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Uniqueness of meromorphic mappings partially shared hypersurfaces

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Abstract. The purpose of this paper is to study uniqueness problem of meromorphic mapping from $\mathbb{C}^m$ into the complex space $\mathbb{P}^n(\mathbb{C})$ sharing partial fixed and moving hypersurfaces. Using the second main theorems due to S. D. Quang and D. P. An [12, 13], we obtain some uniqueness results. Our results are improved some before results in this trend. In our best knowledge, there are not any uniqueness results of meromorphic mapping partially shared hypersurfaces up to now.

1. Introduction and main results

Let f be a nonconstant meromorphic function. A meromorphic function a is said to be small with respect to f if $T(r,a) = o(T(r,f))$, as $r \to +\infty$ possibly outside a set of finite Lebesgue measure. We denote $\mathcal{S}(f)$ by the set of small functions with respect to f and $\widehat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. For a positive integer p and $a \in \widehat{S}(f)$, we denote by $\overline{E}_{p)}(a;f)$ the set of those distinct zeros of $f-a$ whose multiplicities do not exceed p. Here, we mean that a zero of $f-\infty$ is a pole of f. When $p=\infty$, $\overline{E}_{\infty)}(a;f)$ is the set of distinct zeros of $f-a$. For $A \subset \mathbb{C}$, we denote $\overline{N}_A(r,a;f)$(or $\overline{N}_A(r,\dfrac{1}{f-a})$) by the reduced counting function of those zeros of $f-a$ which belong to the set A, where $a \in \widehat{S}(f)$.

In 1926, R. Nevanlinna proved the Five Value Theorem as follows:

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Theorem 1.1. *Let f and g be two non-constant meromorphic functions and $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct values, $j = 1, \dots, 5$. If $\overline{E}_{\infty)}(a_j; f) = \overline{E}_{\infty)}(a_j; g)$ for $j = 1, \dots, 5$, then $f \equiv g$.*

In 2015, I. Lahiri and R. Pal [7] proved the Five Function Theorem which is improved the Five Value Theorem:

Theorem 1.2. *Let f, g be two nonconstant meromorphic functions and $a_j = a_j(z) \in \widehat{S}(f) \cap \widehat{S}(g)$ be distinct for $j = 1, \dots, k, k \geq 5$. Suppose that $p_1 \geq p_2 \geq \dots p_k$ are positive intergers or infinitely and $\delta \geq 0$ is such that*

$$\frac{1}{p_1} + \big(1 + \frac{1}{p_1}\big) \sum_{j=2}^{k} \frac{1}{1 + p_j} + 1 + \delta < (k-2)\big(1 + \frac{1}{p_1}\big).$$

Let $A_j = \overline{E}_{p_j)}(a_j; f) \setminus \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, \dots, k$. If $\sum_{j=1}^{k} \overline{N}_{A_j}(r, a_j; f) \leq \delta T(r, f)$ and

$$\begin{aligned}&\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; g)}\\ &> \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum_{j=2}^{k} \dfrac{1}{1+p_j}},\end{aligned}$$

then $f \equiv g$.

Note that in the proof of Theorem 1.2, we need $\cup_{j=1}^{k} \overline{E}_{p_j)}(a_j; f) \cap \overline{E}_{p_j)}(a_j; g) \neq \emptyset$. Theorem 1.2 is a extension and improvement of some before results in [1, 3, 9].

In 2003, P. C. Hu, P. Li and C. C. Yang gave the extension of Five Value Theorem for meromorphic function several variables.

Theorem 1.3. *[10] Let f and g be two nonconstant meromorphic functions on $\mathbb{C}^m$, let $a_j, j = 1, \dots, q$, be q distinct complex element in $\mathbb{P}^1(\mathbb{C})$ and take $m_j \in \mathbb{Z}^+ \cup \{\infty\} (j = 1, \dots, q)$ satisfying $m_1 \geq m_2 \geq \dots \geq m_q$ and $\nu^1_{f-a_j, \leq m_j} = \nu^1_{g-a_j, \leq m_j} (j = 1, \dots, q)$. If $\sum_{j=3}^{q} \dfrac{m_j}{m_j + 1} > 2$, then $f \equiv g$.*

In 1975, H. Fujimoto [8] generalized Theorem 1.1 to the case of meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and obtained that for two linearly nondegenerate meromorphic mappings f, g of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, if they have the same inverse images of $3n+2$ hyperplane counted with multiplicites in $\mathbb{P}^n(\mathbb{C})$ in general position, then $f \equiv g$. In 1983, L. Smiley considered meromorphic mappings which share $3n + 2$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ without counting multiplicity and proved the following result:

Theorem 1.4. *[15] Let f, g be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $\{H_j\}_{j=1}^q, q \geq 3n+2$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Assume that*
(a) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \leq m-2$ *for all* $1 \leq i < j \leq q$,
(b) $f(z) = g(z)$ *on* $\cup_{j=1}^q f^{-1}(H_j)$ *without counting multiplicity,*
(c) $f^{-1}(H_j) = g^{-1}(H_j)$ *for all* $1 \leq j \leq q$.
Then $f \equiv g$.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}^m) \not\subset D$. Let Q be a homogeneous polynomial defining D. For each positive integer m, we denote $\overline{E}_{m)}(D, f)$ by the set zeros of $Q(\tilde{f})$ with multiplicity at most m, each zero counted only one time. When $m = 1$, we denote $\overline{E}_{1)}(D, f)$ by $\overline{E}(D, f)$.

In 2010, Chen-Yan [4] gave following result for uniqueness of meromorphic mappings partially shared hyperplanes as follows:

Theorem 1.5. *Let f and g be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and $H_j, 1 \leq j \leq q$ be q hyperplanes in general position such that $dim f^{-1}(H_i \cap H_j) \leq m-2$ for $i \neq j$. Assume that*

$$\overline{E}(H_j, f) \subset \overline{E}(H_j, g),\ 1 \leq j \leq q$$

and $f = g$ on $\cup_{j=1}^q f^{-1}(H_j)$. If $q = 2n+3$ and

$$\liminf_{r \to \infty} \frac{\sum_{j=1}^{2n+3} N_f^1(r, H_j)}{\sum_{j=1}^{2n+3} N_g^1(r, H_j)} > \frac{n}{n+1},$$

then $f \equiv g$.

Let V be complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k (k \leq n)$. Let d be positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ defining V, and H_d the vector space consisting of all homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ with degree d. Define

$$I_d(V) := \frac{H_d}{I(V) \cap H_d} \text{ and } H_d(V) := \dim I_d(V).$$

The function $H_V(d)$ is called Hilbert function of V. Each element of $I_d(V)$ which is an equivalent class of an element $Q \in H_d$, will denote by $[Q]$.

Let f be a meromorphic mapping from $\mathbb{C}^m$ to V. Let $\tilde{f}$ be a reduced representation of f. We say that f is degenerate over $I_d(V)$ if there is $[Q] \in I_d(V) \setminus \{0\}$ such that $Q(\tilde{f}) = 0$, otherwise we say that f is nondegenerate over $I_d(V)$. Hence, if f is algebraically nondegenerate then f is nondegenerate over $I_d(V)$ for every $d \geq 1$. Then the condition "*f is nondegenerate over $I_d(V)$*" is meanful.

The family of hypersurfaces $\{D_j\}_{j=1}^q$ is said to be in $N(N \geq k)$-subgeneral position with respective V if for any $1 \leq i_1 \leq \cdots \leq i_{N+1} \leq q$, we have $V \cap (\cap_{j=1}^{N+1} D_{i_j}) = \emptyset$. When $N = k$, we said that $\{D_j\}_{j=1}^q$ is in general position in V.

Motivate from above results, we will prove first result for uniqueness of meromorphic mapping partially shared hypersurfaces as follows:

Theorem 1.6. *Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k(k \leq n)$. Let $\{D_j\}_{j=1}^q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respective to V, which define the homogenous polynomials $\{Q_j\}_{j=1}^q$ with $\deg Q_j = d_j (1 \leq j \leq q)$. Let d be the least common multiple of $d_1, \ldots, d_q$. Let f, g be meromorphic mappings of $\mathbb{C}^m$ into V which are nondegenerate over $I_d(V)$. Let $m_j (j = 1, \ldots, q)$ be positive integers or ∞ with $m_1 \geq \cdots \geq m_q$. Assume that*
(a) $\dim(f^{-1}(D_i) \cap f^{-1}(D_j)) \leq m - 2$ for all $1 \leq i < j \leq q$.
(b) Suppose that $f(z) = g(z)$ on $\cup_{j=1}^q \big(\overline{E}_{m_j)}(D_j, f) \cap \overline{E}_{m_j)}(D_j, g)\big)$, where

$$\cup_{j=1}^q \big(\overline{E}_{m_j)}(D_j, f) \cap \overline{E}_{m_j)}(D_j, g)\big) \neq \emptyset$$

outside a analytic subset with codimension at most 2.
(c) Let $\delta \geq 0$ be a real number. Set $A_j = \overline{E}_{m_j)}(D_j, f) \setminus \overline{E}_{m_j)}(D_j, g)$ for $j = 1, \ldots, q$. Suppose that $\sum_{j=1}^q N_f(r, A_j) \leq \delta T(r, f)$,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^q N_{f,\leq m_j}^{H_V(d)-1}(r, D_j)}{\sum_{j=1}^q N_{g,\leq m_j}^{H_V(d)-1}(r, D_j)}$$
$$> \frac{m_1(H_V(d)-1)}{d'(1+m_1)\Big(q - \frac{(2N-k+1)H_V(d)}{k+1} - \sum_{i=1}^q \frac{H_V(d)-1}{m_i+1}\Big) - (H_V(d)-1)(1+\delta)m_1}$$

and

$$q > \frac{(2N-k+1)H_V(d)}{k+1} + \sum_{i=1}^q \frac{H_V(d)-1}{m_i+1} + \frac{(H_V(d)-1)(2+\delta)m_1}{d'(1+m_1)},$$

then $f \equiv g$, where $d' = \min_{j=1,\ldots,q}\{d_j\}$.

Since f is algebraically nondegenerate implies that f is nondegenerate over $I_d(V)$ for every $d \geq 1$. Then, we obtain the result as follows:

Corollary 1.1. *Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k(k \leq n)$. Let $\{D_j\}_{j=1}^q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respective to V, which define the homogenous polynomials $\{Q_j\}_{j=1}^q$ with*

$\deg Q_j = d_j (1 \leq j \leq q)$. *Let* d *be the least common multiple of* $d_1, \dots, d_q$. *Let* f, g *be meromorphic mappings of* $\mathbb{C}^m$ *into* V *which are algebraically nondegenerate. Let* $m_j (j = 1, \dots, q)$ *be positive integers or* ∞ *with* $m_1 \geq \cdots \geq m_q$. *Assume that*
(*a*) $\dim(f^{-1}(D_i) \cap f^{-1}(D_j)) \leq m-2$ *for all* $1 \leq i < j \leq q$.
(*b*) *Suppose that* $f(z) = g(z)$ *on* $\cup_{j=1}^{q} \big(\overline{E}_{m_j)}(D_j, f) \cap \overline{E}_{m_j)}(D_j, g)\big)$, *where*

$$\cup_{j=1}^{q} \big(\overline{E}_{m_j)}(D_j, f) \cap \overline{E}_{m_j)}(D_j, g)\big) \neq \emptyset$$

outside a analytic subset with codimension at most 2.
(*c*) *Let* $\delta \geq 0$ *be a real number. Set* $A_j = \overline{E}_{m_j)}(D_j, f) \setminus \overline{E}_{m_j)}(D_j, g)$ *for* $j = 1, \dots, q$. *Suppose that* $\sum_{j=1}^{q} N_f(r, A_j) \leq \delta T(r, f)$,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} N_{f,\leq m_j}^{H_V(d)-1}(r, D_j)}{\sum_{j=1}^{q} N_{g,\leq m_j}^{H_V(d)-1}(r, D_j)}$$
$$> \frac{m_1(H_V(d)-1)}{d'(1+m_1)\Big(q - \dfrac{(2N-k+1)H_V(d)}{k+1} - \displaystyle\sum_{i=1}^{q} \dfrac{H_V(d)-1}{m_i+1}\Big) - (H_V(d)-1)(1+\delta)m_1}$$

and

$$q > \frac{(2N-k+1)H_V(d)}{k+1} + \sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1} + \frac{(H_V(d)-1)(2+\delta)m_1}{d'(1+m_1)},$$

then $f \equiv g$.

Remark 1.1. *In Corollary 1.1, when* $V = \mathbb{P}^n(\mathbb{C}), N = n$, *we have* $H_d(V) = \binom{n+d}{n}$ *and* $k = n$. *Suppose that* $E(D_j, f) = E(D_j, g)$ *for all* $j = 1, \dots, q$, *and* $f(z) = g(z)$ *on* $\cup_{j=1}^{q} E(D_j, f)$, *we can choose* $\delta = 0$, *then*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} N_{f,\leq m_j}^{H_V(d)-1}(r, D_j)}{\sum_{j=1}^{q} N_{g,\leq m_j}^{H_V(d)-1}(r, D_j)} = 1$$
$$> \frac{m_1(H_V(d)-1)}{d'(1+m_1)\Big(q - \dfrac{(2N-k+1)H_V(d)}{k+1} - \displaystyle\sum_{i=1}^{q} \dfrac{H_V(d)-1}{m_i+1}\Big) - (H_V(d)-1)(1+\delta)m_1}$$

Hence, all assumptions of Corollary 1.1 are satisfied. Set $m_1 = \cdots = m_q = \infty$, *we get the uniqueness result with* $3\binom{n+d}{n}$ *hypersurfaces. The number hypersurfaces in Corollary 1.1 is smaller than the number hypersurfaces in [6]. Hence, Theorem 1.6 is a improvement the result due to Dulock-Ru [6].*

When $d = 1$, $N = n, V = \mathbb{P}^n(\mathbb{C})$, we get the following result for uniqueness of meromorphic mappings sharing partial hyperplanes from Corollary 1.1:

Corollary 1.2. *Let $\{H_j\}_{j=1}^q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position with respective in $\mathbb{P}^n(\mathbb{C})$. Let f, g be meromorphic mappings nondegenerate algebraically of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $m_j (j = 1, \dots, q)$ be positive integers or ∞ with $m_1 \geq \cdots \geq m_q$. Assume that*
(a) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \leq m - 2$ for all $1 \leq i < j \leq q$.
(b) Suppose that $f(z) = g(z)$ on $\cup_{j=1}^q \big(\overline{E}_{m_j)}(H_j, f) \cap \overline{E}_{m_j)}(H_j, g)\big)$, where

$$\cup_{j=1}^q \big(\overline{E}_{m_j)}(H_j, f) \cap \overline{E}_{m_j)}(H_j, g)\big) \neq \emptyset$$

outside a analytic subset with codimension at most 2.
(c) Let $\delta \geq 0$ be a real number. Set $A_j = \overline{E}_{m_j)}(H_j, f) \setminus \overline{E}_{m_j)}(H_j, g)$ for $j = 1, \dots, q$. Suppose that $\sum_{j=1}^q N_f(r, A_j) \leq \delta T(r, f)$,

$$\begin{aligned} &\liminf_{r\to\infty} \frac{\sum_{j=1}^q N^n_{f,\leq m_j}(r, H_j)}{\sum_{j=1}^q N^n_{g,\leq m_j}(r, H_j)} \\ &> \frac{m_1 n}{(1+m_1)\Big(q - n - 1 - \sum_{i=1}^q \dfrac{n}{m_i+1}\Big) - n(1+\delta)m_1} \end{aligned}$$

and $q > n + 1 + \sum_{i=1}^q \dfrac{n}{m_i+1} + \dfrac{n(2+\delta)m_1}{1+m_1}$, then $f \equiv g$.

Let $\mathcal{M}$ denote the field of all meromorphic function on $\mathbb{C}^m$ and $\mathcal{M}[x_0, \dots, n]$ the $\mathcal{M}$-vector space of all polynomials in variables $x_0, \dots, x_n$ whose coefficients are in $\mathcal{M}$. Denote Q be homogenous polynomials over $\mathbb{C}$ obtained by substituting a specific point $z \in \mathbb{C}^m$ into the coefficients of Q. We will call a moving hypersurface in $\mathbb{P}^n(\mathbb{C})$ each nonzero homogenous polynomial $Q \in \mathcal{M}[x_0, \dots, x_n]$. The moving hypersurface Q is said to be slow with respect to a meromorphic mapping f if all coefficients are small with respect to f. Here a meromorphic function φ is said to be small with respect to f if $\|T(r, a) = o(T(r, f))$.

Let $\{Q_i\}_{i=1}^q$ be a family of moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, $\deg Q_i = d_i, i = 1, \dots, q$. Assume that $Q_i = \sum_{I \in \mathcal{T}_{d_i}} a_{iI} w^I$, where $\mathcal{T}_d$ is the set of all n-tuples $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$ and $i_j \geq 0$ for all j. We may consider Q_i as a meromorphic mapping into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $(\cdots : h_{a_{iI}} : \dots)$ and denote $T(r, Q_i)$ its characteristic function, where h is a suitable holomorphic function and $N = \binom{n+d_i}{n} - 1$. Denote $\mathcal{K} = \mathcal{K}_{\{Q_i\}_{i=1}^q}$ by the smallest subfield of $\mathcal{M}$ which contains $\mathbb{C}$ and all $\dfrac{a_{iI}}{a_{iJ}}$ with $a_{iJ} \not\equiv 0$.

We say that $\{Q_i\}_{i=1}^q$ are in weakly general position if there exists $z \in \mathbb{C}^m$ such that all coefficients of $\{Q_i\}_{i=1}^q$ are holomorphic function at z and for any

$1 \leq i_0 < i_1 < \cdots < i_n \leq q$, the system

$$\begin{cases} Q_{i_j}(z)(w_0, \ldots, w_n) = 0 \\ \qquad 0 \leq j \leq n \end{cases}$$

has only the trivial solution $w = (0, \ldots, 0) \in \mathbb{C}^{n+1}$.

The above family $\{Q_i\}_{i=1}^q$ is said to be in general position with respect to the Veronese embedding if $\{Q_{i_1}, \ldots, Q_{i_{N+1}}\}$ are linearly independent over $\mathcal{M}$ for any $1 \leq i_1 < \cdots < i_{N+1} \leq q$.

With above definitions, we get the results as follows for uniqueness of meromorphic mapping sharing partial moving hypersurfaces.

Theorem 1.7. *Let f and g be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $\tilde{f}$ and $\tilde{g}$ be reduced representation of f and g, respectively. Let $Q_i, i = 1, \ldots, q$, be slow (with respect to f and g) moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in weak general position with $\deg Q_i = d_i$ such that $Q_i(\tilde{f}) \not\equiv 0$ and $Q_i(\tilde{g}) \not\equiv 0$ $(1 \leq i \leq q)$. Put $d = lcm(d_1, \ldots, d_q)$ and $N = \binom{n+d}{n} - 1$. Let $m_j (j = 1, \ldots, q)$ be positive integers or ∞ with $m_1 \geq \cdots \geq m_q$. Assume that*
(a) $\dim(f^{-1}(Q_i) \cap f^{-1}(Q_j)) \leq m - 2$ for all $1 \leq i < j \leq q$.
(b) Suppose that $f(z) = g(z)$ on $\cup_{j=1}^q \big(\overline{E}_{m_j)}(Q_j, f) \cap \overline{E}_{m_j)}(Q_j, g)\big)$, where

$$\cup_{j=1}^q \big(\overline{E}_{m_j)}(Q_j, f) \cap \overline{E}_{m_j)}(Q_j, g)\big) \neq \emptyset$$

outside a analytic subset with codimension at most 2.
(c) Let $\delta \geq 0$ be a real number. Set $A_j = \overline{E}_{m_j)}(Q_j, f) \setminus \overline{E}_{m_j)}(Q_j, g)$ for $j = 1, \ldots, q$. Suppose that $\sum_{j=1}^q N_f(r, A_j) \leq \delta T(r, f)$,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^q N^N_{f,\leq m_j}(r, Q_j)}{\sum_{j=1}^q N^N_{g,\leq m_j}(r, Q_j)} > \frac{m_1 N}{d'(1+m_1)\Big(\dfrac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^q \dfrac{N}{m_i+1}\Big) - N(1+\delta)m_1}$$

and

$$q > (nN+n+1)\Big(\sum_{i=1}^q \frac{N}{m_i+1} + \frac{N(2+\delta)m_1}{d'(1+m_1)}\Big) + (n-1)(N+1),$$

then $f \equiv g$, where $d' = \min_{j=1,\ldots,q}\{d_j\}$.

Remark 1.2. *When $f^{-1}(Q_j) = g^{-1}(Q_j)$ for all $j = 1, \ldots, q$, and $f(z) = g(z)$ on $\cup_{j=1}^q f^{-1}(Q_j)$, we can choose $\delta = 0$ in above theorem. Then we see that all assumptions of Theorem 1.7 are satisfied. Set $m_1 = \cdots = m_q = \infty$, we get* *the uniqueness result with number moving hypersurfaces are smaller than the number moving hypersurfaces in the result of Dethloff-Tan [5]. Furthermore, we do not need the assumption f and g are nondegenerate algebraically over $\mathcal{K}_{\{Q_i\}_{i=1}^q}$. Hence, Theorem 1.7 is an improvement the result due to Dethloff-Tan [5].*

Theorem 1.8. *Let f and g be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $\tilde{f}$ and $\tilde{g}$ be reduced representation of f and g, respectively. Let $Q_i, i = 1, \ldots, q$, be slow (with respect to f and g) moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in general position with respect to the Veronese embedding such that $Q_i(\tilde{f}) \not\equiv 0$ and $Q_i(\tilde{g}) \not\equiv 0$ $(1 \le i \le q)$. Set $\deg Q_i = d_i$ and put $d = lcm(d_1, \ldots, d_q)$, $N = \binom{n+d}{n} - 1$. Let $m_j (j = 1, \ldots, q)$ be positive integers or ∞ with $m_1 \ge \cdots \ge m_q$. Assume that*

(a) $\dim(f^{-1}(Q_i) \cap f^{-1}(Q_j)) \le m - 2$ for all $1 \le i < j \le q$.

(b) Suppose that $f(z) = g(z)$ on $\cup_{j=1}^q \big(\overline{E}_{m_j)}(Q_j, f) \cap \overline{E}_{m_j)}(Q_j, g)\big)$, where

$$\cup_{j=1}^q \big(\overline{E}_{m_j)}(Q_j, f) \cap \overline{E}_{m_j)}(Q_j, g)\big) \neq \emptyset$$

outside a analytic subset with codimension at most 2.

(c) Let $\delta \ge 0$ be a real number. Set $A_j = \overline{E}_{m_j)}(Q_j, f) \setminus \overline{E}_{m_j)}(Q_j, g)$ for $j = 1, \ldots, q$. Suppose that $\sum_{j=1}^q N_f(r, A_j) \le \delta T(r, f)$,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^q N^N_{f,\le m_j}(r, Q_j)}{\sum_{j=1}^q N^N_{g,\le m_j}(r, Q_j)} > \frac{m_1 N}{d'(1+m_1)\Big(\dfrac{q-N+1}{N+2} - \sum_{i=1}^q \dfrac{N}{m_i+1}\Big) - N(1+\delta)m_1}$$

and

$$q > (N+2)\Big(\sum_{i=1}^q \frac{N}{m_i+1} + \frac{N(2+\delta)m_1}{d'(1+m_1)}\Big) + N - 1,$$

then $f \equiv g$, where $d' = \min_{j=1,\ldots,q}\{d_j\}$.

2. Preliminaries

We set $||z|| = (\sum_{j=1}^m |z_j|^2)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$. For $r > 0$, define $B(r) = \{z \in \mathbb{C}^m : ||z|| < r\}$, $S(r) = \{z \in \mathbb{C}^m : ||z|| = r\}$, $d^c = \dfrac{1}{4\pi i}(\partial - \overline{\partial})$, $v = (dd^c||z||^2)^{m-1}$ and $\sigma = d^c \log ||z||^2 \wedge (dd^c||z||^2)^{m-1}$.

Let h be a nonzero entire function on $\mathbb{C}^m$. For $a \in \mathbb{C}^m$, we can write h as $h(z) = \sum_{n=0}^{\infty} P_n(z-a)$, where $P_n(z)$ is either identically zero or a homogeneous polynomial with degree n. The number $\nu_h(a) = \min\{n : P_n \neq 0\}$ is said to be the zero multiplicity of h at a. Set $supp\nu_h := \overline{\{z \in \mathbb{C}^m : \nu_h(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset or empty set.

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, we choose holomorphic functions φ_0 and φ_1 on a neighborhood U of a such that $\varphi = \dfrac{\varphi_0}{\varphi_1}$ on U and $dim((\varphi_0^{-1} \cap \varphi_1^{-1})(0)) \leq m-2$, and we define $\nu_\varphi := \nu_{\varphi_0}, \nu_\varphi^\infty := \nu_{\varphi_1}$, which are independent of the choices of φ_0 and φ_1.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. We can choose holomorphic function $f_0, f_1, \ldots, f_n$ on $\mathbb{C}^m$ such that $I_f := \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $f = (f_0 : \cdots : f_n)$. Usually, $\tilde{f} = (f_0, \ldots, f_n)$ is a reduced representation of f. The characteristic function of f is defined by

$$T(r,f) = \int\limits_{S(r)} \log ||\tilde{f}||\sigma - \int\limits_{S(1)} \log ||\tilde{f}||\sigma \ (r > 1),$$

where $||\tilde{f}|| = (\sum_{j=0}^{n} |f_j|^2)^{1/2}$. Note that $T(r,f)$ is independent, up to an additive constant, of the choice of the reduced representation of f.

For a hyperplane $H = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(\mathbb{C}) : a_0x_0 + \cdots + a_nx_n = 0\}$. We denote $(\tilde{f}, H) = \sum_{j=0}^{n} a_j f_j$. Suppose that $f(\mathbb{C}^m) \not\subset H$, then the proximity function of f with respective to H is defined by

$$m_f(r,H) = \int\limits_{S(r)} \log \frac{||\tilde{f}||.||H||}{|(\tilde{f},H)|}\sigma - \int\limits_{S(1)} \log \frac{||\tilde{f}||.||H||}{|(\tilde{f},H)|}\sigma \ (r > 1),$$

where $||H|| = (\sum_{j=0}^{n} |a_j|^2)^{1/2}$. The function $m_f(r,H)$ is independent, up to an additive constant, of the choice of the reduced representation of f.

On $\mathbb{C}^m$, every norms are equivalent, then we may choose $||\tilde{f}|| = \max\{|f_0|, \ldots, |f_n|\}$ and $||H|| = |a_0| + \cdots + |a_n|$ in above definitions.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial defining D. Under the assumption that $Q(\tilde{f}) \not\equiv 0$, the proximity function $m_f(r,D)$ of f is defined by

$$m_f(r,D) = \int\limits_{S(r)} \log \frac{||\tilde{f}||^d||Q||}{|Q(\tilde{f})|}\sigma - \int\limits_{S(1)} \log \frac{||\tilde{f}||^d||Q||}{|Q(\tilde{f})|}\sigma \ (r > 1),$$

where $||Q||$ is the total absolute the coefficients of Q. We see that $m_f(r,D)$ is independent, up to an additive constant, of the choice of the reduced representation of f.

Let k, M be positive integers or $+\infty$. For a divisor ν on $\mathbb{C}^m$. We define the counting function of ν as follows. Set

$$\nu^M(z) = \min\{\nu(z), M\},\ \nu^M_{\leq k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k \\ \nu^M(z) & \text{if } \nu(z) \leq k \end{cases}$$

and

$$\nu^M_{\geq k}(z) = \begin{cases} 0 & \text{if } \nu(z) < k \\ \nu^M(z) & \text{if } \nu(z) \geq k \end{cases}.$$

We denote

$$n(t) = \begin{cases} \int_{supp\nu \cap B(t)} \nu(z)v & \text{if } n \geq 2 \\ \sum_{|z|\leq t} \nu(z) & \text{if } n = 1 \end{cases}.$$

Similarly, we can define $n^M(t), n^M_{\geq k}(t)$ and $n^M_{\leq k}(t)$. We define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2n-1}} dt \ (r > 1).$$

We can define $N(r, \nu^M), N(r, \nu^M_{\leq k})$ and $N(r, \nu^M_{\geq k})$ similarly and denote by $N^M(r, \nu)$, $N^M_{\leq k}(r, \nu)$ and $N^M_{\geq k}(r, \nu)$, respectively.

For a meromorphic function φ on $\mathbb{C}^m$, we denote by

$$N_\varphi(r) = N(r, \nu_\varphi), N^M_\varphi(r) = N(r, \nu^M_\varphi),$$

$$N^M_{\varphi,\leq k}(r) = N(r, \nu^M_{\varphi,\leq k}), N^M_{\varphi,\geq k}(r) = N(r, \nu^M_{\varphi,\geq k}).$$

Let D be a hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with degree d which is defined by the homogeneous Q. Suppose that $f(\mathbb{C}^m) \not\subset D$, we denote $\nu_{Q(\tilde{f})}$ the map from $\mathbb{C}^m$ into $\mathbb{Z}$ whose value $\nu_{Q(\tilde{f})}(z)$ $(z \in \mathbb{C}^m)$ is the intersection multiplicity of the images of f and Q at $f(z)$. Then $\nu_{Q(\tilde{f})}$ is a divisor on $\mathbb{C}^m$. We denote by

$$N_f(r, D) = N(r, \nu_{Q(\tilde{f})}), N^M_f(r, D) = N(r, \nu^M_{Q(\tilde{f})}),$$

$$N^M_{f,\leq k}(r, D) = N(r, \nu^M_{Q(\tilde{f}),\leq k}), N^M_{f,\geq k}(r, D) = N(r, \nu^M_{Q(\tilde{f}),\geq k}).$$

For a close subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$, we define

$$n_A(t) = \begin{cases} \int\limits_{A \cap B(t)} v & \text{if } m \geq 2 \\ |A \cap B(t)| & \text{if } m = 1 \end{cases}$$

and

$$N(r, A) = \int_1^r \frac{N_A(t)}{t^{2m-1}} dt \ (r > 1).$$

The Poisson-Jensen formula implies the First Main Theorem.

Theorem 2.1. *[14] Let $f = (f_0 : \cdots : f_m) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping, and let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d If $f(\mathbb{C}^m) \not\subset D$, then for every real number r with $1 < r < +\infty$,*

$$dT_f(r) = m_f(r, D) + N_f(r, D) + O(1),$$

where $O(1)$ is a constant independent of r.

As usual, "$\| P$" means the assertion P holds for all $r \in [1, \infty)$ outside a subset with Lebesgue finite. In 2017, S. D. Quang and D. P. An have get the following results.

Theorem 2.2. *[12] Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k (k \leq n)$. Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respective to V, with $\deg Q_i = d_i, 1 \leq i \leq q$. Let d be the least common multiple of $d_1, \ldots, d_q$. Let f be a meromorphic mappings of $\mathbb{C}^m$ into V which is nondegenerate over $I_d(V)$. If $q > \dfrac{(2N - k + 1)H_V(d)}{k + 1}$, then we have*

$$\| \big(q - \frac{(2N - k + 1)H_V(d)}{k + 1}\big) T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N_f^{H_V(d)-1}(r, D_i) + o(T(r, f)).$$

Theorem 2.3. *[13] Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $Q_i (i = 1, \ldots, q)$ be slowly (with respective f) moving hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ with degree $\deg Q_i = d_i$. Set $d = lcm(d_1, \ldots, q_q)$ and $N = \binom{n+d}{n} - 1$. Assume that $Q_i(\tilde{f}) \not\equiv 0 (1 \leq i \leq q)$. If $q \geq 2N + 1$ and $\{Q_i\}_{i=1}^q$ are in general position with respect to the Veronese embedding, then*

$$\| \frac{q - N + 1}{N + 2} T(r, f) \leq \sum_{i=1}^{q} \frac{1}{d_i} N_f^N(r, Q_i) + o(T(r, f)).$$

If $\{Q_i\}_{i=1}^q$ are in weakly general position, then S. D. Quang and D. P. An obtained the result as follows:

Theorem 2.4. *[13] Let f be a meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $Q_i (i = 1, \ldots, q)$ be slowly (with respective f) moving hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in weakly general position with $\deg Q_i = d_i$. Set $d = lcm(d_1, \ldots, d_q)$ and $N = \binom{n+d}{n} - 1$. Assume that $Q_i(\tilde{f}) \not\equiv 0 (1 \leq i \leq q)$ and $q \geq nN + n + 1$, the we have*

$$\| \frac{q - (n - 1)(N + 1)}{nN + n + 1} T(r, f) \leq \sum_{i=1}^{q} \frac{1}{d_i} N_f^N(r, Q_i) + o(T(r, f)).$$

3. Proof of Theorems

3.1. Proof of Theorem 1.6

Suppose that $f \not\equiv g$. Then there exists two distinct indices $s, t \in \{0, \dots, n\}$ such that
$$\Phi := f_s g_t - f_t g_s \not\equiv 0.$$
From assumption (a) and (b), for any z in $\cup_{j=1}^{q}\big(\overline{E}_{m_j)}(D_j, f) \cap \overline{E}_{m_j)}(D_j, g)\big)$ outside a analytic subset with codimension at least 2 is a zero of Φ, and
$$N_{\Phi}(r) \leq T_f(r) + T_g(r) + O(1). \tag{1.1}$$
Apply Theorem 2.2, we have
$$\begin{aligned}
\| \big(q - \frac{(2N-k+1)H_V(d)}{k+1}\big) T_f(r) &\leq \sum_{i=1}^{q} \frac{1}{d_i} N_f^{H_V(d)-1}(r, D_i) + o(T(r,f)) \\
&= \sum_{i=1}^{q} \frac{1}{d_i} \big(N_{f,\leq m_i}^{H_V(d)-1}(r, D_i) + N_{f,\geq m_i+1}^{H_V(d)-1}(r, D_i)\big) + o(T(r,f)) \\
&\leq \sum_{i=1}^{q} \frac{1}{d_i} \big(N_{f,\leq m_i}^{H_V(d)-1}(r, D_i) + (H_V(d)-1) N_{f,\geq m_i+1}^{1}(r, D_i)\big) + o(T(r,f)) \\
&\leq \sum_{i=1}^{q} \frac{1}{d_i} \big(N_{f,\leq m_i}^{H_V(d)-1}(r, D_i) + \frac{H_V(d)-1}{m_i+1} N_{f,\geq m_i+1}(r, D_i)\big) + o(T(r,f)). \qquad (1.2)
\end{aligned}$$
By First Main Theorem, we have
$$N_f(r, D_i) \leq d_i T(r,f) + O(1). \tag{1.3}$$
Then from (1.2) and (1.3), we have
$$\begin{aligned}
\| \Big(q - \frac{(2N-k+1)H_V(d)}{k+1}\Big) T_f(r) \leq \sum_{i=1}^{q} \Big(& \frac{m_i}{d_i(m_i+1)} N_{f,\leq m_i}^{H_V(d)-1}(r, D_i) \\
&+ \frac{H_V(d)-1}{m_i+1} T(r,f)\Big) + o(T(r,f)).
\end{aligned}$$
This implies
$$\begin{aligned}
\| \Big(q - \frac{(2N-k+1)H_V(d)}{k+1} - \sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1}\Big) T(r,f) & \\
\leq \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N_{f,\leq m_i}^{H_V(d)-1}(r, D_i) + o(T(r,f)). & \qquad (1.4)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\Big\| \Big(q - \frac{(2N-k+1)H_V(d)}{k+1} - \sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1}\Big) T(r,g) \\
&\qquad \le \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N^{H_V(d)-1}_{g,\le m_i}(r,D_i) + o(T(r,g)). \qquad (1.5)
\end{aligned}$$

Set $B_j = \overline{E}_{m_j)}(D_j,f) \setminus A_j = \overline{E}_{m_j)}(D_j,f) \cap \overline{E}_{m_j)}(D_j,g) (j = 1,\dots,q)$. From the assumption (c) and (1.1), we have

$$\begin{aligned}
\sum_{j=1}^{q} N^{H_V(d)-1}_{f,\le m_j}(r,D_i) &= \sum_{j=1}^{q} N^{H_V(d)-1}_{f,\le m_j}(r,A_j) + \sum_{j=1}^{q} N^{H_V(d)-1}_{f,\le m_j}(r,B_j) \\
&\le (H_V(d)-1)\sum_{j=1}^{q} N^{1}_{f,\le m_j}(r,A_j) + (H_V(d)-1)\sum_{j=1}^{q} N^{1}_{f,\le m_j}(r,B_j) \\
&\le (H_V(d)-1)\delta T(r,f) + (H_V(d)-1)N_\Phi(r) \\
&\le (H_V(d)-1)(1+\delta)T(r,f) + (H_V(d)-1)T(r,g). \qquad (1.6)
\end{aligned}$$

Hence, from (1.4), (1.5) and (1.6), we obtained

$$\begin{aligned}
&\Big(q - \frac{(2N-k+1)H_V(d)}{k+1} - \sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1}\Big) \sum_{j=1}^{q} N^{H_V(d)-1}_{f,\le m_j}(r,D_i) \\
&\le (H_V(d)-1)(1+\delta) \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N^{H_V(d)-1}_{f,\le m_i}(r,D_i) \\
&+ (H_V(d)-1) \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N^{H_V(d)-1}_{g,\le m_i}(r,D_i) + o(T(r,f)+T(r,g)). \qquad (1.7)
\end{aligned}$$

Set $d' = \min_{i=1,\dots,q}\{d_i\}$. Since $1 \ge \frac{m_1}{1+m_1} \ge \cdots \ge \frac{m_q}{1+m_q} \ge \frac{1}{2}$, then from (1.7), we have

$$\begin{aligned}
&\Big(q - \frac{(2N-k+1)H_V(d)}{k+1} - \sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1} - \frac{(H_V(d)-1)(1+\delta)m_1}{d'(1+m_1)}\Big) \\
&\quad \times \sum_{i=1}^{q} N^{H_V(d)-1}_{f,\le m_i}(r,D_i) \\
&\quad \le \frac{m_1(H_V(d)-1)}{d'(1+m_1)} \sum_{i=1}^{q} N^{H_V(d)-1}_{g,\le m_i}(r,D_i) + o(T(r,f)+T(r,g)). \qquad (1.8)
\end{aligned}$$

From (1.4) and (1.6), we have $T(r,f) \le O(T(r,g))$, then

$$o(T(r,f)) = o(T(r,g)). \tag{1.9}$$

Furthermore, from (1.5), we have

$$\frac{T(r,g)}{\sum_{i=1}^{q} N_{g,\le m_i}^{H_V(d)-1}(r,D_i)} \tag{1.10}$$

is bounded as $r \to \infty$. Thus, combining (1.8) to (1.10), we deduce

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^{q} N_{f,\le m_i}^{H_V(d)-1}(r,D_i)}{\sum_{i=1}^{q} N_{g,\le m_i}^{H_V(d)-1}(r,D_i)}$$
$$\le \frac{m_1(H_V(d)-1)}{d'(1+m_1)\Big(q - \dfrac{(2N-k+1)H_V(d)}{k+1} - \displaystyle\sum_{i=1}^{q} \frac{H_V(d)-1}{m_i+1}\Big) - (H_V(d)-1)(1+\delta)m_1}.$$

This is a contradiction with assumption of Theorem 1.6. Then $f \equiv g$. ■

3.2. Proof of Theorem 1.7

By arguments as Theorem 1.6, there exists two distinct indices $s,t \in \{0,\dots,n\}$ such that

$$\Phi := f_s g_t - f_t g_s \not\equiv 0.$$

From assumption (a) and (b), for any z in $\cup_{j=1}^{q}\big(\overline{E}_{m_j)}(Q_j,f) \cap \overline{E}_{m_j)}(Q_j,g)\big)$ outside a analytic subset with codimension at least 2 is a zero of Φ, and

$$N_\Phi(r) \le T_f(r) + T_g(r) + O(1). \tag{1.11}$$

Apply Theorem 2.4, the inequalities (1.4) and (1.5) are replaced by

$$\begin{aligned} &\|\Big(\frac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^{q} \frac{N}{m_i+1}\Big)T(r,f) \\ &\qquad \le \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N_{f,\le m_i}^{N}(r,Q_i) + o(T(r,f)) \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} &\|\Big(\frac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^{q} \frac{N}{m_i+1}\Big)T(r,g) \\ &\qquad \le \sum_{i=1}^{q} \frac{m_i}{d_i(m_i+1)} N_{g,\le m_i}^{N}(r,Q_i) + o(T(r,g)), \end{aligned} \tag{1.13}$$

respectively. Set $B_j = \overline{E}_{m_j)}(Q_j, f) \setminus A_j = \overline{E}_{m_j)}(Q_j, f) \cap \overline{E}_{m_j)}(Q_j, g)(j = 1, \ldots, q)$. From the assumption (c) and (1.11), we have

$$\begin{aligned}\sum_{j=1}^{q} N^N_{f,\le m_j}(r, Q_i) &= \sum_{j=1}^{q} N^N_{f,\le m_j}(r, A_j) + \sum_{j=1}^{q} N^N_{f,\le m_j}(r, B_j)\\ &\le N\sum_{j=1}^{q} N^1_{f,\le m_j}(r, A_j) + N\sum_{j=1}^{q} N^1_{f,\le m_j}(r, B_j)\\ &\le N\delta T(r,f) + NN_{\Phi}(r)\\ &\le N(1+\delta)T(r,f) + NT(r,g). \qquad (1.14)\end{aligned}$$

Combining (1.12) to (1.14), we get

$$\begin{aligned}&\Big(\frac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^{q}\frac{N}{m_i+1}\Big)\sum_{j=1}^{q} N^N_{f,\le m_j}(r, Q_i)\\ &\le N(1+\delta)\sum_{i=1}^{q}\frac{m_i}{d_i(m_i+1)}N^N_{f,\le m_i}(r, Q_i)\\ &+ N\sum_{i=1}^{q}\frac{m_i}{d_i(m_i+1)}N^N_{g,\le m_i}(r, Q_i) + o(T(r,f)+T(r,g)). \qquad (1.15)\end{aligned}$$

Set $d' = \min_{i=1,\ldots,q}\{d_i\}$. Since $1 \ge \dfrac{m_1}{1+m_1} \ge \cdots \ge \dfrac{m_q}{1+m_q} \ge \dfrac{1}{2}$, then from (1.15), we have

$$\begin{aligned}&\Big(\frac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^{q}\frac{N}{m_i+1} - \frac{N(1+\delta)m_1}{d'(1+m_1)}\Big)\sum_{i=1}^{q} N^N_{f,\le m_i}(r, D_i)\\ &\le \frac{m_1 N}{d'(1+m_1)}\sum_{i=1}^{q} N^N_{g,\le m_i}(r, Q_i) + o(T(r,f)+T(r,g)). \qquad (1.16)\end{aligned}$$

From (1.16), we deduce

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{i=1}^{q} N^N_{f,\le m_i}(r, Q_i)}{\sum_{i=1}^{q} N^N_{g,\le m_i}(r, Q_i)}\\ &\le \frac{m_1 N}{d'(1+m_1)\Big(\dfrac{q-(n-1)(N+1)}{nN+n+1} - \sum_{i=1}^{q}\dfrac{N}{m_i+1}\Big) - N(1+\delta)m_1}.\end{aligned}$$

This is a contradiction with assumption of Theorem 1.7. Then $f \equiv g$. ∎

3.3. Proof of Theorem 1.8

Proof Theorem 1.8 is proved similarly Theorem 1.6 and Theorem 1.7 by using Theorem 2.3. Hence, we omit it here. ■

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