $t \geq t_0$. Consequently,

$$\begin{aligned}|u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^{t}\frac{1}{s}\,ds\\ &\lesssim |u_0(t_0)| + (\log t - \log t_0)\\ &\lesssim \log t\\ &\lesssim \log\frac{1}{|\gamma_0(t)|}\end{aligned}$$

for all $t > t_0$. Therefore there exists a constant $A > 0$ such that $|u_0(t)| \leq A\log\frac{1}{|\gamma_0(t)|}$ for all $t > t_0$. Hence for all $t > t_0$, $\log P(\gamma_0(t)) = 2u(t) \geq -2A\log\frac{1}{|\gamma_0(t)|}$, and thus

$$P(\gamma_0(t)) \geq |\gamma_0(t)|^{2A},\ \forall t \geq t_0.$$

This ensures

$$\lim_{t\to+\infty}\frac{P(\gamma_0(t))}{|\gamma_0(t)|^{2A+1}} = +\infty,$$

which is again impossible since P is flat at 0.

Subcase 3.2.3: $m = 0$.

Let $h(t) := u_0(t) + \mathrm{Re}(a)t$. Recall that in this case we have (E2) which says $\mathrm{Re}\ a \neq 0$. Assume momentarily that $\mathrm{Re}(a) < 0$. (The case that $\mathrm{Re}(a) > 0$ will follow by a similar argument.)

By (2.6), there is a constant $B > 0$ such that

$$|h'(t)| \leq \frac{1}{2}|\mathrm{Re}(a)| + B\frac{1}{t^{1+1/k}}.$$

Therefore,

$$\begin{aligned}|h(t)| &\leq |h(t_0)| + \frac{1}{2}|\mathrm{Re}(a)|(t-t_0) + B\int_{t_0}^{t}\frac{1}{s^{1+1/k}}ds\\ &\leq |h(t_0)| + \frac{1}{2}|\mathrm{Re}(a)|(t-t_0) + kB(\frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}})\end{aligned}$$

for all $t > t_0$. Thus

$$\begin{aligned}u_0(t) &\geq -\mathrm{Re}(a)t - |h(t)|\\ &\geq |\mathrm{Re}(a)|t - |h(t_0)| - \frac{1}{2}|\mathrm{Re}(a)|(t-t_0) - kB(\frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}})\\ &\gtrsim t\end{aligned}$$

for all $t > t_0$. It means that $u_0(t) \to +\infty$ as $t \to +\infty$, and it is hence absurd.

Subcase 3.2.4: $0 < \frac{m}{k} < 1$. Assume for a moment that m and k are relatively prime. (In the end, it will become obvious that this assumption can be taken without loss of generality.) Then τ^m is a primitive k-th root of unity. Therefore there exist $j_0, j_1 \in \{1, \cdots, k-1\}$ such that $\pi/2 < arg(\tau^{mj_0}) \leq \pi$ and $-\pi \leq arg(\tau^{mj_1}) < -\pi/2$. Hence, it follows that there exists $j \in \{0, \cdots, k-1\}$ such that $\cos\big(arg(a/b) + \frac{k-m}{k}arg(-b) - 2\pi mj/k\big) > 0$. Denote by

$$A := \frac{|a|}{(k-m)|b|}\cos\Big(arg(a/b) + \frac{k-m}{k}arg(-b) - 2\pi mj/k\Big) > 0,$$

a positive constant. Now let

$$h_j(t) := u_j(t) + \mathrm{Re}(\tau^{-mj}\frac{a}{-b(k-m)}(c-kbt)^{1-m/k}).$$

Note that $arg\big(c-kbt)\big) \to arg(-b)$ as $t \to +\infty$ and $\delta > 0$ can be chosen so small that there exists $t_1 > t_0$ big enough such that

$$\begin{aligned}\Big|\gamma_j^m(t) - \tau^{-mj}\Big(\frac{1}{c-kbt}\Big)^{m/k}\Big| &= \Big|\frac{1}{\big(c-kb(t+\epsilon_j(t))\big)^{m/k}}\Big[1 - \Big(1 - \frac{kbt\epsilon_j(t)}{c-kbt}\Big)^{m/k}\Big]\Big|\\ &\leq \frac{k-m}{8k|a|}A(k|b|)^{1-m/k}\frac{1}{t^{m/k}}\end{aligned}$$

for every $t > t_1$. Hence it follows from (2.6) that there exist positive constants B and t_2 $(t_2 > t_1)$ such that

$$|h_j'(t)| \leq \frac{k-m}{4k} A(k|b|)^{1-m/k} \frac{1}{t^{m/k}} + \frac{B}{t^{1+1/k}}$$

and

$$\cos\Big(\arg(a/b) + \frac{k-m}{k}\arg(c-kbt) - 2mj\pi/k\Big) \\ \geq \frac{1}{2}\cos\Big(\arg(a/b) + \frac{k-m}{k}\arg(-b) - 2mj\pi/k\Big)$$

for every $t \geq t_2$. Thus we have

$$\begin{aligned} |h_j(t)| &\leq |h_j(t_2)| + A(k|b|)^{1-m/k}\frac{k-m}{4k}\int_{t_2}^{t} s^{-m/k}ds + B\int_{t_2}^{t} s^{-1-1/k}ds \\ &\leq |h_j(t_2)| + \frac{A}{4}(k|b|)^{1-m/k}(t^{1-m/k} - t_2^{1-m/k}) + kB(t_2^{-1/k} - t^{-1/k}) \end{aligned}$$

for $t > t_2$. Hence

$$\begin{aligned} u_j(t) &\geq -\mathrm{Re}\Big(\frac{a\tau^{-mj}}{-kb(1-m/k)}(c-kbt)^{1-m/k}\Big) - |h_j(t)| \\ &\geq \frac{|a|}{|b|(k-m))}|c-kbt|^{1-m/k}\cos\Big(arg(a/b) \\ &\quad + \frac{(k-m)arg(c-kbt) - 2mj\pi}{k}\Big) - |h_j(t_2)| \\ &\quad - \frac{A}{4}(k|b|)^{1-m/k}(t^{1-m/k} - t_2^{1-m/k}) - kB(t_2^{-1/k} - t^{-1/k}) \\ &\geq \frac{A}{2}|c-kbt|^{1-m/k} - |h_j(t_2)| \\ &\quad - \frac{A}{4}(k|b|)^{1-m/k}(t^{1-m/k} - t_2^{1-m/k}) - kB(t_2^{-1/k} - t^{-1/k}) \\ &\gtrsim t^{1-m/k} \end{aligned}$$

for $t > t_2$. This implies that $u_j(t) \to +\infty$ as $t \to +\infty$, which is absurd since $\log P(z) \to -\infty$ as $z \to 0$.

Hence all the cases are covered, and the proof of Lemma 2.1 is finally complete. ■

Following the proof of Lemma 2.1, we have the following lemma.

Lemma 2.2. *Let $P : \Delta_{\epsilon_0} \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function satisfying that $P(z) > 0$ for any $z \in \Delta^*_{\epsilon_0}$ and that P is flat at 0. If b is a complex number and if g is a $\mathcal{C}^1$-smooth function defined on Δ_{ϵ_0} satisfying:*

(B1) $g(z) = O(|z|^{k+1})$, *and*

(B2) $Re\Big[\big(bz^k + g(z)\big)P'(z)\Big] = 0$ *for every* $z \in \Delta_{\epsilon_0}$

for some nonnegative integer k, *except the case* $k = 1$ *and* $Re(b) = 0$, *then* $b = 0$.

3. The vector space of tangential holomorphic vector fields

This section is devoted to the proof of Theorem 1.1. First of all, we need the following theorem.

Theorem 3.1. *If a holomorphic vector field germ* $(H, 0)$ *vanishing at the origin which contains no nonzero term* $i\beta z_2 \frac{\partial}{\partial z_2}$ $(\beta \in \mathbb{R}^*)$ *and is tangent to a real* $\mathcal{C}^1$*-smooth hypersurface germ* $(M, 0)$ *defined by the equation* $\rho(z) := \rho(z_1, z_2) = \text{Re } z_1 + P(z_2) + \text{Im } z_1 Q(z_2, \text{Im } z_1) = 0$ *satisfying the conditions:*

(1) P, Q *are* $\mathcal{C}^1$*-smooth with* $P(0) = Q(0, 0) = 0$,

(2) $P(z_2) > 0$ *for any* $z_2 \neq 0$, *and*

(3) $P(z_2), P'(z_2)$ *are flat at* $z_2 = 0$,

then $H = 0$.

Proof. The CR hypersurface germ $(M, 0)$ at the origin in $\mathbb{C}^2$ under consideration is defined by the equation $\rho(z_1, z_2) = 0$, where

$$\rho(z_1, z_2) = \text{Re } z_1 + P(z_2) + (\text{Im } z_1)\ Q(z_2, \text{Im } z_1) = 0,$$

where P, Q are $\mathcal{C}^1$-smooth functions satisfying the three conditions specified in the hypothesis of our lemma. Recall that $P(z_2), P'(z_2)$ are flat at $z_2 = 0$ in particular.

Then we consider a holomorphic vector field $H = h_1(z_1, z_2)\frac{\partial}{\partial z_1} + h_2(z_1, z_2)\frac{\partial}{\partial z_2}$ defined on a neighborhood of the origin satisfying that $H(0) = 0$ and that H contains no nonzero term $i\beta z_2 \frac{\partial}{\partial z_2}$ $(\beta \in \mathbb{R}^*)$. We only consider H that is tangent to M, which means that they satisfy the identity

$$(\text{Re } H)\rho(z) = 0,\ \forall z \in M. \tag{3.1}$$

Expand h_1 and h_2 into the Taylor series at the origin so that

$$h_1(z_1, z_2) = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \text{ and } h_2(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{jk} z_1^j z_2^k,$$

where $a_{jk}, b_{jk} \in \mathbb{C}$. We note that $a_{00} = b_{00} = 0$ since $h_1(0,0) = h_2(0,0) = 0$.

By a simple computation, we have

$$\rho_{z_1}(z_1, z_2) = \frac{1}{2} + \frac{Q(z_2, \text{Im } z_1)}{2i} + (\text{Im } z_1) Q_{z_1}(z_2, \text{Im } z_1),$$
$$\rho_{z_2}(z_1, z_2) = P'(z_2) + (\text{Im } z_1) Q_{z_2}(z_2, \text{Im } z_1),$$

and the equation (3.1) can thus be re-written as

$$\begin{aligned}\text{Re}\Big[\Big(\frac{1}{2} + \frac{Q(z_2, \text{Im } z_1)}{2i} + (\text{Im } z_1) Q_{z_1}(z_2, \text{Im } z_1)\Big) h_1(z_1, z_2) \\ + \Big(P'(z_2) + (\text{Im } z_1) Q_{z_2}(z_2, \text{Im } z_1)\Big) h_2(z_1, z_2)\Big] = 0\end{aligned} \tag{3.2}$$

for all $(z_1, z_2) \in M$.

Since $\Big(it - P(z_2) - tQ(z_2, t), z_2\Big) \in M$ for any $t \in \mathbb{R}$ with t small enough, the above equation again admits a new form

$$\begin{aligned}\text{Re}\Big[\Big(\frac{1}{2} + \frac{Q(z_2, t)}{2i} + tQ_{z_1}(z_2, t)\Big) \sum_{j,k=0}^{\infty} a_{jk} \big(it - P(z_2) - tQ(z_2, t)\big)^j z_2^k \\ + \Big(P'(z_2) + tQ_{z_2}(z_2, t)\Big) \sum_{m,n=0}^{\infty} b_{mn} \big(it - P(z_2) - tQ(z_2, t)\big)^m z_2^n\Big] = 0\end{aligned} \tag{3.3}$$

for all $z_2 \in \mathbb{C}$ and for all $t \in \mathbb{R}$ with $|z_2| < \epsilon_0$ and $|t| < \delta_0$, where $\epsilon_0 > 0$ and $\delta_0 > 0$ are small enough.

The goal is to show that $H \equiv 0$. Indeed, striving for a contradiction, suppose that $H \not\equiv 0$. We notice that if $h_2 \equiv 0$ then (3.2) shows that $h_1 \equiv 0$. So, we must have $h_2 \not\equiv 0$.

We now divide the argument into two cases as follows.

Case 1. $h_1 \not\equiv 0$. In this case let us denote by j_0 the smallest integer such that $a_{j_0 k} \neq 0$ for some integer k. Then let k_0 be the smallest integer such that $a_{j_0 k_0} \neq 0$. Similarly, let m_0 be the smallest integer such that $b_{m_0 n} \neq 0$ for some integer n. Then denote by n_0 the smallest integer such that $b_{m_0 n_0} \neq 0$. We can see that $j_0 \geq 1$ if $k_0 = 0$ and $m_0 \geq 1$ if $n_0 = 0$.

Since $P(z_2) = o(|z_2|^j)$ for any $j \in \mathbb{N}$, inserting $t = \alpha P(z_2)$ into (3.3), where $\alpha \in \mathbb{R}$ will be chosen later, one has

$$\begin{aligned}&\text{Re}\Big[\frac{1}{2} a_{j_0 k_0} (i\alpha - 1)^{j_0} (P(z_2))^{j_0} \big(z_2^{k_0} + o(|z_2|^{k_0})\big) + b_{m_0 n_0} (i\alpha - 1)^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}) \\ &\times (P(z_2))^{m_0} \Big(P'(z_2) + \alpha P(z_2) Q_{z_2}(z_2, \alpha P(z_2))\Big)\Big] = 0\end{aligned} \tag{3.4}$$

for all $z_2 \in \Delta_{\epsilon_0}$. We note that in the case $k_0 = 0$ and $\mathrm{Re}(a_{j_0 0}) = 0$, α can be chosen in such a way that $\mathrm{Re}\big((i\alpha - 1)^{j_0} a_{j_0 0}\big) \neq 0$. Then (3.4) yields that $j_0 > m_0$ by virtue of the fact that $P'(z_2), P(z_2)$ are flat at $z_2 = 0$. Hence, we conclude from Lemma 2.1 that $m_0 = 0, n_0 = 1$, and $b_{0,1} = i\beta z_2$ for some $\beta \in \mathbb{R}^*$. This is a contradiction with the assumption H contains no nonzero term $i\beta z_2 \frac{\partial}{\partial z_2}$.

Case 2. $\boldsymbol{h_1 \equiv 0}$. Let m_0, n_0 be as in the Case 1. Since $P(z_2) = o(|z_2|^{n_0})$, letting $t = 0$ in (3.3) one obtains that

$$\mathrm{Re}\Big[b_{m_0 n_0}\big(z_2^{n_0} + o(|z_2|^{n_0}\big)P'(z_2)\Big] = 0 \tag{3.5}$$

for all $z_2 \in \Delta_{\epsilon_0}$. Therefore, Lemma 2.2 yields that $m_0 = 0, n_0 = 1$, and $b_{0,1} = i\beta z_2$ for some $\beta \in \mathbb{R}^*$, which is again impossible.

Altogether, the proof of our theorem is complete. ■

Now we are ready to prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] Let $H_1, H_2 \in \mathrm{hol}_0(M, p)$ be arbitrary. Then by Theorem 3.1 we have that H_j contains term $i\beta_j z_2 \frac{\partial}{\partial z_2}$ $(j = 1, 2)$ for some $\beta_1, \beta_2 \in \mathbb{R}$. Therefore, $\beta_2 H_1 - \beta_1 H_2$ does not contain a term $i\beta z_2 \frac{\partial}{\partial z_2}$. Hence, Theorem 3.1 again yields that $\beta_2 H_1 - \beta_1 H_2 = 0$, which proves the theorem. ■

4. Appendix

We recall the following theorem that gives examples of holomorphic vector fields and real hypersurfaces which are tangent.

Theorem 4.1 (see Theorem 3 in [9])**.** *Let $\alpha \in \mathbb{R}$ and let $a(z) = \sum_{n=1}^{\infty} a_n z^n$ be a non-zero holomorphic function defined on a neighborhood of $0 \in \mathbb{C}$, where $a_n \in \mathbb{C}$ for all $n \geq 1$. Then there exist positive numbers $\epsilon_0, \delta_0 > 0$ such that the holomorphic vector field*

$$H^{a,\alpha}(z_1, z_2) = L^{\alpha}(z_1)a(z_2)\frac{\partial}{\partial z_1} + iz_2\frac{\partial}{\partial z_2},$$

where

$$L^{\alpha}(z_1) = \begin{cases} \frac{1}{\alpha}\big(\exp(\alpha z_1) - 1\big) & \text{if } \alpha \neq 0 \\ z_1 & \text{if } \alpha = 0, \end{cases}$$

is tangent to the $\mathcal{C}^1$-smooth hypersurface M given by

$$M = \big\{(z_1, z_2) \in \Delta_{\delta_0} \times \Delta_{\epsilon_0} : \rho(z_1, z_2) := \mathrm{Re}\ z_1 + P(z_2) + f(z_2, \mathrm{Im}\ z_1) = 0\big\},$$

where f and P are respectively defined on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ and Δ_{ϵ_0} by

$$f(z_2,t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos\left(R(z_2)+\alpha t\right)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where $R(z_2) = q(|z_2|) - \mathrm{Re}\big(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n\big)$ for all $z_2 \in \Delta_{\epsilon_0}$, and

$$P(z_2) = \begin{cases} \frac{1}{\alpha} \log \Big[1 + \alpha P_1(z_2)\Big] & \text{if } \alpha \neq 0 \\ P_1(z_2) & \text{if } \alpha = 0, \end{cases}$$

where

$$P_1(z_2) = \exp\Big[p(|z_2|) + \mathrm{Re}\Big(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n\Big) - \log\big|\cos\big(R(z_2)\big)\big|\Big]$$

for all $z_2 \in \Delta^*_{\epsilon_0}$ and $P_1(0) = 0$, and q, p are reasonable functions defined on $[0, \epsilon_0)$ and $(0, \epsilon_0)$ respectively with $q(0) = 0$ so that P, R are $\mathcal{C}^1$-smooth in Δ_{ϵ_0}.

Proof. First of all, it is easy to show that there is $\epsilon_0 > 0$ such that we can choose a function q so that the function R defined as in the theorem is $\mathcal{C}^1$-smooth and $|R(z_2)| \leq 1$ on Δ_{ϵ_0}. Choose $\delta_0 = \frac{1}{2|\alpha|}$ if $\alpha \neq 0$ and $\delta_0 = +\infty$ if otherwise. Then the function $f(z_2,t)$ given in the theorem is $\mathcal{C}^1$-smooth on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$. Moreover, $f(z_2,t)$ is real analytic in t and $\frac{\partial^m f}{\partial t^m}$ is $\mathcal{C}^1$-smooth on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ for each $m \in \mathbb{N}$.

Next, let P_1, P, R be functions defined as in the theorem and let $Q_0(z_2) := \tan(R(z_2))$ for all $z_2 \in \Delta_{\epsilon_0}$. By a direct computation, we have the following equations.

(i) $\mathrm{Re}\Big[iz_2 Q_{0_{z_2}}(z_2) + \frac{1}{2}\Big(1 + Q_0^2(z_2)\Big) ia(z_2)\Big] \equiv 0;$

(ii) $\mathrm{Re}\Big[iz_2 P_{1_{z_2}}(z_2) - \Big(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\Big) a(z_2) P_1(z_2)\Big] \equiv 0;$

(iii) $\mathrm{Re}\Big[iz_2 P_{z_2}(z_2) + \frac{\exp\left(-\alpha P(z_2)\right) - 1}{\alpha}\Big(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\Big) a(z_2)\Big] \equiv 0$ for $\alpha \neq 0$;

(iv) $\Big(i + f_t(z_2,t)\Big) \exp\Big(\alpha\big(it - f(z_2,t)\big)\Big) \equiv i + Q_0(z_2);$

(v) $\mathrm{Re}\Big[2i\alpha z_2 f_{z_2}(z_2,t) + \Big(f_t(z_2,t) - Q_0(z_2)\Big) ia(z_2)\Big] \equiv 0$

on Δ_{ϵ_0} for any $t \in (-\delta_0, \delta_0)$.

We now prove that the holomorphic vector field $H^{a,\alpha}$ is tangent to the hypersurface M. Indeed, by a calculation we get

$$\begin{aligned}\rho_{z_1}(z_1,z_2) &= \frac{1}{2}+\frac{f_t(z_2,\text{Im }z_1)}{2i},\\ \rho_{z_2}(z_1,z_2) &= P_{z_2}(z_2)+f_{z_2}(z_2,\text{Im }z_1).\end{aligned}$$

We divide the proof into two cases.

a) $\alpha=0$**.** In this case, $f(z_2,t)=Q_0(z_2)t$ for all $(z_2,t)\in\Delta_{\epsilon_0}\times(-\delta_0,\delta_0)$. Therefore, by (i) and (ii) one obtains that

$$\begin{aligned}\text{Re }H^{a,\alpha}(\rho(z_1,z_2)) &= \text{Re}\Big[\Big(\frac{1}{2}+\frac{Q_0(z_2)}{2i}\Big)z_1a(z_2)+\Big(P_{1z_2}(z_2)+(\text{Im }z_1)Q_{0z_2}(z_2)\Big)iz_2\Big]\\ &= \text{Re}\Big[\Big(\frac{1}{2}+\frac{Q_0(z_2)}{2i}\Big)\Big(i(\text{Im }z_1)-P_1(z_2)-(\text{Im }z_1)Q_0(z_2)\Big)a(z_2)\\ &\quad+\Big(P_{1z_2}(z_2)+(\text{Im }z_1)Q_{0z_2}(z_2)\Big)iz_2\Big]\\ &= \text{Re}\Big[iz_2P_{1z_2}(z_2)-\Big(\frac{1}{2}+\frac{Q_0(z_2)}{2i}\Big)a(z_2)P_1(z_2)\Big]\\ &\quad+(\text{Im }z_1)\text{Re}\Big[iz_2Q_{0z_2}(z_2)+\frac{1}{2}\Big(1+Q_0(z_2)^2\Big)ia(z_2)\Big]=0\end{aligned}$$

for every $(z_1,z_2)\in M$, which proves the theorem for $\alpha=0$.

b) $\alpha\neq 0$**.** It follows from (iii), (iv), and (v) that

$$\begin{aligned}&\text{Re }H^{a,\alpha}(\rho(z_1,z_2))\\ &= \text{Re}\Big[\Big(\frac{1}{2}+\frac{f_t(z_2,\text{Im }z_1)}{2i}\Big)L(z_1)a(z_2)+\Big(P_{z_2}(z_2)+f_{z_2}(z_2,\text{Im }z_1)\Big)iz_2\Big]\\ &= \text{Re}\Big[\Big(\frac{1}{2}+\frac{f_t(z_2,\text{Im }z_1)}{2i}\Big)\frac{1}{\alpha}\Big(\exp\Big(\alpha\big(i\text{Im }z_1-P(z_2)-f(z_2,\text{Im }z_1)\big)\Big)-1\Big)a(z_2)\\ &\quad+\Big(P_{z_2}(z_2)+f_{z_2}(z_2,\text{Im }z_1)\Big)iz_2\Big]\\ &= \text{Re}\Big[\frac{1}{\alpha}\frac{i+f_t(z_2,\text{Im }z_1)}{2i}\exp\Big(\alpha\big(i\text{Im }z_1-f(z_2,\text{Im }z_1)\big)\Big)\exp(-\alpha P(z_2))a(z_2)\\ &\quad-\frac{1}{\alpha}\Big(\frac{1}{2}+\frac{f_t(z_2,\text{Im }z_1)}{2i}\Big)a(z_2)+\Big(P_{z_2}(z_2)+f_{z_2}(z_2,\text{Im }z_1)\Big)iz_2\Big]\\ &= \text{Re}\Big[\frac{1}{\alpha}\frac{i+Q_0(z_2)}{2i}\exp(-\alpha P(z_2))a(z_2)-\frac{1}{\alpha}\Big(\frac{1}{2}+\frac{f_t(z_2,\text{Im }z_1)}{2i}\Big)a(z_2)\\ &\quad+\Big(P_{z_2}(z_2)+f_{z_2}(z_2,\text{Im }z_1)\Big)iz_2\Big]\\ &= \text{Re}\Big[iz_2P_{z_2}(z_2)+\Big(\frac{1}{2}+\frac{Q_0(z_2)}{2i}\Big)\frac{\exp(-\alpha P(z_2))-1}{\alpha}a(z_2)\Big]\\ &\quad+\text{Re}\Big[iz_2f_{z_2}(z_2,\text{Im }z_1)+\frac{1}{2\alpha}\Big(f_t(z_2,\text{Im }z_1)-Q_0(z_2)\Big)ia(z_2)\Big]=0\end{aligned}$$

for every $(z_1, z_2) \in M$, which ends the proof. ∎

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